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Fernando Albiac
Nigel J. Kalton

# Topics in Banach Space Theory

*Second Edition*



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# Topics in Banach Space Theory

Second Edition



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Nigel J. Kalton
(Deceased)

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*To the living memory of Nigel J. Kalton*
*(1946–2010)*

# Foreword

Ten years ago, Fernando Albiac and Nigel Kalton completed their book *Topics in Banach Space Theory*. Sometimes, the mathematical community recognizes the true value of a fine work. Fortunately, that is what happened with this book, which has been successful and influential, to such an extent that a second edition is necessary.

It is appropriate to ponder this success. As explained in the preface to the first edition, this book grew out of two graduate courses delivered by Nigel Kalton at the University of Missouri–Colombia. It therefore reflects the enlightening vision of a master in this field: a cascade falling from so high is a powerful force, and a beautiful sight. But after the cascade, the stream and the river flow. Once the work was put into motion, much remained to be done. And the authors' wisdom led them to gather their "topics in Banach space theory" according to the following rules: choose proofs that are accessible to graduate students, pick positive results in the various fields where Banach space is the operative word, make every attempt to address the largest possible audience, explain concisely counterexamples without fully displaying the technicalities. Their efforts resulted in a well-balanced, instructive, and reader-friendly masterpiece.

Fate had in store that Fernando Albiac faced alone the task of producing the second edition of this masterpiece. Nigel Kalton passed away on August 31, 2010. This disaster left his friends and coauthors alike devastated, but aware of their duty to his mathematical legacy. Fernando Albiac met this challenge in the best possible way, with comprehensive work on the text and the addition of two completely new chapters on recently explored fields of research. The style of the book, however, remains the same: a pleasant walk from basic functional analysis to the border of present knowledge.

Are you ready for this walk? Hopefully yes, since it will be nice. Please read, and enjoy.

Paris, France
November 2015

Gilles Godefory

# Preface to the Second Edition

On a very cold morning in early January 2006, I stopped by the Math Department at the University of Missouri–Colombia, where I was a visiting assistant professor at that time. In my mailbox was the first printed copy of *Topics in Banach Space Theory* [6], the book that Nigel Kalton and I had started working on in 2001. Full of excitement, I drove to Nigel's place to share the surprise with him. Jenny, his wife, opened the door and called Nigel, who was working in his basement study. I could see that my turning up unannounced had taken him aback, and when I handed the book to him, after a brief gesture of inner satisfaction that illuminated his face, he grabbed it with both hands and in a dramatic theatrical voice exclaimed, "But we didn't realize on time that it contained an error that everyone will notice!" To which I replied, "Don't worry, we'll have the opportunity to fix it for the second edition." I was joking, of course. And we laughed, because we were feeling happy enough with one edition of the book, and also because nothing seemed more remote to us in that moment than selling out the first printing, or working toward a second edition after all those years of intensive dedication that the first one had required.

Ten years have passed now since *Topics in Banach Space Theory* was released. The last time I saw Nigel Kalton was at the functional analysis conference held in Valencia in June 2010. We had met the very first day of the event right before the inaugural speech. Kind of early for his accustomed schedule, I thought when I saw him arrive. Later on, he explained that in fact, it was kind of late, since he had not gone to sleep yet. That summer was particularly hectic for Nigel. He was traveling back and forth across the Atlantic to participate in invited conferences in Europe and the United States. On that occasion, to avoid the jet lag that was awaiting him back home upon his return right at the end of the week, he had decided to stay up at night. When he saw me hanging out by the registration desk, Nigel approached me with two bags of presents that he and Jenny had purchased for the birth of my son Julen two months earlier, and that he had carried all the way from Missouri. I perfectly remember his face of relief when he finally handed over the load: "All yours!" After that, unlike other times that we had met at conferences, we didn't interact much. He was keeping his own parallel schedule, and would withdraw to get some rest shortly after the first afternoon sessions. Despite his somewhat tired look, he delivered three

memorable lectures that evidenced his excellent mathematical shape, the vastness of the scope of his research, the depth of his insights, and the beauty of the mathematics he produced. On one of the few occasions when we had a chat at the conference hall, I told him that Springer was interested in hearing our proposal for a second edition of our book. The project attracted his interest, and we briefly sketched out the themes that we would like to propose to Springer for the new edition. At the end of the meeting we agreed to take up our discussion after the summer in order to submit the book proposal as soon as possible. Unfortunately, that was not to happen. Nigel left us suddenly and unexpectedly three months afterward, on August 31, 2010. He was 64 years old.

Nigel's passing left an irreparable void in many of us. But our memories of him and the inspiration he transmitted will always be our companions. In the process of completing this edition of *Topics in Banach Space Theory* in Nigel's absence, I have faced the hard task to give form to his vision while maintaining his standards of quality. My apologies in advance to any readers who feel I have fallen short.

**Contents and Arrangement** This edition will maintain the chapter flow and the contents of the first edition through Chapter 9. Chapters 10, 11, 12, and 13 of the first edition are now Chapters 11, 12, 13, and 15, respectively. The present edition differs from the first mainly in that two new chapters, devoted to greedy bases and to the nonlinear geometry of Banach spaces, have been added. These two chapters are the lengthiest of the book, and the corresponding topics are so vast that each of them would require a whole fat book to be properly covered.

- GREEDY BASES were tackled very briefly in the last section of Chapter 9 of the first edition. The material it contained has been widely expanded to include other forms of greediness and relations among them in the new Chapter 10. The subject of finding estimates for the rate of approximation of a function by means of essentially nonlinear algorithms with respect to biorthogonal systems and, in particular, the greedy approximation algorithm using bases, has attracted much attention over the last 15 years, on the one hand from researchers interested in the applied nature of nonlinear approximation, and on the other hand from researchers with a more classical Banach space theory background. Although the basic idea behind the concept of a greedy basis had been around for some time, the formal development of a theory of greedy bases was initiated in 1999 by Konyagin and Temlyakov in the important paper [176]. Subsequently, the theory of greedy bases and its derivates developed very rapidly as many fundamental results were discovered, leading to new ramifications. As a result, this is an area with a fruitful interplay between abstract methods from classical Banach space theory and other, more concrete, techniques from approximation theory; we refer to [293] for a recent textbook in this applied direction. In Chapter 10 we will concentrate on the Banach space aspects of this theory, where, rather unexpectedly, the theory of greedy bases has links to old classical results and also to some open problems. The idea of studying greedy bases and related greedy algorithms from a more abstract point of view seems to have originated with the work of Wojtaszczyk [305] and the work of Dilworth et al. [66].

- Chapter 14 focuses on the NONLINEAR GEOMETRY OF BANACH SPACES. This is a topic in which the most significant progress has been made in recent years and is of much current interest among functional analysts. It is also an area of research that Nigel was particularly keen on and productive in for the last years of his life. His posthumous papers [157–160] have contributed to advancing the state of the art of the subject and are a source of inspiration to delve deeper into the matter. The fundamental question of nonlinear geometry is to determine to what extent the metric structure of a Banach space already determines its linear structure. The subject is an old one, dating back almost to the origins of Banach space theory. Perhaps the first result in the area is the celebrated Mazur–Ulam theorem on isometries from 1932 [217]. Later, the work of Lindenstrauss [191], Kadets [145], and Enflo [84–86] in the 1960s gave a tremendous impetus to the study of Banach spaces as metric spaces. The explosion of interest in the linear theory of Banach spaces between 1960 and 2000 spurred some significant advances in nonlinear theory. However, it is since about 2000 that there has been a quite marked increase of activity in this area. There are several reasons for this. First, there is the appearance of the authoritative book *Geometric Nonlinear Functional Analysis* of Benyamini and Lindenstrauss [23]. This book finally gave a definitive form to the subject and highlighted both what we know and what we do not know. Then there has been an upsurge of interest from other areas of mathematics in the problem of determining how well a metric space can be embedded in a particular type of Banach space. This sort of problem is of interest to theoretical computer scientists (see, e.g., [35, 207]) in connection with data mining and to specialists in $C^*$-algebras in connection with the Novikov conjecture (see, e.g., [166, 307]). Thus the amount of available information by now is so vast that here we can hope to provide only an introduction to the theory. We will include an exposition of the basic tools that have arisen in the study of the Lipschitz and uniform geometries of the classical spaces, covering the uniqueness of Lipschitz structure of $\ell_p$ and $L_p$ for $1 < p < \infty$ and also the uniqueness of uniform structure of $\ell_p$ for $1 < p < \infty$. The reader is referred to [108] for an account of recent developments in the theory.

**Other Changes and Improvements** This new edition has given me the opportunity to implement the comments that our readers have been sending and make it more error-free. I have rewritten parts of Sections 3.1, 3.2, 4.1, 6.1, and 12.1 (was Section 11.1 in the first edition). Chapter 5 contains the new Section 5.5 on the emergence of the Radon–Nikodym property, and I have revised thoroughly what now is Chapter 13 in order to clarify and correct certain details that appeared to be garbled in the first edition. One of the items in my to-do list for this edition was to fill in the appendices with some more introductory stuff and complete them with a few proofs. Finally, I just added four new appendix chapters in which I gathered some background on duality in $L_p(\mu)$, probability theory, ultraproducts, and Bochner integration. The rest is wonderfully taken care of in the recently published book *An Introductory Course in Functional Analysis* [33], jointly written by Adam Bowers and Nigel Kalton.

**Acknowledgements** I would like to express my most sincere appreciation to all of those who have helped me in one way or another in the process of completing this second edition in the absence of Nigel. Without them, this book that you now have in your hands would not be a reality.

First of all, I owe Prof. Gilles Godefroy my deepest gratitude. Gilles convinced me that I should rise to the challenge of a second edition and offered me his invaluable knowledge and collaboration to write Chapter 14 together. He also agreed to write the foreword for the present edition. Nigel would have been moved by Gilles's beautiful words.

I was very glad when another friend and collaborator of Nigel, Prof. Przemysław Wojtaszczyk, accepted my invitation to write a first draft of what in time would become Chapter 10. Przemek supervised the evolution of the contents of the chapter, proposed a list of problems, and gave me hints that improved its final version.

I shall always be indebted to my colleague and friend Prof. José Luis Ansorena for his involvement and great support for the duration of this project. *Anso* helped me to complete and edit the contents of the two new chapters and assisted me in polishing countless details from the first edition.

I feel obliged to the editorial board of *Revista Matemática Complutense*, and to Prof. Paco Hernández in particular, for granting me permission to use excerpts from the introduction, and Sections 1 and 2 of Nigel's survey article [156] in Chapter 14. It meant a lot to me to be able to graft this chapter, devoted to a subject that he loved so much, with Nigel's own words.

I thank all the colleagues who used the book for their courses and who sent us comments or corrections that contributed to reducing the number of errors in this edition. In particular, I want to thank George Androulakis, Pablo Berná, Óscar Blasco, Piotr Koszmider, Iwo Labuda, Gilles Lancien, Antonio Martínez-Abejón, Mikhail Ostrovskii, Colin Petitjean, Grzegorz Plebanek, and especially Florent Baudier.

I am also grateful to the staff at Springer. To former associate editor Kaitlin Leach for conceiving a second edition of *Topics in Banach Space Theory* while Nigel was still with us, for bearing with me through a long process of not knowing what to do, for waiting for three years until I was ready to take the step of submitting the proposal by myself, and for accepting it with enthusiasm so quickly. And also to executive editor Elizabeth Loew for her kind assistance in making the last stages of the production of the book as smooth and easy as possible.

I would like to acknowledge the support from the government of Spain's grants MTM2014-53009-P and MTM2012-31286. P. Wojtaszczyk's contribution to Chapter 10 was supported by Polish NCN grant DEC2011/03/B/ST1/04902.

At a crucial point in my career I was blessed to cross paths with Nigel Kalton. Working closely with him, getting to know him personally, and sharing a bit of his vast mathematical wisdom, which he carried with so much humility, have been some of the best gifts in my life. Thank you Nigel, wherever you are.

Pamplona, Spain
January 2016

Fernando Albiac

# Preface to the First Edition

This book grew out of a one-semester course given by the second author in 2001 and a subsequent two-semester course in 2004–2005, both at the University of Missouri-Colombia. The text is intended for a graduate student who has already had a basic introduction to functional analysis; the aim is to give a reasonably brief and self-contained introduction to classical Banach space theory.

Banach space theory has advanced dramatically in the last 50 years and we believe that the techniques that have been developed are very powerful and should be widely disseminated among analysts in general and not restricted to a small group of specialists. Therefore we hope that this book will also prove of interest to an audience who may not wish to pursue research in this area but still would like to understand what is known about the structure of the classical spaces.

Classical Banach space theory developed as an attempt to answer very natural questions on the structure of Banach spaces; many of these questions date back to the work of Banach and his school in Lvov. It enjoyed, perhaps, its golden period between 1950 and 1980, culminating in the definitive books by Lindenstrauss and Tzafriri [203] and [204], in 1977 and 1979 respectively. The subject is still very much alive, but the reader will see that much of the basic groundwork was done in this period.

We will be interested specifically in questions of the following type: given two Banach spaces $X$ and $Y$, when can we say that they are linearly isomorphic, or that $X$ is linearly isomorphic to a subspace of $Y$? Such questions date back to Banach's book in 1932 [18], where they are treated as *problems of linear dimension*. We want to study these questions particularly for the classical Banach spaces, that is, the spaces $c_0$, $\ell_p$ $(1 \leq p \leq \infty)$, spaces $\mathcal{C}(K)$ of continuous functions, and the Lebesgue spaces $L_p$, for $1 \leq p \leq \infty$.

At the same time, our aim is to introduce the student to the fundamental techniques available to a Banach space theorist. As an example, we spend much of the early chapters discussing the use of Schauder bases and basic sequences in the theory. The simple idea of extracting basic sequences in order to understand subspace structure has become second nature in the subject, and so the importance of this notion is too easily overlooked.

It should be pointed out that this book is intended as a text for graduate students, not as a reference work, and we have selected material with an eye to what we feel can be appreciated relatively easily in a quite leisurely two-semester course. Two of the most spectacular discoveries in this area during the last 50 years are Enflo's solution of the basis problem [88] and the Gowers–Maurey solution of the unconditional basic sequence problem [116]. The reader will find discussion of these results but no presentation. Our feeling, based on experience, is that detouring from the development of the theory to present lengthy and complicated counterexamples tends to break up the flow of the course. We prefer therefore to present only relatively simple and easily appreciated counterexamples such as the James space and Tsirelson's space. We also decided, to avoid disruption, that some counterexamples of intermediate difficulty should be presented only in the last optional chapter and not in the main body of the text.

Let us describe the contents of the book in more detail. Chapters 1, 2 and 3 are intended to introduce the reader to the methods of bases and basic sequences and to study the structure of the sequence spaces $\ell_p$ for $1 \le p < \infty$ and $c_0$. We then turn to the structure of the classical function spaces. Chapters 4 and 5 concentrate on $\mathcal{C}(K)$-spaces and $L_1(\mu)$-spaces; much of the material in these chapters is very classical indeed. However, we do include Miljutin's theorem that all $\mathcal{C}(K)$-spaces for $K$ uncountable compact metric are linearly isomorphic in Chapter 4; this section (Section 4.4) and the following one (Section 4.5) on $\mathcal{C}(K)$-spaces for $K$ countable can be skipped if the reader is more interested in the $L_p$-spaces, as they are not used again. Chapters 6 and 7 deal with the basic theory of $L_p$-spaces. In Chapter 6 we introduce the notions of type and cotype. In Chapter 7 we present the fundamental ideas of Maurey–Nikishin factorization theory. This leads into the Grothendieck theory of absolutely summing operators in Chapter 8. Chapter 9 is devoted to problems associated with the existence of certain types of bases. In Chapter 11 we introduce Ramsey theory and prove Rosenthal's $\ell_1$-theorem; we also cover Tsirelson space, which shows that not every Banach space contains a copy of $\ell_p$ for some $p$, $1 \le p < \infty$, or $c_0$. Chapters 12 and 13 introduce the reader to local theory from two different directions. In Chap. 12 we use Ramsey theory and infinite-dimensional methods to prove Krivine's theorem and Dvoretzky's theorem, while in Chapter 13 we use computational methods and the concentration of measure phenomenon to prove again Dvoretzky's theorem. Finally, Chapter 15 covers, as already noted, some important examples which we removed from the main body of the text.

The reader will find all the prerequisites we assume (without proofs) in the appendices. In order to make the text flow rather more easily we decided to make a default assumption that all Banach spaces are real. That is, unless otherwise stated, we treat only real scalars. In practice, almost all the results in the book are equally valid for real or complex scalars, but we leave to the reader the extension to the complex case when needed.

There are several books which cover some of the same material from somewhat different viewpoints. Perhaps the closest relatives are the books by Diestel [61] and Wojtaszczyk [303], both of which share some common themes. Two very recent books, namely, Carothers [40] and Li and Queffélec [189], also cover some similar

topics. We feel that the student will find it instructive to compare the treatments in these books. Some other texts which are highly relevant are [22, 123, 219], and [90]. If, as we hope, the reader is inspired to learn more about some of the topics, a good place to start is the *Handbook of the Geometry of Banach Spaces*, edited by Johnson and Lindenstrauss [136, 138], which is a collection of articles on the development of the theory; this has the advantage of being (almost) up to date at the turn of the century. Included is an article by the editors [137] which gives a condensed summary of the basic theory.

The first author gratefully acknowledges Gobierno de Navarra for funding, and wants to express his deep gratitude to Sheila Johnson for all her patience and unconditional support for the duration of this project. The second author acknowledges support from the National Science Foundation and wishes to thank his wife, Jennifer, for her tolerance while he was working on this project.

Colombia, MO, USA Fernando Albiac
Colombia, MO, USA Nigel J. Kalton
November 2005

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# Chapter 1
# Bases and Basic Sequences

In this chapter we are going to introduce the fundamental notion of a Schauder basis of a Banach space and the corresponding notion of a basic sequence. One of the key ideas in the isomorphic theory of Banach spaces is to use the properties of bases and basic sequences as a tool to understanding the differences and similarities between spaces. The systematic use of basic sequence arguments also turns out to simplify some classical theorems, and we illustrate this with the Eberlein–Šmulian theorem on weakly compact subsets of a Banach space.

Before proceeding, let us remind the reader that our convention will be that all Banach spaces are real, unless otherwise stated. In fact, there is very little change in the theory in switching to complex scalars, but to avoid keeping track of minor notational changes it is convenient to restrict ourselves to the real case. Occasionally, we will give proofs in the complex case when it appears to be useful to do so. In other cases the reader is invited to convince himself that he can obtain the same result in the complex case.

## 1.1 Schauder Bases

The basic idea of functional analysis is to combine the techniques of linear algebra with topological considerations of convergence. It is therefore very natural to look for a concept to extend the notion of a basis of a finite-dimensional vector space.

In the context of Hilbert spaces, orthonormal bases have proved a very useful tool in many areas of analysis. We recall that if $(e_n)_{n=1}^{\infty}$ is an orthonormal basis of a Hilbert space $H$, then for every $x \in H$ there is a unique sequence of scalars $(a_n)_{n=1}^{\infty}$ given by $a_n = \langle x, e_n \rangle$ such that



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$$x = \sum_{n=1}^{\infty} a_n e_n.$$

The usefulness of orthonormal bases stems partly from the fact that they are relatively easy to find; indeed, every separable Hilbert space has an orthonormal basis. Procedures such as the Gram–Schmidt process allow very easy constructions of new orthonormal bases.

There are several possible extensions of the basis concept to Banach spaces, but the following definition is the most useful.

**Definition 1.1.1.** A sequence of elements $(e_n)_{n=1}^{\infty}$ in an infinite-dimensional Banach space $X$ is said to be a *basis* of $X$ if for each $x \in X$ there is a *unique* sequence of scalars $(a_n)_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} a_n e_n.$$

This means that we require that the sequence $(\sum_{n=1}^{N} a_n e_n)_{N=1}^{\infty}$ converge to $x$ in the norm topology of $X$.

It is clear from the definition that a basis consists of linearly independent, and in particular nonzero, vectors. If $X$ has a basis $(e_n)_{n=1}^{\infty}$ then its closed linear span, $[e_n]$, coincides with $X$ and therefore $X$ is separable (the rational finite linear combinations of $(e_n)$ will be dense in $X$). Let us stress that the order of the basis is important; if we permute the elements of the basis then the new sequence can very easily fail to be a basis. We will discuss this phenomenon in much greater detail later, in Chapter 3.

The reader should not confuse the notion of basis in an infinite-dimensional Banach space with the purely algebraic concept of Hamel basis or vector space basis. A *Hamel basis* $(e_i)_{i \in \mathcal{I}}$ for $X$ is a collection of linearly independent vectors in $X$ such that each $x$ in $X$ is uniquely representable as a *finite* linear combination of $e_i$. From the Baire category theorem it is easy to deduce that if $(e_i)_{i \in \mathcal{I}}$ is a Hamel basis for an infinite-dimensional Banach space $X$ then $(e_i)_{i \in \mathcal{I}}$ must be uncountable. Henceforth, whenever we refer to a basis for an infinite-dimensional Banach space $X$ it will be in the sense of Definition 1.1.1.

We also note that if $(e_n)_{n=1}^{\infty}$ is a basis of a Banach space $X$, the maps $x \mapsto a_n$ are linear functionals on $X$. Let us write, for the time being, $e_n^{\#}(x) = a_n$. However, it is by no means immediate that the linear functionals $(e_n^{\#})_{n=1}^{\infty}$ are actually continuous. Let us make the following definition:

**Definition 1.1.2.** Let $(e_n)_{n=1}^{\infty}$ be a sequence in a Banach space $X$. Suppose there is a sequence $(e_n^*)_{n=1}^{\infty}$ in $X^*$ such that

*(i)* $e_k^*(e_j) = 1$ if $j = k$, and $e_k^*(e_j) = 0$ otherwise, for every $k$ and $j$ in $\mathbb{N}$,
*(ii)* $x = \sum_{n=1}^{\infty} e_n^*(x) e_n$ for each $x \in X$.

Then $(e_n)_{n=1}^{\infty}$ is called a *Schauder basis* for $X$ and the functionals $(e_n^*)_{n=1}^{\infty}$ are called the *biorthogonal functionals* (or the *coordinate functionals*) associated to $(e_n)_{n=1}^{\infty}$.

If $(e_n)_{n=1}^{\infty}$ is a Schauder basis for $X$ and $x = \sum_{n=1}^{\infty} e_n^*(x)e_n \in X$, the *support of* $x$ is the subset of integers $n$ such that $e_n^*(x) \neq 0$. We denote it by $\operatorname{supp}(x)$. If $|\operatorname{supp}(x)| < \infty$ we say that $x$ is *finitely supported*.

The name *Schauder* in the previous definition is in honor of J. Schauder, who first introduced the concept of a basis in 1927 [279]. In practice, nevertheless, every basis of a Banach space is a Schauder basis, and the concepts are not distinct (the distinction is important, however, in more general locally convex spaces).

The proof of the equivalence between the concepts of basis and Schauder basis is an early application of the closed graph theorem [18, p. 111]. Although this result is a very nice use of some of the basic principles of functional analysis, it has to be conceded that it is essentially useless in the sense that in all practical situations we are able to prove that $(e_n)_{n=1}^{\infty}$ is a basis only by showing the formally stronger conclusion that it is already a Schauder basis. Thus the reader can safely skip the next theorem.

**Theorem 1.1.3.** *Let $X$ be a (separable) Banach space. A sequence $(e_n)_{n=1}^{\infty}$ in $X$ is a Schauder basis for $X$ if and only if $(e_n)_{n=1}^{\infty}$ is a basis for $X$.*

*Proof.* Let us assume that $(e_n)_{n=1}^{\infty}$ is a basis for $X$ and introduce the *partial sum projections* $(S_n)_{n=0}^{\infty}$ associated to $(e_n)_{n=1}^{\infty}$ defined by $S_0 = 0$ and for $n \geq 1$,

$$S_n(x) = \sum_{k=1}^{n} e_k^{\#}(x)e_k.$$

Of course, we do not yet know that these operators are bounded! Let us consider a new norm on $X$ defined by the formula

$$|||x||| = \sup_{n\geq 1} \|S_n x\|.$$

Since $\lim_{n\to\infty} \|x - S_n x\| = 0$ for each $x \in X$, it follows that $|||\cdot||| \geq \|\cdot\|$. We will show that $(X, |||\cdot|||)$ is complete.

Suppose that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(X, |||\cdot|||)$. Of course, $(x_n)_{n=1}^{\infty}$ is convergent to some $x \in X$ in the original norm. Our goal is to prove that $\lim_{n\to\infty} |||x_n - x||| = 0$.

Notice that for each fixed $k$ the sequence $(S_k x_n)_{n=1}^{\infty}$ is convergent in the original norm to some $y_k \in X$, and note also that $(S_k x_n)_{n=1}^{\infty}$ is contained in the finite-dimensional subspace $[e_1, \dots, e_k]$. Certainly, the functionals $e_j^{\#}$ are continuous on every finite-dimensional subspace; hence if $1 \leq j \leq k$ we have

$$\lim_{n\to\infty} e_j^{\#}(x_n) = e_j^{\#}(y_k) := a_j.$$

Next we argue that $\sum_{j=1}^{\infty} a_j e_j = x$ for the original norm.

Given $\epsilon > 0$, pick an integer $n$ such that if $m \geq n$ then $|||x_m - x_n||| \leq \frac{1}{3}\epsilon$, and take $k_0$ such that $k \geq k_0$ implies $\|x_n - S_k x_n\| \leq \frac{1}{3}\epsilon$. Then for $k \geq k_0$ we have

$$\|y_k - x\| \leq \lim_{m\to\infty} \|S_k x_m - S_k x_n\| + \|S_k x_n - x_n\| + \lim_{m\to\infty} \|x_m - x_n\| \leq \epsilon.$$

Thus $\lim_{k\to\infty} \|y_k - x\| = 0$ and, by the uniqueness of the expansion of $x$ with respect to the basis, $S_k x = y_k$.

Now,

$$|||x_n - x||| = \sup_{k\geq 1} \|S_k x_n - S_k x\| \leq \limsup_{m\to\infty} \sup_{k\geq 1} \|S_k x_n - S_k x_m\|,$$

so $\lim_{n\to\infty} |||x_n - x||| = 0$ and $(X, |||\cdot|||)$ is complete.

By the closed graph theorem (or the open mapping theorem), the identity map $(X, \|\cdot\|) \to (X, |||\cdot|||)$ is bounded, i.e., there exists $K$ such that $|||x||| \leq K\|x\|$ for $x \in X$. This implies that

$$\|S_n x\| \leq K\|x\|, \qquad x \in X,\ n \in \mathbb{N}.$$

In particular,

$$|e_n^{\#}(x)|\|e_n\| = \|S_n x - S_{n-1}x\| \leq 2K\|x\|;$$

hence $e_n^{\#} \in X^*$ and $\|e_n^{\#}\| \leq 2K\|e_n\|^{-1}$. □

Let $(e_n)_{n=1}^{\infty}$ be a basis for a Banach space $X$. The preceding theorem tells us that $(e_n)_{n=1}^{\infty}$ is actually a Schauder basis; hence we use $(e_n^*)_{n=1}^{\infty}$ for the biorthogonal functionals.

As above, we consider the partial sum operators $S_n : X \to X$, given by $S_0 = 0$ and, for $n \geq 1$,

$$S_n\left(\sum_{k=1}^{\infty} e_k^*(x)e_k\right) = \sum_{k=1}^{n} e_k^*(x)e_k.$$

$S_n$ is a continuous linear operator, since each $e_k^*$ is continuous. That the operators $(S_n)_{n=1}^{\infty}$ are uniformly bounded was already proved in Theorem 1.1.3, but we note it for further reference:

**Proposition 1.1.4.** *Let $(e_n)_{n=1}^{\infty}$ be a Schauder basis for a Banach space $X$ and $(S_n)_{n=1}^{\infty}$ the natural projections associated with it. Then*

$$\sup_n \|S_n\| < \infty.$$

*Proof.* For a Schauder basis the operators $(S_n)_{n=1}^{\infty}$ are bounded a priori. Since $S_n(x) \to x$ for every $x \in X$, we have $\sup_n \|S_n(x)\| < \infty$ for each $x \in X$. Then the uniform boundedness principle yields that $\sup_n \|S_n\| < \infty$. □

**Definition 1.1.5.** If $(e_n)_{n=1}^{\infty}$ is a basis for a Banach space $X$ then the number

$$\mathsf{K}_\mathsf{b} = \sup_n \|S_n\|$$

is called the *basis constant*. In the optimal case that $\mathsf{K}_\mathsf{b} = 1$, the basis $(e_n)_{n=1}^{\infty}$ is said to be *monotone*.

*Remark 1.1.6.* We can always renorm a Banach space $X$ with a basis in such a way that the given basis is monotone. Just put

$$|||x||| = \sup_{n\geq 1} \|S_n x\|.$$

Then $\|x\| \leq |||x||| \leq \mathsf{K}_\mathsf{b}\|x\|$, so the new norm is equivalent to the old one and it is quickly verified that $|||S_n||| = 1$ for $n \in \mathbb{N}$.

The next result establishes a method for constructing a basis for a Banach space $X$, provided we have a family of projections enjoying the properties of the partial sum operators.

**Proposition 1.1.7.** *Suppose $S_n : X \to X$, $n \in \mathbb{N}$, is a sequence of bounded linear projections on a Banach space $X$ such that*

*(i)* $\dim S_n(X) = n$ *for each $n$;*
*(ii)* $S_n S_m = S_m S_n = S_{\min\{m,n\}}$, *for all integers $m$ and $n$; and*
*(iii)* $S_n(x) \to x$ *for every $x \in X$.*

*Then every nonzero sequence of vectors $(e_k)_{k=1}^{\infty}$ in $X$ chosen inductively so that $e_1 \in S_1(X)$ and $e_k \in S_k(X) \cap S_{k-1}^{-1}(0)$ if $k \geq 2$ is a basis for $X$ with partial sum projections $(S_n)_{n=1}^{\infty}$.*

*Proof.* Let $0 \neq e_1 \in S_1(X)$ and define $e_1^* : X \to \mathbb{R}$ by $e_1^*(x)e_1 = S_1(x)$. Next we pick $0 \neq e_2 \in S_2(X) \cap S_1^{-1}(0)$ and define the functional $e_2^* : X \to \mathbb{R}$ by $e_2^*(x)e_2 = S_2(x) - S_1(x)$. This gives us by induction a procedure to extract the basis and its biorthogonal functionals: for each integer $n$, we pick $0 \neq e_n \in S_n(X) \cap S_{n-1}^{-1}(0)$ and define $e_n^* : X \to \mathbb{R}$ by $e_n^*(x)e_n = S_n(x) - S_{n-1}(x)$. Then

$$|e_n^*(x)| = \|S_n(x) - S_{n-1}(x)\| \, \|e_n\|^{-1} \leq 2 \sup_n \|S_n\| \, \|e_n\|^{-1} \, \|x\|;$$

hence $e_n^* \in X^*$. It is immediate to check that $e_k^*(e_j) = \delta_{kj}$ for any two integers $k, j$.

On the other hand, if we let $S_0(x) = 0$ for all $x$, we can write

$$S_n(x) = \sum_{k=1}^{n}(S_k(x) - S_{k-1}(x)) = \sum_{k=1}^{n} e_k^*(x)e_k,$$

which, by (*iii*) in the hypothesis, converges to $x$ for every $x \in X$. Therefore, the sequence $(e_n)_{n=1}^{\infty}$ is a basis and $(S_n)_{n=1}^{\infty}$ its natural projections. □

In the next definition we relax the assumption that a basis must span the entire space.

**Definition 1.1.8.** A sequence $(e_k)_{k=1}^{\infty}$ in a Banach space $X$ is called a *basic sequence* if it is a basis for $[e_k]$, the closed linear span of $(e_k)_{k=1}^{\infty}$.

As the reader will quickly realize, basic sequences are of fundamental importance in the theory of Banach spaces and will be exploited throughout this volume. To recognize a sequence of elements in a Banach space as a basic sequence we use the following test, also known as *Grunblum's criterion* [122]:

**Proposition 1.1.9.** *A sequence $(e_k)_{k=1}^{\infty}$ of nonzero elements of a Banach space $X$ is basic if and only if there is a positive constant $K$ such that*

$$\Big\| \sum_{k=1}^{m} a_k e_k \Big\| \leq K \Big\| \sum_{k=1}^{n} a_k e_k \Big\| \tag{1.1}$$

*for every sequence of scalars $(a_k)$ and all integers $m$, $n$ such that $m \leq n$.*

*Proof.* Assume $(e_k)_{k=1}^{\infty}$ is basic, and let $S_m \colon [e_k] \to [e_k]$, $m = 1, 2, \dots$, be its partial sum projections. If $m \leq n$ we have

$$\Big\| \sum_{k=1}^{m} a_k e_k \Big\| = \Big\| S_m\Big(\sum_{k=1}^{n} a_k e_k\Big) \Big\| \leq \sup_m \|S_m\| \Big\| \sum_{k=1}^{n} a_k e_k \Big\|,$$

so (1.1) holds with $K = \sup_m \|S_m\|$.

For the converse, let $E$ be the linear span of $(e_k)_{k=1}^{\infty}$. Condition (1.1) implies that the vectors $(e_k)_{k=1}^{\infty}$ are linearly independent. This permits us to define unambiguously for each $m$ the finite-rank operator $s_m : E \to [e_k]_{k=1}^{m}$ by

$$s_m\left(\sum_{k=1}^{n} a_j e_j\right) = \sum_{k=1}^{\min(m,n)} a_k e_k, \qquad m, n \in \mathbb{N}.$$

By density each $s_m$ extends to $S_m : [e_k] \to [e_k]_{k=1}^{m}$ with $\|S_m\| = \|s_m\| \leq K$.

Notice that for each $x \in E$ we have

$$S_n S_m(x) = S_m S_n(x) = S_{\min(m,n)}(x), \qquad m, m \in \mathbb{N}, \tag{1.2}$$

so by density, (1.2) holds for all $x \in [e_k]$.

For each $x \in [e_k]$ the sequence $(S_n(x))_{n=1}^{\infty}$ converges to $x$, since the set $\{x \in [e_k]\colon S_m(x) \to x\}$ is closed (see E.14 in the appendix) and contains $E$, which is dense in $[e_k]$. Now Proposition 1.1.7 yields that $(e_k)_{k=1}^{\infty}$ is a basis for $[e_k]$ with partial sum projections $(S_m)_{m=1}^{\infty}$. □

## 1.2 Examples: Fourier Series

Some of the classical Banach spaces come with a naturally given basis. For example, in the spaces $\ell_p$ for $1 \leq p < \infty$ and $c_0$ there is a canonical basis given by the sequence $e_n = (0, \dots, 0, 1, 0, \dots)$, where the only nonzero entry is in the $n$th coordinate. We leave the verification of these simple facts to the reader. In this section we will discuss an example from Fourier analysis and also Schauder's original construction of a basis in $\mathcal{C}[0, 1]$.

Let $\mathbb{T}$ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. We denote a typical element of $\mathbb{T}$ by $e^{i\theta}$ and then we can identify the space $\mathcal{C}_{\mathbb{C}}(\mathbb{T})$ of continuous complex-valued functions on $\mathbb{T}$ with the space of continuous $2\pi$-periodic functions on $\mathbb{R}$. Let us note that in the context of Fourier series it is more natural to consider complex function spaces than real spaces.

For every $n \in \mathbb{Z}$ let $e_n \in \mathcal{C}_{\mathbb{C}}(\mathbb{T})$ be the function such that $e_n(\theta) = e^{in\theta}$. The question we wish to tackle is whether the sequence $(e_0, e_1, e_{-1}, e_2, e_{-2}, \dots)$ (in this particular order) is a basis of $\mathcal{C}_{\mathbb{C}}(\mathbb{T})$. In fact, we shall see that it is not. This is a classical result in Fourier analysis (a good reference is Katznelson [167]) that is equivalent to the statement that there is a continuous function $f$ whose Fourier series does not converge uniformly. The stronger statement that there is a continuous function whose Fourier series does not converge at some point is due to Du Bois-Reymond, and a nice treatment can be found in Körner [178]; we shall prove this below.

That $[e_n]_{n\in\mathbb{Z}} = \mathcal{C}_{\mathbb{C}}(\mathbb{T})$ follows directly from the Stone–Weierstrass theorem (see Theorem 4.1.2), but we shall also prove this directly.

The *Fourier coefficients* of $f \in \mathcal{C}_{\mathbb{C}}(\mathbb{T})$ are defined by the formula

$$\hat{f}(n) = \int_{-\pi}^{\pi} f(t)e^{-int}\frac{dt}{2\pi}, \qquad n \in \mathbb{Z}.$$

The linear functionals

$$e_n^* : \mathcal{C}_{\mathbb{C}}(\mathbb{T}) \to \mathbb{C}, \quad f \mapsto e_n^*(f) = \hat{f}(n)$$

are biorthogonal to the sequence $(e_n)_{n\in\mathbb{Z}}$.

The *Fourier series* of $f$ is the formal series

$$\sum_{-\infty}^{\infty} \hat{f}(n)e^{in\theta}.$$

For each integer $n$ let $T_n : \mathcal{C}_{\mathbb{C}}(\mathbb{T}) \to \mathcal{C}_{\mathbb{C}}(\mathbb{T})$ be the operator

$$T_n(f) = \sum_{k=-n}^{n} \hat{f}(k) e_k,$$

which gives us the $n$th partial sum of the Fourier series of $f$. Then

$$\begin{aligned} T_n(f)(\theta) &= \sum_{k=-n}^{n} \int_{\theta-\pi}^{\theta+\pi} f(t) e^{ik(\theta-t)} \frac{dt}{2\pi} \\ &= \int_{-\pi}^{\pi} f(\theta - t) \sum_{k=-n}^{n} e^{ikt} \frac{dt}{2\pi} \\ &= \int_{-\pi}^{\pi} f(\theta - t) \frac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{2}} \frac{dt}{2\pi}. \end{aligned}$$

The function

$$D_n(t) = \frac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{2}}$$

is known as the *Dirichlet kernel*.

Let us also consider the operators

$$A_n = \frac{1}{n}(T_0 + \cdots + T_{n-1}), \quad n = 2, 3, \ldots.$$

Then

$$\begin{aligned} A_n f(\theta) &= \frac{1}{n} \int_{-\pi}^{\pi} f(\theta - t) \sum_{k=0}^{n-1} \frac{\sin(k + \frac{1}{2})t}{\sin \frac{t}{2}} \frac{dt}{2\pi} \\ &= \frac{1}{n} \int_{-\pi}^{\pi} f(\theta - t) \left( \frac{\sin(\frac{nt}{2})}{\sin \frac{t}{2}} \right)^2 \frac{dt}{2\pi}. \end{aligned}$$

The function

$$F_n(t) = \frac{1}{n} \left( \frac{\sin(\frac{nt}{2})}{\sin \frac{t}{2}} \right)^2$$

is called the *Fejér kernel*. Note that

$$\int_{-\pi}^{\pi} D_n(t) \frac{dt}{2\pi} = \int_{-\pi}^{\pi} F_n(t) \frac{dt}{2\pi} = 1.$$

Nevertheless, a crucial difference is that $F_n$ is a positive function, whereas $D_n$ is not.

Let us now show that if $f \in \mathcal{C}_{\mathbb{C}}(\mathbb{T})$ then $\|A_n f - f\| \to 0$. Since $f$ is uniformly continuous, given $\epsilon > 0$ we can find $0 < \delta < \pi$ such that $|\theta - \theta'| < \delta$ implies $|f(\theta) - f(\theta')| \leq \epsilon$. Then for every $\theta$ we have

$$A_n f(\theta) - f(\theta) = \int_{-\pi}^{\pi} F_n(t)(f(\theta - t) - f(\theta)) \frac{dt}{2\pi}.$$

Hence

$$\|A_n f - f\| \leq \|f\| \int_{\delta < |t| \leq \pi} F_n(t) \frac{dt}{2\pi} + \epsilon \int_{-\delta}^{\delta} F_n(t) \frac{dt}{2\pi}.$$

Now

$$\int_{\delta < |t| \leq \pi} F_n(t) \frac{dt}{2\pi} \leq \frac{1}{n} \sin^{-2}(\delta/2)$$

and so

$$\limsup \|A_n f - f\| \leq \epsilon.$$

This shows that $[e_n]_{n \in \mathbb{Z}} = \mathcal{C}_{\mathbb{C}}(\mathbb{T})$.

Since the biorthogonal functionals are given by the Fourier coefficients, it follows that if $(e_0, e_1, e_{-1}, \dots)$ is a basis then the partial sum operators $(S_n)$ satisfy $S_{2n+1} = T_n$ for all $n$. To show that it is not a basis it therefore suffices to show that the sequence of operators $(T_n)_{n=1}^{\infty}$ is not uniformly bounded.

Let $\varphi \in \mathcal{C}_{\mathbb{C}}(\mathbb{T})^*$ be given by

$$\varphi(f) = f(0).$$

Then

$$\varphi(T_n f) = \int_{-\pi}^{\pi} D_n(t) f(-t) \frac{dt}{2\pi};$$

hence

$$\|T_n^* \varphi\| = \int_{-\pi}^{\pi} |D_n(t)| \frac{dt}{2\pi}.$$

Thus, since $|\sin x| \leq |x|$ for all real $x$,

$$\|T_n\| \geq \int_{-\pi}^{\pi} |D_n(t)| \frac{dt}{2\pi}$$

$$= \frac{1}{\pi} \int_0^{\pi} \left| \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin \frac{t}{2}} \right| dt$$

$$\geq \frac{2}{\pi} \int_0^{(n+1/2)\pi} \left| \frac{\sin t}{\sin \frac{t}{2n+1}} \right| \frac{dt}{2n+1}$$

$$\geq \frac{2}{\pi} \int_0^{(n+1/2)\pi} \frac{|\sin t|}{t}\, dt.$$

Hence,

$$\liminf_{n\to\infty} \|T_n\| \geq \frac{2}{\pi} \int_0^{\infty} \frac{|\sin x|}{x} dx = \infty.$$

Let us remark that we have actually proved that $\sup_n \|T_n^* \varphi\| = \infty$; therefore by the uniform boundedness principle there must exist $f \in \mathcal{C}_{\mathbb{C}}(\mathbb{T})$ such that $(T_n f(0))_{n=1}^{\infty}$ is unbounded. Notice also that this is not an explicit example; see [178] for such an example.

If we prefer to deal with the space of continuous real-valued functions $\mathcal{C}(\mathbb{T})$, exactly the same calculations show that the (real) trigonometric system

$$\{1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \dots\}$$

fails to be a basis. Indeed, the operators $(T_n)_{n=1}^{\infty}$ are unbounded on the space $\mathcal{C}(\mathbb{T})$ and correspond to the partial sum operators $(S_{2n+1})_{n=1}^{\infty}$ as before.

However, $\mathcal{C}(\mathbb{T})$ and $\mathcal{C}_{\mathbb{C}}(\mathbb{T})$ do have a basis. This can easily be shown in a very similar way to Schauder's original construction of a basis in $\mathcal{C}[0,1]$, which we now describe. Let $(q_n)_{n=1}^{\infty}$ be a sequence that is dense in $[0,1]$ and such that $q_1 = 0$ and $q_2 = 1$. We construct inductively a sequence of operators $(S_n)_{n=1}^{\infty}$, defined on $\mathcal{C}[0,1]$, by $S_1 f(t) = f(q_1)$ for $0 \leq t \leq 1$, and subsequently $S_n f$ is the piecewise linear function defined by $S_n f(q_k) = f(q_k)$ for $1 \leq k \leq n$ and linear on all the intervals of $[0,1] \setminus \{q_1, \dots, q_n\}$. It is then easy to see that $\|S_n\| = 1$ for all $n$ and that the assumptions of Proposition 1.1.7 are satisfied. In this way we obtain a monotone basis for $\mathcal{C}[0,1]$. The basis elements are given by $e_1(t) = 1$ for all $t$, and then $e_n$ is defined recursively by $e_n(q_n) = 1$, $e_n(q_k) = 0$ for $1 \leq k \leq n-1$, and $e_n$ is linear on each interval in $[0,1] \setminus \{q_1, \dots, q_n\}$.

To modify this for the case of the circle we identify $\mathcal{C}(\mathbb{T})$ [respectively, $\mathcal{C}_{\mathbb{C}}(\mathbb{T})$] with the functions in $\mathcal{C}[0,2\pi]$ [respectively, $\mathcal{C}_{\mathbb{C}}[0,2\pi]$] such that $f(0) = f(2\pi)$. Let $q_1 = 0$ and suppose $(q_n)_{n=1}^{\infty}$ is dense in $[0,2\pi)$. Then $S_n f$ for $n > 1$ is defined by $S_n f(q_k) = f(q_k)$ for $1 \leq k \leq n$ and $S_n f(2\pi) = f(q_1)$ and to be affine on each interval in $[0,2\pi) \setminus \{q_1, \dots, q_n\}$.

In both cases this procedure constructs a monotone basis. We summarize this in the following theorem.

**Theorem 1.2.1.** *The spaces* $\mathcal{C}[0,1]$, $\mathcal{C}_{\mathbb{C}}(\mathbb{T})$ *each have a monotone basis. The complex trigonometric system* $(1, e^{i\theta}, e^{-i\theta}, \dots)$ *fails to be a basis of* $\mathcal{C}_{\mathbb{C}}(\mathbb{T})$.

## 1.3 Equivalence of Bases and Basic Sequences

If we select a basis in a finite-dimensional vector space, then we are, in effect, selecting a system of coordinates. Bases in infinite-dimensional Banach spaces play the same role. Thus, if we have a basis $(e_n)_{n=1}^{\infty}$ of $X$, then we can specify $x \in X$ by its coordinates $(e_n^*(x))_{n=1}^{\infty}$. Of course, it is not true that every scalar sequence $(a_n)_{n=1}^{\infty}$ defines an element of $X$. Thus $X$ is coordinatized by a certain sequence space, i.e., a linear subspace of the vector space of all sequences. This leads us naturally to the following definition.

**Definition 1.3.1.** Two bases (or basic sequences) $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in the respective Banach spaces $X$ and $Y$ are *equivalent* if whenever we take a sequence of scalars $(a_n)_{n=1}^{\infty}$, then $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges.

Hence if the bases $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent, then the corresponding sequence spaces associated to $X$ by $(x_n)_{n=1}^{\infty}$ and to $Y$ by $(y_n)_{n=1}^{\infty}$ coincide. It is an easy consequence of the closed graph theorem that if $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent, then the spaces $X$ and $Y$ must be isomorphic. More precisely, we have the following theorem.

**Theorem 1.3.2.** *Two bases (or basic sequences)* $(x_n)_{n=1}^{\infty}$ *and* $(y_n)_{n=1}^{\infty}$ *are equivalent if and only if there is an isomorphism* $T : [x_n] \to [y_n]$ *such that* $Tx_n = y_n$ *for each* $n$
.

*Proof.* Let $X = [x_n]$ and $Y = [y_n]$. It is obvious that $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent if there is an isomorphism $T$ from $X$ onto $Y$ such that $Tx_n = y_n$ for each $n$.

Suppose conversely that $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent. Let us define $T : X \to Y$ by $T(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{\infty} a_n y_n$. Then $T$ is one-to-one and onto. To prove that $T$ is continuous we use the closed graph theorem. Suppose $(u_j)_{j=1}^{\infty}$ is a sequence such that $u_j \to u$ in $X$ and $Tu_j \to v$ in $Y$. Let us write $u_j = \sum_{n=1}^{\infty} x_n^*(u_j)x_n$ and $u = \sum_{n=1}^{\infty} x_n^*(u)x_n$. It follows from the continuity of the biorthogonal functionals associated respectively with $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ that $x_n^*(u_j) \to x_n^*(u)$ and $y_n^*(Tu_j) = x_n^*(u_j) \to y_n^*(v)$ for all $n$. By the uniqueness of limit, $x_n^*(u) = y_n^*(v)$, for all $n$. Therefore $Tu = v$ and so $T$ is continuous. □

**Corollary 1.3.3.** *Let* $(x_n)_{n=1}^{\infty}$ *and* $(y_n)_{n=1}^{\infty}$ *be two bases for the Banach spaces* $X$ *and* $Y$ *respectively. Then* $(x_n)_{n=1}^{\infty}$ *is equivalent to* $(y_n)_{n=1}^{\infty}$ *if and only if there exists a constant* $C > 0$ *such that for all finitely nonzero sequences of scalars* $(a_i)_{i=1}^{\infty}$ *we have*

$$C^{-1}\Big\|\sum_{i=1}^{\infty} a_i y_i\Big\| \le \Big\|\sum_{i=1}^{\infty} a_i x_i\Big\| \le C\Big\|\sum_{i=1}^{\infty} a_i y_i\Big\|. \tag{1.3}$$

If $C = 1$ in (1.3), then the basic sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are said to be *isometrically equivalent*.

Equivalence of basic sequences (and in particular of bases) will become a powerful technique for studying the isomorphic structure of Banach spaces.

Let us now introduce a special type of basic sequence:

**Definition 1.3.4.** Let $(e_n)_{n=1}^{\infty}$ be a basis for a Banach space $X$. Suppose that $(p_n)_{n=1}^{\infty}$ is a strictly increasing sequence of integers with $p_0 = 0$ and that $(a_n)_{n=1}^{\infty}$ are scalars. Then a sequence of nonzero vectors $(u_n)_{n=1}^{\infty}$ in $X$ of the form

$$u_n = \sum_{j=p_{n-1}+1}^{p_n} a_j e_j$$

is called a *block basic sequence* of $(e_n)_{n=1}^{\infty}$.

**Lemma 1.3.5.** *Suppose $(e_n)_{n=1}^{\infty}$ is a basis for the Banach space $X$ with basis constant $\mathsf{K_b}$. Let $(u_k)_{k=1}^{\infty}$ be a block basic sequence of $(e_n)_{n=1}^{\infty}$. Then $(u_k)_{k=1}^{\infty}$ is a basic sequence with basis constant less than or equal to $\mathsf{K_b}$.*

*Proof.* Suppose that $u_k = \sum_{j=p_{k-1}+1}^{p_k} a_j e_j$, $k \in \mathbb{N}$, is a block basic sequence of $(e_n)_{n=1}^{\infty}$. Then, for any scalars $(b_k)$ and integers $m, n$ with $m \le n$ we have

$$\begin{aligned}
\Big\| \sum_{k=1}^{m} b_k u_k \Big\| &= \Big\| \sum_{k=1}^{m} b_k \sum_{j=p_{k-1}+1}^{p_k} a_j e_j \Big\| \\
&= \Big\| \sum_{k=1}^{m} \sum_{j=p_{k-1}+1}^{p_k} b_k a_j e_j \Big\| \\
&= \Big\| \sum_{j=1}^{p_m} c_j e_j \Big\|, \text{ where } c_j = a_j b_k \text{ if } p_{k-1} + 1 \le j \le p_k \\
&\le \mathsf{K_b} \Big\| \sum_{j=1}^{p_n} c_j e_j \Big\| \\
&= \mathsf{K_b} \Big\| \sum_{k=1}^{n} b_k u_k \Big\|.
\end{aligned}$$

That is, $(u_k)_{k=1}^{\infty}$ satisfies Grunblum's condition (Proposition 1.1.9); hence $(u_k)_{k=1}^{\infty}$ is a basic sequence with basis constant at most $\mathsf{K_b}$. □

**Definition 1.3.6.** A basic sequence $(x_n)_{n=1}^{\infty}$ in $X$ is *complemented* if $[x_n]$ is a complemented subspace of $X$.

*Remark 1.3.7.* Suppose $(x_n)_{n=1}^{\infty}$ is a complemented basic sequence in a Banach space $X$. Let $Y = [x_n]$ and $P : X \to Y$ be a projection. If $(x_n^*)_{n=1}^{\infty} \subset Y^*$ are the biorthogonal functionals associated to $(x_n)_{n=1}^{\infty}$, then using the Hahn–Banach theorem we can obtain a biorthogonal sequence $(\hat{x}_n^*)_{n=1}^{\infty} \subset X^*$ such that each $\hat{x}_n^*$ is an extension of $x_n^*$ to $X$ with preservation of norm. But since we have a projection, $P$, we can also extend each $x_n^*$ to the whole of $X$ by putting $u_n^* = x_n^* \circ P$. Then for $x \in X$, we will have

$$\sum_{n=1}^{\infty} u_n^*(x) x_n = P(x).$$

Conversely, if we can make a sequence $(u_n^*)_{n=1}^{\infty} \subset X^*$ such that $u_n^*(x_m) = \delta_{nm}$ and the series $\sum_{n=1}^{\infty} u_n^*(x) x_n$ converges for all $x \in X$, then the subspace $[x_n]$ is complemented by the projection $X \to [x_n]$, $x \mapsto \sum_{n=1}^{\infty} u_n^*(x) x_n$.

**Definition 1.3.8.** Let $X$ and $Y$ be Banach spaces. We say that two sequences $(x_n)_{n=1}^{\infty} \subset X$ and $(y_n)_{n=1}^{\infty} \subset Y$ are *congruent with respect to* $(X, Y)$ if there is an invertible operator $T : X \to Y$ such that $T(x_n) = y_n$ for all $n \in \mathbb{N}$. When $(x_n)$ and $(y_n)$ satisfy this condition in the particular case that $X = Y$, we will simply say that they are *congruent*.

Let us suppose that the sequences $(x_n)_{n=1}^{\infty}$ in $X$ and $(y_n)_{n=1}^{\infty}$ in $Y$ are congruent with respect to $(X, Y)$. The operator $T$ of $X$ onto $Y$ that exists by the previous definition preserves every isomorphic property of $(x_n)_{n=1}^{\infty}$. For example, if $(x_n)_{n=1}^{\infty}$ is a basis of $X$, then $(y_n)_{n=1}^{\infty}$ is a basis of $Y$; if $\mathsf{K}_\mathsf{b}$ is the basis constant of $(x_n)_{n=1}^{\infty}$, then the basis constant of $(y_n)_{n=1}^{\infty}$ is at most $\mathsf{K}_\mathsf{b}\|T\|\|T^{-1}\|$.

The following stability result dates back to 1940 [180]. Roughly speaking, it says that if $(x_n)_{n=1}^{\infty}$ is a basic sequence in a Banach space $X$ and $(y_n)_{n=1}^{\infty}$ is another sequence in $X$ such that $(\|x_n - y_n\|)_{n=1}^{\infty}$ converges fast enough to 0, then $(y_n)_{n=1}^{\infty}$ and $(x_n)_{n=1}^{\infty}$ are congruent.

**Theorem 1.3.9 (Principle of small perturbations).** *Let $(x_n)_{n=1}^{\infty}$ be a basic sequence in a Banach space $X$ with basis constant $\mathsf{K}_\mathsf{b}$. If $(y_n)_{n=1}^{\infty}$ is a sequence in $X$ such that*

$$2\mathsf{K}_\mathsf{b} \sum_{n=1}^{\infty} \frac{\|x_n - y_n\|}{\|x_n\|} = \theta < 1,$$

*then $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are congruent. In particular:*

*(i) $(y_n)_{n=1}^{\infty}$ is a basic sequence with basis constant at most $\mathsf{K}_\mathsf{b}(1+\theta)(1-\theta)^{-1}$.*
*(ii) If $(x_n)_{n=1}^{\infty}$ is a basis, so is $(y_n)_{n=1}^{\infty}$.*
*(iii) If $[x_n]$ is complemented in $X$, then so is $[y_n]$.*

*Proof.* For $n \geq 2$ and $x \in [x_n]$ we have

$$x_n^*(x)x_n = \sum_{k=1}^{n} x_k^*(x)x_k - \sum_{k=1}^{n-1} x_k^*(x)x_k,$$

where $(x_n^*)_{n=1}^\infty \subset [x_n]^*$ are the biorthogonal functionals of $(x_n)_{n=1}^\infty$. Then

$$\|x_n^*(x)x_n\| \leq 2\mathsf{K_b}\|x\|,$$

and so

$$\|x_n^*\|\|x_n\| \leq 2\mathsf{K_b}.$$

For $n = 1$ it is clear that $\|x_1^*\|\|x_1\| \leq \mathsf{K_b}$. These inequalities still hold if we replace $x_n^*$ by its Hahn–Banach extension $\hat{x}_n^*$ to $X$.

For each $x \in X$ put

$$T(x) = x + \sum_{n=1}^{\infty} \hat{x}_n^*(x)(y_n - x_n).$$

Then $T$ is a bounded operator from $X$ to $X$ with $T(x_n) = y_n$ and with norm

$$\|T\| \leq 1 + \sum_{n=1}^{\infty} \|\hat{x}_n^*\|\|y_n - x_n\| \leq 1 + 2\mathsf{K_b}\sum_{n=1}^{\infty} \frac{\|y_n - x_n\|}{\|x_n\|} = 1 + \theta.$$

Moreover,

$$\|T - I_X\| \leq \sum_{n=1}^{\infty} \|\hat{x}_n^*\|\|y_n - x_n\| = \theta < 1,$$

which implies that $T$ is invertible and $\|T^{-1}\| \leq (1-\theta)^{-1}$. □

As an application we obtain the following result, known as the *Bessaga–Pełczyński selection principle*. It was first formulated in [24]. The technique used in its proof has come to be called the *gliding hump* (or *sliding hump*) argument; the reader will see this type of argument in other contexts.

**Proposition 1.3.10 (The Bessaga–Pełczyński Selection Principle).** *Let* $(e_n)_{n=1}^\infty$ *be a basis for a Banach space* $X$ *with basis constant* $\mathsf{K_b}$ *and dual functionals* $(e_n^*)_{n=1}^\infty$. *Suppose* $(x_n)_{n=1}^\infty$ *is a sequence in* $X$ *such that*

*(i)* $\inf_n \|x_n\| > 0$, *but*
*(ii)* $\lim_{n\to\infty} e_k^*(x_n) = 0$ *for all* $k \in \mathbb{N}$.

*Then* $(x_n)_{n=1}^\infty$ *contains a subsequence* $(x_{n_k})_{k=1}^\infty$ *that is congruent to some block basic sequence* $(y_k)_{k=1}^\infty$ *of* $(e_n)_{n=1}^\infty$. *Furthermore, for every* $\epsilon > 0$ *it is possible to choose* $(n_k)_{k=1}^\infty$ *such that* $(x_{n_k})_{k=1}^\infty$ *has basis constant at most* $\mathsf{K_b} + \epsilon$. *In particular, the same result holds if* $(x_n)_{n=1}^\infty$ *converges to* 0 *weakly but not in the norm topology.*

*Proof.* Let $\alpha = \inf_n \|x_n\| > 0$ and suppose $0 < \nu < \frac{1}{4}$. Pick $n_1 = 1$, $r_0 = 0$. There exists $r_1 \in \mathbb{N}$ such that

$$\|x_{n_1} - S_{r_1} x_{n_1}\| < \frac{\nu\alpha}{2\mathsf{K_b}}.$$

Here, as usual, $S_m$ denotes the $m$th-partial sum operator with respect to the basis $(e_n)_{n=1}^\infty$. We know that $\lim_{n\to\infty} \|S_{r_1} x_n\| = 0$; therefore, there is $n_2 > n_1$ such that

$$\|S_{r_1} x_{n_2}\| < \frac{\nu^2\alpha}{2\mathsf{K_b}}.$$

Pick $r_2 > r_1$ such that

$$\|x_{n_2} - S_{r_2} x_{n_2}\| < \frac{\nu^2\alpha}{2\mathsf{K_b}}.$$

Again, since $\lim_{n\to\infty} \|S_{r_2} x_n\| = 0$, there exists $n_3 > n_2$ such that

$$\|S_{r_2} x_{n_3}\| < \frac{\nu^3\alpha}{2\mathsf{K_b}}.$$

In this way, we get a sequence $(x_{n_k})_{k=1}^\infty \subset X$ and a sequence of integers $(r_k)_{k=0}^\infty$ with $r_0 = 0$ such that

$$\|S_{r_{k-1}} x_{n_k}\| < \frac{\nu^k\alpha}{2\mathsf{K_b}} \quad \text{and} \quad \|x_{n_k} - S_{r_k} x_{n_k}\| < \frac{\nu^k\alpha}{2\mathsf{K_b}}.$$

For each $k \in \mathbb{N}$ let $y_k = S_{r_k} x_{n_k} - S_{r_{k-1}} x_{n_k}$. The sequence $(y_k)_{k=1}^\infty$ is a block basic sequence of the basis $(e_n)_{n=1}^\infty$. Hence, by Lemma 1.3.5, $(y_k)_{k=1}^\infty$ is a basic sequence with basis constant not bigger than $\mathsf{K_b}$. Notice that for each $k$,

$$\|y_k - x_{n_k}\| < \frac{\nu^k\alpha}{\mathsf{K_b}};$$

hence

$$\|y_k\| > \alpha - \frac{\nu\alpha}{\mathsf{K_b}} \geq (1-\nu)\alpha.$$

Then

$$2\mathsf{K}_{\mathsf{b}} \sum_{k=1}^{\infty} \frac{\|y_k - x_{n_k}\|}{\|y_k\|} < 2(1-\nu)^{-1} \sum_{k=1}^{\infty} \nu^k = 2\nu(1-\nu)^{-2} < \frac{8}{9}.$$

By Theorem 1.3.9, $(x_{n_k})_{k=1}^{\infty}$ is a basic sequence equivalent to $(y_k)_{k=1}^{\infty}$. Since $\nu$ can be made arbitrarily small, we can arrange the basis constant for $(x_{n_k})_{k=1}^{\infty}$ to be as close to $\mathsf{K}_{\mathsf{b}}$ as we wish. Moreover, if $(y_k)_{k=1}^{\infty}$ is complemented in $X$, so is $(x_{n_k})_{k=1}^{\infty}$. □

## 1.4 Bases and Basic Sequences: Discussion

The abstract concept of a Banach space grew very naturally from work in the early part of the twentieth century by Fredholm, Hilbert, F. Riesz, and others on concrete function spaces such as $\mathcal{C}[0,1]$ and $L_p$ for $1 \leq p < \infty$. The original motivation of these authors was to study linear differential and integral equations using the methods of linear algebra with analysis. By the end of the First World War the definition of a Banach space was almost demanding to be made, and it is therefore not surprising that it was independently discovered by Norbert Wiener and Stefan Banach around the same time. The axioms for a Banach space were introduced in Banach's thesis (1920), published in *Fundamenta Mathematicae* in 1922 in French.

The initial results of functional analysis are the underlying principles (uniform boundedness, closed graph and open mapping theorems and the Hahn–Banach theorem) which crystallized the common theme in so many arguments in analysis of the early twentieth century. However, after this, it was Banach and the school (Steinhaus, Mazur, Orlicz, Schauder, Ulam, et al.) in Lwów (then in Poland but now in the Ukraine) that developed the program of studying the isomorphic theory of Banach spaces. This school flourished until the time of the Second World War. In 1939, under the terms of the Nazi–Soviet pact, shortly after Germany invaded Poland, the Soviet Union occupied eastern Poland, including Lwów. After the Soviet invasion Banach was able to continue working, but the German invasion of 1941 effectively and tragically ended the work of his group. Banach himself suffered great hardship during the German occupation and died shortly after the end of the war, in 1945.

Given two classical Banach spaces $X$ and $Y$, one can ask questions such as whether $X$ is isomorphic to $Y$, or whether $X$ is isomorphic to a [complemented] subspace of $Y$. For these sorts of questions, bases and basic sequences are an invaluable tool.

In 1932 Banach formulated in his book [18, p. 111] the following:

**The basis problem:** *Does every separable Banach space have a basis?*

This problem motivated a great deal of research over the next forty years. Undoubtedly, the Lwów school knew much more about this problem than was ever published, but unfortunately, their research came to an untimely end with the

German invasion of the Soviet Union in 1941. In particular, Mazur, in the Scottish Book (an informal collection of problems kept in Lwów), formulated a very closely related problem that has come to be known as the *approximation problem*. Both problems were eventually solved by Per Enflo in 1973 [88], when he gave an example of a separable Banach space failing to have the approximation property and hence also failing to have a basis (see Problem 1.4). This solution is beyond the scope of this book (see [203]), but we can at least present two facts that were known to Banach: Theorem 1.4.4 and Theorem 1.4.5. To that end, let us first record the following lemma, which will be required many times.

**Lemma 1.4.1.** *Let $X$ be a Banach space.*

*(i) If $X$ is separable, then the closed unit ball $B_{X^*}$ of $X^*$ is (compact and) metrizable for the weak$^*$ topology.*
*(ii) Suppose $X^*$ contains a separating (or total) sequence $(x_n^*)_{n=1}^\infty$ for $X$; that is, $x_n^*(x) = 0$ for all $n \in \mathbb{N}$ implies $x = 0$. Then every weakly compact subset of $X$ is metrizable for the weak topology.*

*The conditions of (ii) hold when $X$ is separable.*

*Proof.* The proofs of both (*i*) and (*ii*) rely on the following simple observation. If $K$ is a compact set for some topology $\tau$, and $\tau'$ is any Hausdorff topology on $K$ that is weaker than $\tau$, then $\tau$ and $\tau'$ coincide. Indeed, suppose $A$ is a $\tau$-closed subset of $K$. Then $A$ is $\tau$-compact, and so its continuous image in $(K, \tau')$ under the identity map $(K, \tau) \to (K, \tau')$ is also compact, i.e., $A$ is $\tau'$-compact. Since $\tau'$ is Hausdorff, $A$ is $\tau'$-closed.

For (*i*), let us take $(x_n)_{n=1}^\infty$ dense in the unit ball $B_X$ of $X$. We define the topology $\rho$ induced on $X^*$ by convergence on each $x_n$. Precisely, a base of neighborhoods for $\rho$ at a point $x_0^* \in X^*$ is given by sets of the form

$$V_\epsilon(x_0^*; x_1, \dots, x_N) = \{x^* \in X^* : |x^*(x_n) - x_0^*(x_n)| < \epsilon,\ n = 1, \dots, N\},$$

where $\epsilon > 0$ and $N \in \mathbb{N}$. This topology is metrizable, and a metric inducing $\rho$ may be defined by

$$d(x^*, y^*) = \sum_{n=1}^\infty 2^{-n} \min(1, |x^*(x_n) - y^*(x_n)|), \qquad x^*, y^* \in X^*.$$

The topology $\rho$ is Hausdorff and weaker than the weak$^*$ topology, so it coincides with the weak$^*$ topology on the weak$^*$ compact set $B_{X^*}$.

To prove (*ii*) we choose for $\rho$ the topology on $X$ induced by convergence in each $x_n^*$. The details are very similar; the point separation property is equivalent to $\rho$ being Hausdorff.

Finally, if $X$ is separable, let $(x_n)_{n=1}^\infty$ be a sequence of nonzero vectors that is dense in $X$. For each $n$, using the Hahn–Banach theorem, pick $x_n^* \in X^*$ such that $x_n^*(x_n) = \|x_n\|$ and $\|x_n^*\| = 1$. Suppose $x_n^*(x) = 0$ for all $n$. Then if $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $\|x - x_m\| < \epsilon$. Thus $\|x_m\| = x_m^*(x_m) < \epsilon$, and so $\|x\| < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, we have $x = 0$. □

*Remark 1.4.2. (a)* Note that if $X = \ell_\infty$, then the conditions of Lemma 1.4.1 (*ii*) hold (use the coordinate functionals) but $X$ is not separable. Thus, every weakly compact subset of $\ell_\infty$ is metrizable.

*(b)* Let us observe as well that if $X$ is separable, then not only is the sequence $(x_n^*)_{n=1}^\infty$ in Lemma 1.4.1 (*ii*) separating for $X$, but the norm of every $x \in X$ is completely determined by this countably infinite set of functionals:

$$\|x\| = \sup_n |x_n^*(x)|, \qquad x \in X.$$

**Definition 1.4.3.** Suppose that $X$ is a normed space and that $Y$ is a subspace of $X^*$. Let us consider a new norm on $X$ given by

$$\|x\|_Y = \sup\big\{|y^*(x)| : y^* \in Y, \|y^*\| = 1\big\}, \qquad x \in X.$$

If there is a constant $c \leq 1$ such that

$$c\,\|x\| \leq \|x\|_Y \leq \|x\|\,, \qquad x \in X,$$

then $Y$ is said to be a *c-norming* subspace for $X$ in $X^*$.

The next theorem is in [18, p. 185]. The proof uses the Cantor set and some of its topological properties. By the *Cantor set*,[1] $\Delta$, we mean the topological space $\{0,1\}^{\mathbb{N}}$, the countable product of the two-point space $\{0,1\}$, endowed with the product topology. Sometimes, for convenience, we will equivalently realize the Cantor set as $\Delta = \{-1,1\}^{\mathbb{N}}$.

Among the features of the Cantor set we single out the following:

• $\Delta$ *embeds homeomorphically as a closed subspace of* $[0,1]$.
The map

$$\Delta \to [0,1], \qquad (t_n) \mapsto \sum_{n=1}^{\infty} \frac{2t_n}{3^n},$$

does the job.

• $[0,1]$ *is the continuous image of* $\Delta$.

Indeed, the function $\varphi : \Delta \to [0,1]$ defined by $\varphi((t_n)_{n=1}^\infty) = \sum_{n=1}^\infty t_n/2^n$ is continuous and surjective (but not one-to-one).

• $\Delta$ *is homeomorphic to the countable product of Cantor sets,* $\Delta^{\mathbb{N}}$.

[1] On the other hand, the *Cantor middle third set*, $\mathcal{C}$, consists of all real numbers $x$ in $[0,1]$ such that when we write $x$ in ternary form $x = \sum_{i=1}^\infty a_i/3^i$, then none of the numbers $a_1, a_2, \ldots$ equals 1 (i.e., either $a_i = 0$ or $a_i = 2$). The ternary correspondence from $\mathcal{C}$ onto $\Delta$, $\sum_{i=1}^\infty a_i/3^i \mapsto (a_1/2, a_2/2, \ldots)$, is a homeomorphism.

This follows from the fact that if $(A_i, \tau_i)_{i\in\mathbb{N}}$ is a countable family of topological spaces each of which is homeomorphic to the countable product of two-point spaces, $\{0, 1\}^{\mathbb{N}}$, then the topological product space $\prod_{i\in\mathbb{N}} A_i$ is homeomorphic to $\{0, 1\}^{\mathbb{N}}$.

• $[0, 1]^{\mathbb{N}}$ *is the continuous image of* $\Delta$.
Since $\Delta$ is homeomorphic to $\Delta^{\mathbb{N}}$, a point in $\Delta$ can be assumed to be of the form $(x_1, x_2, \dots)$, where $x_i \in \Delta$ for each $i$. If $\varphi : \Delta \to [0, 1]$ is a continuous surjection, then $\psi : \Delta \to [0, 1]^{\mathbb{N}}$ defined by $\psi(x_1, x_2, \dots) = (\varphi(x_1), \varphi(x_2), \dots)$ is continuous and surjective as well.

**Theorem 1.4.4 (The Banach–Mazur Theorem).** *If X is a separable Banach space then X embeds isometrically into* $\mathcal{C}[0, 1]$ *(and hence embeds isometrically in a space with a monotone basis).*

*Proof.* The proof will be a direct consequence of the following two facts:

**Fact 1.** *If X is a separable Banach space, then there exists a compact metrizable Hausdorff space K such that X embeds isometrically into* $\mathcal{C}(K)$.

Indeed, take $K = B_{X^*}$ with the relative weak$^*$ topology. If $X$ is separable, then $B_{X^*}$ is compact and metrizable, as we saw in Lemma 1.4.1. The isometric embedding of $X$ into $\mathcal{C}(B_{X^*})$ is easily checked to be achieved by the mapping $x \mapsto f_x$ where $f_x(x^*) = x^*(x)$ for all $x^* \in B_{X^*}$.

**Fact 2.** *If K is a compact metrizable space, then* $\mathcal{C}(K)$ *embeds isometrically into* $\mathcal{C}[0, 1]$. We split the proof of this statement into several steps:

• *If K is a compact metrizable space, then K embeds homeomorphically into* $[0, 1]^{\mathbb{N}}$.
Being compact and metrizable, $K$ contains a countable dense set, $(s_n)_{n=1}^{\infty}$. Let $\rho$ be a metric on $K$ inducing its topology. Without loss of generality we can assume that $0 \le \rho \le 1$. Now we define

$$\theta: K \to [0, 1]^{\mathbb{N}}, \qquad x \mapsto \theta(x) = (\rho(x, s_n))_{n=1}^{\infty}.$$

The map $\theta$ is continuous, since $x \mapsto \rho(x, s_n)$ is continuous for each $n$. Furthermore, $\theta$ is injective because if $x$ and $y$ are two different points in $K$, then there exists some $s_n$ such that $\rho(x, s_n) < \rho(y, s_n)$ (or the other way round), and therefore, $\theta(x)$ and $\theta(y)$ will differ in the $n$th coordinate. Since $K$ is compact and $[0, 1]^{\mathbb{N}}$ is Hausdorff, it follows that $\theta$ maps $K$ homeomorphically into its image.

• *If E is a closed subset of* $[0, 1]$*, then* $\mathcal{C}(E)$ *embeds isometrically into* $\mathcal{C}[0, 1]$. To show this, we need only define a norm-one extension operator $T: \mathcal{C}(E) \to \mathcal{C}[0, 1]$, i.e., a norm-one linear map such that $Tf|_E = f$ for all $f \in \mathcal{C}(E)$. Note that $[0, 1] \setminus E$ is a countable disjoint union of relatively open intervals; thus, we may extend $f$ to be affine on each such interval interior to $[0, 1]$ and to be constant on every such interval containing an endpoint of $[0, 1]$. This procedure clearly gives a linear extension operator.

We are ready now to complete the proof of Fact 2 and, therefore, of the theorem. Let $\psi : \Delta \to [0,1]^{\mathbb{N}}$ be a continuous surjection and let us consider $K$ a closed subset of $[0,1]^{\mathbb{N}}$. It follows that if $E = \psi^{-1}(K)$, then $E$ is homeomorphic to a (closed) subset of $[0,1]$. Then $\mathcal{C}(E)$ embeds isometrically into $\mathcal{C}[0,1]$. Finally, the map $f \mapsto f \circ \psi$ embeds $\mathcal{C}(K)$ isometrically into $\mathcal{C}(E)$, and therefore, $\mathcal{C}(K)$ embeds isometrically into $\mathcal{C}[0,1]$. □

Theorem 1.4.5 was also known to Banach's school in their approach to tackling the *basis problem*, and it is mentioned without proof by Banach in [18, p. 238]. Several proofs have been given since; for example, a proof due to Mazur is presented on p. 4 of [203], and we shall revisit this theorem in the next section (Corollary 1.5.3). The proof we include here is due to Bessaga and Pełczyński [24].

**Theorem 1.4.5.** *Every separable infinite-dimensional Banach space contains a basic sequence (i.e., a closed infinite-dimensional subspace with a basis). Furthermore, if* $\epsilon > 0$*, we may find a basic sequence with basis constant at most* $1 + \epsilon$.

*Proof.* By the Banach–Mazur theorem (Theorem 1.4.4) we can consider the case in which the separable Banach space $X$ is a closed subspace of $\mathcal{C}[0,1]$. Let $(e_n)_{n=1}^{\infty}$ be a monotone basis for $\mathcal{C}[0,1]$ with biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$. Since $X$ is infinite-dimensional, we may pick a sequence $(f_n)_{n=1}^{\infty}$ in $X$ with $\|f_n\| = 1$ and $e_k^*(f_n) = 0$ for $1 \le k \le n$. By Proposition 1.3.10 we can find a subsequence $(f_{n_k})_{k=1}^{\infty}$ that is basic with constant at most $1 + \epsilon$. □

## 1.5 Constructing Basic Sequences

The study of the isomorphic theory of Banach spaces went into retreat after the Second World War and was revived with the emergence of a new Polish school in Warsaw around 1958. There were some profound advances in Banach space theory between 1941 and 1958 (for example, the work of James and Grothendieck), but it seems that only after 1958 was there a concerted attack on problems of isomorphic structure. The prime mover in this direction was Pełczyński. Pełczyński, together with his collaborators, developed the theory of bases and basic sequences into a subtle and effective tool in Banach space theory. One nice aspect of the new theory was that basic sequences could be used to establish some classical results. In this section we are going to look deeper into the problem of constructing basic sequences and then show in the next section how this theory gives a nice and quite brief proof of the Eberlein–Šmulian theorem on weakly compact sets.

We will now present a refinement of the Mazur method for constructing basic sequences. We work in the dual $X^*$ of a Banach space for purely technical reasons; ultimately we will apply Lemma 1.5.1 and Theorem 1.5.2 to $X^{**}$.

**Lemma 1.5.1.** *Suppose that $S$ is a subset of $X^*$ such that $0 \in \overline{S}^{\text{weak}^*}$ but $0 \notin \overline{S}^{\|\cdot\|}$. Let $E$ be a finite-dimensional subspace of $X^*$. Then given $\epsilon > 0$ there exists $x^* \in S$ such that*

$$\|e^* + \lambda x^*\| \geq (1-\epsilon)\,\|e^*\|$$

*for all $e^* \in E$ and $\lambda \in \mathbb{R}$.*

*Proof.* Let us notice that such a set $S$ exists because the weak* topology and the norm topology of an infinite-dimensional Banach space do not coincide. The fact that $0 \notin \overline{S}^{\|\cdot\|}$ implies $\alpha \leq \|x^*\|$ for all $x^* \in S$, for some $0 < \alpha < \infty$.

Given $\epsilon > 0$ put $\overline{\epsilon} = \alpha\epsilon/2(1+\alpha)$. Let $U_E = \{e^* \in E \colon \|e^*\| = 1\}$. Since $E$ is finite-dimensional, $U_E$ is norm-compact. Take $y_1^*, y_2^*, \dots, y_N^* \in U_E$ such that given any $e^* \in U_E$ there is $y_k^*$ in that set with $\|e^* - y_k^*\| < \overline{\epsilon}$.

For each $j = 1, \dots, N$ pick $x_j \in B_X$ such that $y_j^*(x_j) > 1 - \overline{\epsilon}$. Since $0 \in \overline{S}^{\text{weak}^*}$ each neighborhood of $0$ in the weak* topology of $X^*$ contains at least one point of $S$ distinct from $0$. In particular, there is $x^* \in S$ such that $|x^*(x_j)| < \overline{\epsilon}$ for each $j = 1, \dots, N$.

If $e^* \in U_E$ and $|\lambda| \geq 2/\alpha$, we have

$$\|e^* + \lambda x^*\| \geq |\lambda|\alpha - 1 \geq 1.$$

If $|\lambda| < 2/\alpha$, we pick $y_k^*$ such that $\left\|e^* - y_k^*\right\| < \overline{\epsilon}$. Then

$$\left\|y_k^* + \lambda x^*\right\| \geq y_k^*(x_k) + \lambda x^*(x_k) > (1-\overline{\epsilon}) + \lambda x^*(x_k) \geq (1-\overline{\epsilon}) - |\lambda|\overline{\epsilon} \geq 1 - \Big(1 + \frac{2}{\alpha}\Big)\overline{\epsilon},$$

and therefore,

$$\|e^* + \lambda x^*\| \geq \Big|\, \|e^* - y_k^*\| - \|y_k^* + \lambda x^*\| \,\Big| \geq 1 - \Big(1 + \frac{2}{\alpha}\Big)\overline{\epsilon} - \overline{\epsilon} = 1 - \epsilon.$$

□

**Theorem 1.5.2.** *Suppose that $S$ is a subset of $X^*$ such that $0 \in \overline{S}^{\text{weak}^*}$ but $0 \notin \overline{S}^{\|\cdot\|}$. Then for every $\epsilon > 0$, $S$ contains a basic sequence with basis constant $\leq 1 + \epsilon$.*

*Proof.* Fix a decreasing sequence of positive numbers $(\epsilon_n)_{n=1}^\infty$ with $\sum_{n=1}^\infty \epsilon_n < \infty$ and such that $\prod_{n=1}^\infty (1-\epsilon_n) > (1+\epsilon)^{-1}$. Pick $x_1^* \in S$ and consider the 1-dimensional space $E_1 = [x_1^*]$. By Lemma 1.5.1 there is $x_2^* \in S$ such that

$$\|e^* + \lambda x_2^*\| \geq (1-\epsilon_1)\,\|e^*\|$$

for all $e^* \in E_1$ and $\lambda \in \mathbb{R}$.

Next let $E_2$ be the 2-dimensional space spanned by $x_1^*, x_2^*$. Lemma 1.5.1 yields $x_3^* \in S$ such that

$$\|e^* + \lambda x_3^*\| \geq (1 - \epsilon_2)\, \|e^*\|$$

for all $e^* \in E_2$ and $\lambda \in \mathbb{R}$. Repeating this process, we produce a sequence $(x_n^*)_{n=1}^\infty$ in $S$ such that for each $n \in \mathbb{N}$ and any scalars $(a_k)$,

$$\Big\| \sum_{k=1}^{n+1} a_k x_k^* \Big\| \geq (1 - \epsilon_n) \Big\| \sum_{k=1}^{n} a_k x_k^* \Big\|.$$

Therefore, given any integers $m, n$ with $m \leq n$, we have

$$\Big\| \sum_{k=1}^{m} a_k x_k^* \Big\| \leq \frac{1}{\prod_{j=1}^{n-1}(1 - \epsilon_j)} \Big\| \sum_{k=1}^{n} a_k x_k^* \Big\|.$$

Applying the Grunblum condition (Proposition 1.1.9), we conclude that $(x_n^*)_{n=1}^\infty$ is a basic sequence with basis constant at most $1 + \epsilon$. □

**Corollary 1.5.3.** *Every infinite-dimensional Banach space contains, for every $\epsilon > 0$, a basic sequence with basis constant $\leq 1 + \epsilon$.*

*Proof.* Let $X$ be an infinite-dimensional Banach space. Consider $S = \partial B_X = \{x \in X : \|x\| = 1\}$. We claim that 0 belongs to the weak closure of $S$; therefore, it belongs to the weak$^*$ closure of $S$ as a subspace of $X^{**}$.

If our claim failed, then there would exist some $\epsilon > 0$ and linear functionals $x_1^*, \ldots, x_n^*$ in $X^*$ such that the weak neighborhood of 0

$$V = \big\{x \in X : |x_k^*(x)| < \epsilon, \text{ for } k = 1, \ldots, n\big\}$$

satisfies $V \cap S = \emptyset$. This is impossible, because the intersection of the null subspaces of the $x_k^*$'s is a nontrivial subspace of $X$ contained in $V$ with points in $S$. Now Theorem 1.5.2 yields the existence of a basic sequence $(x_n)_{n=1}^\infty$ in $S$ with basis constant as close to 1 as we wish. □

The following proposition is often stated as a special case of Theorem 1.5.2. It may also be deduced equally easily using Theorem 1.4.5.

**Proposition 1.5.4.** *If $(x_n)_{n=1}^\infty$ is a weakly null sequence in an infinite-dimensional Banach space $X$ such that $\inf_n \|x_n\| > 0$, then for every $\epsilon > 0$, $(x_n)_{n=1}^\infty$ contains a basic subsequence with basis constant $\leq 1 + \epsilon$.*

*Proof.* Consider $S = \{x_n : n \in \mathbb{N}\}$. Since $(x_n)_{n=1}^\infty$ is weakly convergent, the set $S$ is norm bounded. Furthermore, $0 \in \overline{S}^{\text{weak}}$; hence by Theorem 1.5.2, $S$ contains a basic

sequence with basis constant at most $1+\epsilon$. To finish the proof we just have to prune this basic sequence by extracting terms in increasing order, and we obtain a basic subsequence of $(x_n)_{n=1}^{\infty}$. □

The next technical lemma will be required for our main result on basic sequences.

**Lemma 1.5.5.** *Let $(x_n)_{n=1}^{\infty}$ be a basic sequence in $X$. Suppose that there exists a linear functional $x^* \in X^*$ such that $x^*(x_n) = 1$ for all $n \in \mathbb{N}$. If $u \notin [x_n]$, then the sequence $(x_n + u)_{n=1}^{\infty}$ is basic.*

*Proof.* Since $u \notin [x_n]$, without loss of generality we can assume $x^*(u) = 0$. Let $T : X \to X$ be the operator given by $T(x) = x^*(x)u$. Then $I_X + T$ is invertible with inverse $I_X - T$. Since $(I_X + T)(x_n) = x_n + u$, the sequences $(x_n)_{n=1}^{\infty}$ and $(x_n + u)_{n=1}^{\infty}$ are congruent; hence $(x_n + u)_{n=1}^{\infty}$ is basic. □

We are now ready to give a criterion for a subset of a Banach space to contain a basic sequence. This criterion is due to Kadets and Pełczyński [148].

**Theorem 1.5.6.** *Let $S$ be a bounded subset of a Banach space $X$ such that $0 \notin \overline{S}^{\|\cdot\|}$. Then the following are equivalent:*

*(i) $S$ fails to contain a basic sequence,*
*(ii) $\overline{S}^{\text{weak}}$ is weakly compact and fails to contain $0$.*

*Proof.* $(ii) \Rightarrow (i)$ Suppose $(x_n)_{n=1}^{\infty} \subset S$ is a basic sequence. Since $\overline{S}^{\text{weak}}$ is weakly compact, $(x_n)_{n=1}^{\infty}$ has a weak cluster point, say $x$, in $\overline{S}^{\text{weak}}$. By Mazur's theorem, $x$ belongs to $[x_n]$, so we can write $x = \sum_{n=1}^{\infty} x_n^*(x)x_n$. By the continuity of the coefficient functionals $(x_n^*)_{n=1}^{\infty}$, it follows that for each $n$, $x_n^*(x)$ is a cluster point of the scalar sequence $(x_n^*(x_m))_{m=1}^{\infty}$, which converges to 0. Therefore, $x_n^*(x) = 0$ for all $n$, and as a consequence, $x = 0$. This contradicts the hypothesis, so $S$ contains no basic sequences.

$(i) \Rightarrow (ii)$ Assume $S$ contains no basic sequences. We can apply Theorem 1.5.2 to $S$ considered as a subset of $X^{**}$ with the weak* topology to conclude that 0 cannot be a weak closure point of $S$. It remains to show that $S$ is relatively weakly compact. To achieve this, we simply need to show that every weak* cluster point of $S$ in $X^{**}$ is already contained in $X$. Let us suppose $x^{**}$ is a weak* cluster point of $S$ and that $x^{**} \in X^{**} \setminus X$. Consider the set $S - x^{**} = \{s - x^{**} : s \in S\}$ in $X^{**}$. By Theorem 1.5.2 there exists $(x_n)_{n=1}^{\infty}$ in $S$ such that the sequence $(x_n - x^{**})_{n=1}^{\infty}$ is basic. We can suppose that $x^{**} \notin [x_n - x^{**} : n \geq 1]$, because it is certainly true that $x^{**} \notin [x_n - x^{**} : n \geq N]$ for some choice of $N$. By the Hahn–Banach theorem there exists $x^{***} \in X^{***}$ such that $x^{***} \in X^{\perp}$ and $x^{***}(x^{**}) = -1$. This implies that $x^{***}(x_n - x^{**}) = 1$ for all $n \in \mathbb{N}$. Now Lemma 1.5.5 applies, and we deduce that $(x_n)_{n=1}^{\infty}$ is also basic, contrary to our assumption on $S$. □

## 1.6 The Eberlein–Šmulian Theorem

Let $\mathcal{M}$ be a topological space. Let us recall that a subset $A$ of $\mathcal{M}$ is said to be *sequentially compact* [respectively, *relatively sequentially compact*] if every sequence in $A$ has a subsequence convergent to a point in $A$ [respectively, to a point in $\mathcal{M}$] and that $A$ is *countably compact* [respectively, *relatively countably compact*] if every sequence in $A$ has a cluster point in $A$ [respectively, in $\mathcal{M}$].

Countable compactness is implied by both compactness and sequential compactness. If $\mathcal{M}$ is a metrizable topological space, these three concepts certainly coincide, but if $\mathcal{M}$ is instead a general topological space, these equivalences are no longer valid. The easiest counterexample is obtained by considering $B_{\ell_\infty^*}$, the unit ball in $\ell_\infty^*$ with the weak* topology. The ball $B_{\ell_\infty^*}$ is, of course, weak* compact but fails to be weak* sequentially compact: the sequence of functionals $(e_n^*)$ given by $e_n^*(\xi) = \xi(n)$ has no weak* convergent subsequence.

In this section we will prove the Eberlein–Šmulian theorem, which asserts that in a Banach space the weak topology behaves like a metrizable topology in this respect although it need not be metrizable even on compact sets (except in the case of separable Banach space; see Lemma 1.4.1). That weak compactness implies weak sequentially compactness was discovered by Šmulian in 1940 [285]; the more difficult converse direction was obtained by Eberlein in 1947 [82]. This result is rather hard, and the original proof did not use the concept of a basic sequence, as the result predates the development of basic sequence techniques. The proof via basic sequences is due to Pełczyński [244]. Basic sequences seem to provide a conceptual simplification of the idea of the proof.

The lemmas we will need are the following:

**Lemma 1.6.1.** *If $(x_n)_{n=1}^\infty$ is a basic sequence in a Banach space and $x$ is a weak cluster point of $(x_n)_{n=1}^\infty$, then $x = 0$.*

*Proof.* Since $x$ is in the weak closure of the convex set $\langle x_n : n \in \mathbb{N}\rangle$ (the linear span of the sequence $(x_n)_{n=1}^\infty$), Mazur's theorem yields that $x$ belongs to $[x_n]$, the norm-closed linear span of $(x_n)_{n=1}^\infty$. Hence $x$ can be written as $x = \sum_{n=1}^\infty x_n^*(x)x_n$, where $(x_n^*)_{n=1}^\infty$ are the biorthogonal functionals of $(x_n)_{n=1}^\infty$. For each $n$, $x_n^*(x)$ is a cluster point of $(x_n^*(x_m))_{m=1}^\infty$, hence it is forced to be zero. Thus $x = 0$. □

**Lemma 1.6.2.** *Let $A$ be a relatively weakly countably compact subset of a Banach space $X$. Suppose that $x \in X$ is the only weak cluster point of a sequence $(x_n)_{n=1}^\infty$ contained in $A$. Then $(x_n)_{n=1}^\infty$ converges weakly to $x$.*

*Proof.* Assume that $(x_n)_{n=1}^\infty$ does not converge weakly to $x$. Then $(x^*(x_n))_{n=1}^\infty$ fails to converge to $x^*(x)$ for some $x^* \in X^*$. Hence we may pick a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $\inf_k |x^*(x) - x^*(x_{n_k})| > 0$. But this prevents $x$ from being a weak cluster point of $(x_{n_k})_{k=1}^\infty$, contradicting the hypothesis. □

**Theorem 1.6.3 (The Eberlein–Šmulian Theorem).** *Let $A$ be a subset of a Banach space $X$. The following are equivalent:*

*(i) $A$ is [relatively] weakly compact,*
*(ii) $A$ is [relatively] weakly sequentially compact,*
*(iii) $A$ is [relatively] weakly countably compact.*

*Proof.* Since $(i)$ and $(ii)$ both imply $(iii)$, we need only show that $(iii)$ implies both $(ii)$ and $(i)$. We will prove the relativized versions; minor modifications can be made to prove the nonrelativized versions. Note that each of the statements of the theorem implies that $A$ is bounded.

Let us first do the case $(iii)$ implies $(ii)$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $A$. By hypothesis there is a weak cluster point $x$ of $(x_n)_{n=1}^{\infty}$. If $x$ is a point in the norm-closure of the set $\{x_n\}_{n=1}^{\infty}$, then there is a subsequence that converges in norm, and we are done. If not, using Theorem 1.5.6, we can extract a subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $(y_n - x)_{n=1}^{\infty}$ is a basic sequence. But $(y_n)_{n=1}^{\infty}$ has a weak cluster point, $y$; hence $y - x$ is a weak cluster point of the basic sequence $(y_n - x)_{n=1}^{\infty}$. By Lemma 1.6.1 we have $y = x$. Thus $x$ is the *only* weak cluster point of $(y_n)_{n=1}^{\infty}$. Then $(y_n)_{n=1}^{\infty}$ converges weakly to $x$ by Lemma 1.6.2.

Let us turn to the case $(iii)$ implies $(i)$. Suppose $A$ fails to be relatively weakly compact. Since the weak$^*$ closure $W$ of $A$ in $X^{**}$ is necessarily weak$^*$ compact by the Banach–Alaoglu theorem, we conclude that this set cannot be contained in $X$. Thus there exists $x^{**} \in W \setminus X$. Pick $x^* \in X^*$ such that $x^{**}(x^*) > 1$. Then consider the set $A_0 = \{x \in A : \ x^*(x) > 1\}$. The set $A_0$ is not relatively weakly compact, since $x^{**}$ is in its weak$^*$ closure. Theorem 1.5.6 gives us a basic sequence $(x_n)_{n=1}^{\infty}$ contained in $A_0$. Appealing to countable compactness, we see that $(x_n)_{n=1}^{\infty}$ has a weak cluster point, $x$, which by Lemma 1.6.1 must be $x = 0$. This is a contradiction, since by construction, $x^*(x) \geq 1$. □

Combining Theorem 1.6.3 with Proposition H.2 in the appendix yields the following corollary.

**Corollary 1.6.4.** *A Banach space $X$ is reflexive if and only if every bounded sequence has a weakly convergent subsequence.*

The Eberlein–Šmulian theorem was probably the deepest result of earlier (pre-1950) Banach space theory. Not surprisingly, it inspired more examination, and it is far from the end of the story. In [119] the Eberlein–Šmulian theorem is extended to bounded subsets of $\mathcal{C}(K)$ ($K$ a compact Hausdorff space) with the weak topology replaced by the topology of pointwise convergence. This does not follow from basic sequence techniques, because it is no longer true that a cluster point of a basic sequence for pointwise convergence is necessarily zero. Later, Bourgain, Fremlin, and Talagrand [30] proved similar results for subsets of the Baire class-one functions on a compact metric space. A function is of *Baire class one* if it is a pointwise limit of a sequence of continuous functions.

## Problems

**1.1 (Mazur's Weak Basis Theorem).** A sequence $(e_n)_{n=1}^\infty$ is called a *weak basis* of a Banach space $X$ if for each $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^\infty$ such that $x = \sum_{n=1}^\infty a_n x_n$ in the weak topology. Show that every weak basis is a basis. [*Hint*: Try to imitate Theorem 1.1.3.]

**1.2 (Krein–Milman–Rutman Theorem).** Let $X$ be a Banach space with a basis and let $D$ be a dense subset of $X$. Show that $D$ contains a basis for $X$.

**1.3.** Let $(e_n)$ be a normalized basis for a Banach space $X$ and suppose there exists $x^* \in X^*$ with $x^*(e_n) = 1$ for all $n$. Show that the sequence $(e_n - e_{n-1})_{n=1}^\infty$ is also a basis for $X$ (we let $e_0 = 0$ in this definition).

**1.4 (The Approximation Property).** A Banach space $X$ is said to have the *approximation property*, (AP) for short, if for every compact set $K$ in $X$ and every $\epsilon > 0$ there is a finite-rank operator $T: X \to X$ such that $\|T(x) - x\| < \epsilon$ for every $x \in K$. In other words, a Banach space $X$ has (AP) if and only if the identity operator on $X$ can be approximated uniformly on every compact set by finite-rank operators.

Show that every Banach space $X$ with a basis has (AP).

**1.5 (The Bounded Approximation Property).** A separable Banach space $X$ has the *bounded approximation property* (BAP) if there is a sequence $(T_n)_{n=1}^\infty$ of finite-rank operators such that

$$\lim_{n\to\infty} \|x - T_n x\| = 0, \qquad x \in X. \tag{1.4}$$

(a) Show that (1.4) implies $\sup_n \|T_n\| < \infty$ and hence (BAP) implies (AP).
(b) Show that every complemented subspace of a space with a basis has (BAP).

**1.6.** Let $X$ be a Banach space and $A : X \to X$ a finite-rank operator. Show that for $\epsilon > 0$ there is a finite sequence of rank-one operators $(B_n)_{n=1}^N$ such that $A = B_1 + \cdots + B_N$ and

$$\sup_{1\le n\le N} \left\| \sum_{k=1}^n B_k \right\| < \|A\| + \epsilon.$$

**1.7.** Show that if $X$ has (BAP), then there is a sequence of rank-one operators $(B_n)_{n=1}^\infty$ such that $x = \sum_{n=1}^\infty B_n x$ for each $x \in X$. [*Hint*: Apply Problem 1.6 to $A = T_1$ and $A = T_n - T_{n-1}$ for $n = 2, 3, \ldots$.]

**1.8.** If $X$ has (BAP), let $(B_n)_{n=1}^\infty$ be the sequence of rank-one operators given in Problem 1.7. Let $B_n x = x_n^*(x) x_n$, where $x_n^* \in X^*$ and $x_n \in X$. Define $Y$ to be the

space of all sequences $\xi = (\xi(n))_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \xi(n)x_n$ converges under the norm

$$\|\xi\|_Y = \sup_n \left\| \sum_{k=1}^{n} \xi(k)x_k \right\|.$$

(a) Show that $(Y, \|\cdot\|_Y)$ is a Banach space and that the canonical basis vectors $(e_n)_{n=1}^{\infty}$ form a basis of $Y$.
(b) Show further that $X$ is isomorphic to a complemented subspace of $Y$.

Thus $X$ has (BAP) if and only if it is isomorphic to a complemented subspace of a space with a basis. This is due independently to Johnson et al. [142] and Pełczyński [247]. In 1987, Szarek [291] gave an example to show that not every space with (BAP) has a basis; this is very difficult! We refer to [41] for a full discussion of the problems associated with the bounded approximation property. See also Chapter 15 for the construction of Pełczyński's universal basis space $U$.

**1.9.** Suppose $X$ is a separable Banach space with the property that there is a sequence of finite-rank operators $(T_n)$ such that $\lim_{n\to\infty}\langle T_n x, x^*\rangle = \langle x, x^*\rangle$ for all $x \in X$, $x^* \in X^*$. Show that $X$ has (BAP).

**1.10.** Suppose that $X$ is a Banach space and that $(T_n)_{n=1}^{\infty}$ is a sequence of finite-rank operators such that $\lim_{n\to\infty}\langle T_n^* x^*, x^{**}\rangle = \langle x^*, x^{**}\rangle$ for every $x^* \in X^*$.

(a) Show that $(T_n)_{n=1}^{\infty}$ is a weakly Cauchy sequence in the space $\mathcal{K}(X)$ of compact operators on $X$ and that $(T_n)_{n=1}^{\infty}$ converges weak* to an element $\chi \in \mathcal{K}(X)^{**}$ where $\|\chi\| = 1$.
[*Hint:* Consider $B_{X^*}$ and $B_{X^{**}}$ with their respective weak* topologies. Embed $\mathcal{K}(X)$ into $\mathcal{C}(B_{X^*} \times B_{X^{**}})$ via the embedding $T \mapsto f_T$, where $f_T(x^*, x^{**}) = \langle T^*x^*, x^{**}\rangle$.]
(b) Using Goldstine's theorem, deduce the existence of a sequence of finite-rank operators $(S_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty}\|S_n\| = 1$ and $\lim_{n\to\infty}\|S_n x - x\| = 0$ for $x \in X$. [*Hint:* Choose each $S_n$ as a convex combination of $\{T_n, T_{n+1}, \dots\}$.]
Thus if $X$ is reflexive and has (BAP), we can choose the operators $T_n$ to have $\|T_n\| \le 1$; thus $X$ has the *metric approximation property* (MAP).

**1.11.** Consider $\mathbb{T}$ with the normalized measure $\frac{d\theta}{2\pi}$.

(a) Show that the exponentials$(e_0, e_1, e_{-1}, \dots)$ (see Section 1.2) do not form a basis of the complex space $L_1(\mathbb{T})$. [*Hint*: Prove that the partial sum operators $S_n f = \sum_{k=-n}^{n} \hat{f}(k)e_k$ are not uniformly bounded.]
(b) Show that if $1 < p < \infty$, then $(e_0, e_1, e_{-1}, \dots)$ form a basis of $L_p(\mathbb{T})$. (You may assume that the *Riesz projection* is bounded on $L_p(\mathbb{T})$, i.e., that there is a bounded linear operator $R : L_p(\mathbb{T}) \to L_p(\mathbb{T})$ such that $Re_k = 0$ when $k \le 0$ and $Re_k = e_k$ for $k \ge 0$. This is equivalent to the boundedness of the Hilbert transform; see for example [167, Theorem 1.8] or the original source [270].)

# Chapter 2
# The Classical Sequence Spaces

We now turn to the classical sequence spaces $\ell_p$ for $1 \leq p < \infty$ and $c_0$. The techniques developed in the previous chapter will prove very useful in this context. These Banach spaces are, in a sense, the simplest of all Banach spaces, and their structure has been well understood for many years. However, if $p \neq 2$, there can still be surprises, and there remain intriguing open questions.

Let us note at once that the spaces $\ell_p$ and $c_0$ are equipped with a canonical monotone Schauder basis $(e_n)_{n=1}^{\infty}$ given by $e_n(k) = 1$ if $k = n$ and 0 otherwise. One feature of the canonical basis of the $\ell_p$-spaces and $c_0$ that is useful to know is that $(e_n)_{n=1}^{\infty}$ is equivalent to the basis $(a_n e_n)_{n=1}^{\infty}$ whenever $0 < \inf_n |a_n| \leq \sup_n |a_n| < \infty$. This property is equivalent to the *unconditionality* of the basis, but we will not formally introduce this concept until the next chapter.

To avoid some complicated notation we will write a typical element of $\ell_p$ or $c_0$ as $\xi = (\xi(n))_{n=1}^{\infty}$. It is useful, and now fairly standard, to use $c_{00}$ to denote the subspace of all sequences of scalars $\xi = (\xi(n))_{n=1}^{\infty}$ such that $\xi(n) = 0$ except for finitely many $n$.

## 2.1 The Isomorphic Structure of the $\ell_p$-Spaces and $c_0$

We first ask ourselves a very simple question: are the spaces $\ell_p$ distinct (i.e., mutually nonisomorphic) Banach spaces? This question may seem absurd, because they look different, but recall that $L_2[0, 1]$ and $\ell_2$ are actually the same space in two different disguises. We can observe, for instance, that $c_0$ and $\ell_1$ are nonreflexive, while the spaces $\ell_p$ for $1 < p < \infty$ are reflexive; further, the dual of $c_0$ (i.e., $\ell_1$) is separable, but the dual of $\ell_1$ (i.e., $\ell_\infty$) is nonseparable.



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To help answer our question we need the following lemma:

**Lemma 2.1.1.** *Let $(y_k)_{k=1}^{\infty}$ be a normalized block basic sequence in $c_0$ or in $\ell_p$ for some $1 \le p < \infty$ . Then $(y_k)_{k=1}^{\infty}$ is isometrically equivalent to the canonical basis of the space, and $[y_k]$ is the range of a contractive projection.*

*Proof.* Let us treat the case that $(y_k)_{k=1}^{\infty}$ is a block basic sequence in $\ell_p$ for $1 \le p < \infty$ and leave the modifications for the $c_0$ case to the reader. Suppose that

$$y_k = \sum_{j=r_{k-1}+1}^{r_k} a_j e_j, \qquad k \in \mathbb{N},$$

where $0 = r_0 < r_1 < r_2 < \cdots$ are positive integers and $(a_j)_{j=1}^{\infty}$ are scalars such that

$$\|y_k\|^p = \sum_{j=r_{k-1}+1}^{r_k} |a_j|^p = 1, \qquad k \in \mathbb{N}.$$

Then, given any $m \in \mathbb{N}$ and any scalars $b_1, \dots, b_m$, we have

$$\begin{aligned}\Big\| \sum_{k=1}^{m} b_k y_k \Big\| &= \Big\| \sum_{k=1}^{m} \sum_{j=r_{k-1}+1}^{r_k} b_k a_j e_j \Big\| \\ &= \Big( \sum_{k=1}^{m} |b_k|^p \sum_{j=r_{k-1}+1}^{r_k} |a_j|^p \Big)^{1/p} \\ &= \Big( \sum_{k=1}^{m} |b_k|^p \Big)^{1/p}.\end{aligned}$$

This establishes isometric equivalence. We shall construct a contractive projection onto $[y_k]_{k=1}^{\infty}$. Here we suppose $1 < p < \infty$ and leave both cases $c_0$ and $\ell_1$ to the reader. For each $k$ we select scalars $(b_j)_{j=r_{k-1}+1}^{r_k}$ such that $\sum_{j=r_{k-1}+1}^{r_k} |b_j|^q = 1$ and $\sum_{j=r_{k-1}+1}^{r_k} b_j a_j = 1$. Put

$$y_k^* = \sum_{j=r_{k-1}+1}^{r_k} b_j e_j^*.$$

Clearly, $(y_k^*)_{k=1}^{\infty}$ is biorthogonal to $(y_k)_{n=1}^{\infty}$ and $\|y_k^*\| = \|y_k\| = 1$. Let us see that

$$P(\xi) = \sum_{k=1}^{\infty} y_k^*(\xi) y_k, \qquad \xi \in \ell_p,$$

defines a norm-one projection from $\ell_p$ onto $[y_k]$. We will show that $\|P\xi\| \le \|\xi\|$ when $\xi \in c_{00}$ and then observe that $P$ extends by density to a contractive projection.

For each $\xi = (\xi(j))_{j=1}^{\infty} \in c_{00}$,

$$
\begin{aligned}
|y_k^*(\xi)| &= \Big| \sum_{j=r_{k-1}+1}^{r_k} b_j \xi(j) \Big| \\
&\le \Big( \sum_{j=r_{k-1}+1}^{r_k} |b_j|^q \Big)^{1/q} \Big( \sum_{j=r_{k-1}+1}^{r_k} |\xi(j)|^p \Big)^{1/p} = \Big( \sum_{j=r_{k-1}+1}^{r_k} |\xi(j)|^p \Big)^{1/p}.
\end{aligned}
$$

Then, using the isometric equivalence of $(y_k)_{k=1}^{\infty}$ and $(e_k)_{k=1}^{\infty}$, we obtain

$$
\|P(\xi)\| = \Big( \sum_{k=1}^{\infty} |y_k^*(\xi)|^p \Big)^{1/p} \le \Big( \sum_{k=1}^{\infty} \sum_{j=r_{k-1}+1}^{r_k} |\xi(j)|^p \Big)^{1/p} = \|\xi\|.
$$

□

*Remark 2.1.2.* Notice that if $(y_k)_{k=1}^{\infty}$ is not normalized but satisfies instead an inequality

$$
0 < a \le \|y_k\| \le b < \infty, \quad k \in \mathbb{N},
$$

for some constants $a$, $b$ (in which case $(y_k)_{k=1}^{\infty}$ is said to be *seminormalized*), then we can apply the previous lemma to $(y_k/\|y_k\|)_{k=1}^{\infty}$, and we obtain that $(y_k)_{k=1}^{\infty}$ is equivalent to $(e_k)_{k=1}^{\infty}$ (but not isometrically) and $[y_k]$ is complemented by a contractive projection.

Although the preceding lemma was quite simple, it already leads to a powerful conclusion:

**Proposition 2.1.3.** *Suppose $1 \le p < \infty$. Let $(x_n)_{n=1}^{\infty}$ be a normalized sequence in $\ell_p$ [respectively, $c_0$] such that for each $j \in \mathbb{N}$ we have $\lim_{n\to\infty} x_n(j) = 0$ (for example, suppose $(x_n)_{n=1}^{\infty}$ is weakly null). Then there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ that is a basic sequence equivalent to the canonical basis of $\ell_p$ [respectively, $c_0$] and such that $[x_{n_k}]_{k=1}^{\infty}$ is complemented in $\ell_p$ [respectively, $c_0$].*

*Proof.* The *gliding hump* technique (see Proposition 1.3.10) yields a subsequence $(x_{n_k})_{k=1}^{\infty}$ and a block basic sequence $(y_k)_{k=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$ such that $(x_{n_k})_{k=1}^{\infty}$ is basic, equivalent to $(y_k)_{k=1}^{\infty}$, and such that $[x_{n_k}]_{k=1}^{\infty}$ is complemented whenever $[y_k]_{k=1}^{\infty}$ is. By Lemma 2.1.1 we are done. □

Now let us prove a classical result from the 1930s (Pitt [261]).

**Theorem 2.1.4 (Pitt's Theorem).** *Suppose $1 \le p < r < \infty$. If $X$ is a closed subspace of $\ell_r$ and $T : X \to \ell_p$ is a bounded operator, then $T$ is compact.*

*Proof.* Since $\ell_r$ is reflexive, the subspace $X$ is reflexive, and so $B_X$ is weakly compact. Therefore in order to prove that $T$ is compact, it suffices to show that $T|_{B_X}$ is weak-to-norm continuous. Since the weak topology of $X$ restricted to $B_X$ is metrizable (Lemma 1.4.1 (*ii*)), it suffices to see that whenever $(x_n)_{n=1}^{\infty} \subset B_X$ is weakly convergent to some $x$ in $B_X$, then $(T(x_n))_{n=1}^{\infty}$ converges in norm to $Tx$.

We need only show that if $(x_n)_{n=1}^{\infty}$ is a weakly null sequence in $X$, then $\lim \|Tx_n\| = 0$. If this fails, we may suppose the existence of a weakly null sequence $(x_n)_{n=1}^{\infty}$ with $\|x_n\| = 1$ such that $\|Tx_n\| \geq \delta > 0$ for all $n$. By passing to a subsequence, we may suppose that $(x_n)_{n=1}^{\infty}$ is a basic sequence equivalent to the canonical $\ell_r$-basis (Proposition 2.1.3). But then, since $(Tx_n)_{n=1}^{\infty}$ is also weakly null, by passing to a further subsequence we may suppose that $(Tx_n/\|Tx_n\|)_{n=1}^{\infty}$, and hence $(Tx_n)_{n=1}^{\infty}$, is basic and equivalent to the canonical $\ell_p$-basis. Since $T$ is bounded, we have effectively shown that the identity map $\iota : \ell_r \to \ell_p$ is bounded, which is absurd. Or alternatively, there exist constants $C_1$ and $C_2$ such that the following inequalities hold simultaneously for all $n$:

$$\Big\| \sum_{k=1}^{n} x_k \Big\|_r \leq C_1 n^{1/r} \quad \text{and} \quad \Big\| \sum_{k=1}^{n} Tx_k \Big\|_p \geq C_2 n^{1/p},$$

which contradicts the boundedness of $T$. □

*Remark 2.1.5. (a)* Essentially the same proof of Theorem 2.1.4 works with $c_0$ replacing $\ell_r$. Although $c_0$ is nonreflexive, Lemma 1.4.1 can still be used to show that $B_X$ is at least weakly metrizable, and the weak-to-norm continuity of $T|_{B_X}$ is enough to show that the image is relatively norm-compact.

*(b)* We would like to single out the following crucial ingredient in the proof of Pitt's theorem. *Suppose $T : \ell_r \to \ell_p$ is a bounded operator with $1 \leq p < r < \infty$. Then whenever $(x_n)_{n=1}^{\infty}$ is a weakly null sequence in $\ell_r$, we have $\|Tx_n\|_p \to 0$. In particular, $\|Te_n\|_p \to 0$. The same is true for any operator $T\colon c_0 \to \ell_p$.*

**Corollary 2.1.6.** *The spaces of the set $\{c_0\} \cup \{\ell_p : 1 \leq p < \infty\}$ are mutually nonisomorphic. In fact, if $X$ is an infinite-dimensional subspace of one of the spaces $\{c_0\} \cup \{\ell_p : 1 \leq p < \infty\}$, then $X$ is not isomorphic to a subspace of any other.*

This suggests the following definition:

**Definition 2.1.7.** Two infinite-dimensional Banach spaces $X$, $Y$ are said to be *totally incomparable* if they have no infinite-dimensional subspaces in common (up to isomorphism).

What can be said for bounded operators $T : \ell_p \longrightarrow \ell_r$ for $p < r$? First, notice that in this case, Pitt's theorem is not true. Take, for example, the natural inclusion $\iota : \ell_p \hookrightarrow \ell_r$. The map $\iota$ is a norm-one operator that is not compact, since the image of the canonical basis of $\ell_p$ is a sequence contained in $\iota(B_{\ell_p})$ with no convergent subsequences.

**Definition 2.1.8.** A bounded operator $T$ from a Banach space $X$ into a Banach space $Y$ is *strictly singular* if there is no infinite-dimensional subspace $E \subset X$ such that $T|_E$ is an isomorphism onto its range.

**Theorem 2.1.9.** *If $p \neq r$, every bounded operator $T: \ell_p \longrightarrow \ell_r$ is strictly singular.*

*Proof.* This is immediate from Corollary 2.1.6. □

## 2.2 Complemented Subspaces of $\ell_p$ $(1 \le p < \infty)$ and $c_0$

The results of this section are due to Pełczyński [241]; they demonstrate the power of basic sequence techniques.

**Proposition 2.2.1.** *Every infinite-dimensional closed subspace $Y$ of $\ell_p$ $(1 \le p < \infty)$ [respectively, $c_0$] contains a closed subspace $Z$ such that $Z$ is isomorphic to $\ell_p$ [respectively, $c_0$] and complemented in $\ell_p$ [respectively, $c_0$].*

*Proof.* Since $Y$ is infinite-dimensional, for every $n$ there is $x_n \in Y$ with $\|x_n\| = 1$ such that $e_k^*(x_n) = 0$ for $1 \le k \le n$. If not, for some $N \in \mathbb{N}$ the projection $S_N(\sum_{n=1}^{\infty} a_n e_n) = \sum_{n=1}^{N} a_n e_n$ restricted to $Y$ would be injective (since $0 \neq y \in Y$ would imply $S_N(y) \neq 0$), and so $S_N|_Y$ would be an isomorphism onto its image, which is impossible because $Y$ is infinite-dimensional. By Proposition 2.1.3 the sequence $(x_n)_{n=1}^{\infty}$ has a subsequence $(x_{n_k})_{k=1}^{\infty}$ that is basic, equivalent to the canonical basis of the space and such that the subspace $Z = [x_{n_k}]$ is complemented. □

Since $c_0$ and $\ell_1$ are nonreflexive and every closed subspace of a reflexive space is reflexive, using Proposition 2.2.1 we obtain the following result.

**Proposition 2.2.2.** *Let $Y$ be an infinite-dimensional closed subspace of either $c_0$ or $\ell_1$. Then $Y$ is not reflexive.*

Suppose now that $Y$ is itself complemented in $\ell_p$ $(1 \le p < \infty)$ [respectively, $c_0$]. Proposition 2.2.1 tells us that $Y$ contains a complemented copy of $\ell_p$ [respectively, $c_0$]. Can we say more? Remarkably, Pełczyński discovered a trick that enables us, by rather "soft" arguments, to do quite a bit better. This trick is nowadays known as the *Pełczyński decomposition technique* and has proved very useful in different contexts.

The situation is this: we have two Banach spaces $X$ and $Y$ such that $Y$ is isomorphic to a complemented subspace of $X$, and $X$ is isomorphic to a complemented subspace of $Y$. We would like to deduce that $X$ and $Y$ are isomorphic. This is known (by analogy with a similar result for cardinals) as the *Schroeder–Bernstein problem* for Banach spaces. It was not until 1996 that Gowers [115] showed that it is not true in general, using a space constructed by himself and Maurey that contains no unconditional basic sequences (see the mention of *the unconditional basic sequence*

*problem* in Section 3.5); one year later, Gowers and Maurey [117] gave an example of a Banach space $X$ that is isomorphic to $X^3$ but not to $X^2$ (thereby failing to have the Schroeder–Bernstein property).

The Pełczyński decomposition technique gives two criteria for which the Schroeder–Bernstein problem has a positive solution.

We need to introduce the spaces $\ell_p(X)$ for $1 \leq p < \infty$ and $c_0(X)$, where $X$ is a given Banach space.

For $1 \leq p < \infty$, the space $\ell_p(X) = (X \oplus X \oplus \cdots)_p$, called the *infinite direct sum of $X$ in the sense of $\ell_p$*, consists of all sequences $x = (x(n))_{n=1}^{\infty}$ with values in $X$ such that $(\|x(n)\|_X)_{n=1}^{\infty} \in \ell_p$, with the norm

$$\|x\| = \|(\|x(n)\|_X)_{n=1}^{\infty}\|_p = \left(\sum_{n=1}^{\infty} \|x(n)\|_X^p\right)^{1/p}.$$

Similarly, the space $c_0(X) = (X \oplus X \oplus \cdots)_{c_0}$, called the *infinite direct sum of $X$ in the sense of $c_0$*, consists of all $X$-valued sequences $x = (x(n))_{n=1}^{\infty}$ such that $\lim_n \|x(n)\|_X = 0$ under the norm

$$\|x\| = \max_{1 \leq n < \infty} \|x(n)\|_X.$$

Notice that $\ell_p(\ell_p)$ can be identified with $\ell_p(\mathbb{N} \times \mathbb{N})$, and hence it is isometric to $\ell_p$. Analogously, $c_0(c_0)$ is isometric to $c_0$.

**Theorem 2.2.3 (The Pełczyński decomposition technique [241]).** *Let $X$ and $Y$ be Banach spaces such that $X$ is isomorphic to a complemented subspace of $Y$, and $Y$ is isomorphic to a complemented subspace of $X$. Suppose further that either*

*(a) $X \approx X^2 = X \oplus X$ and $Y \approx Y^2$, or*
*(b) $X \approx c_0(X)$ or $X \approx \ell_p(X)$ for some $1 \leq p \leq \infty$.*

*Then $X$ is isomorphic to $Y$.*

*Proof.* Let us put $X \approx Y \oplus E$ and $Y \approx X \oplus F$. If $(a)$ holds, then we have

$$X \approx Y \oplus Y \oplus E \approx Y \oplus X,$$

and by a symmetric argument $Y \approx X \oplus Y$. Hence $Y \approx X$.

If $X$ satisfies $(b)$ in particular, we have $X \approx X^2$, so as in part $(a)$ we obtain $Y \approx X \oplus Y$. On the other hand, then

$$\ell_p(X) \approx \ell_p(Y \oplus E) \approx \ell_p(Y) \oplus \ell_p(E).$$

Hence if $X \approx \ell_p(X)$,

$$X \approx Y \oplus \ell_p(Y) \oplus \ell_p(E) \approx Y \oplus \ell_p(X) \approx Y \oplus X.$$

The proof is analogous if $X \approx c_0(X)$. □

We are ready to prove a beautiful theorem due to Pełczyński [241] that had a profound influence on the development of Banach space theory.

**Theorem 2.2.4.** *Suppose $Y$ is a complemented infinite-dimensional subspace of $\ell_p$ where $1 \le p < \infty$ [respectively, $c_0$]. Then $Y$ is isomorphic to $\ell_p$ [respectively, $c_0$].*

*Proof.* Proposition 2.2.1 gives an infinite-dimensional subspace $Z$ of $Y$ such that $Z$ is isomorphic to $\ell_p$ [respectively, $c_0$] and $Z$ is complemented in $\ell_p$ [respectively, $c_0$]. Obviously $Z$ is also complemented in $Y$; therefore, $\ell_p$ [respectively, $c_0$] is (isomorphic to) a complemented subspace in $Y$. Since $\ell_p(\ell_p) = \ell_p$ [respectively, $c_0(c_0) = c_0$], $(b)$ of Theorem 2.2.3 applies and we are done. □

At this point let us discuss where this theorem leads. First, the alert reader may ask whether it is true that *every* subspace of $\ell_p$ is actually complemented. Certainly this is true when $p = 2$! This is a special case of what is known as the complemented subspace problem.

**The complemented subspace problem.** *If $X$ is a Banach space such that every closed subspace is complemented, is $X$ isomorphic to a Hilbert space?*

This problem was settled positively by Lindenstrauss and Tzafriri in 1971 [200]. We will later discuss its general solution, but at the moment, let us point out that it is not so easy to demonstrate the answer even for the $\ell_p$-spaces when $p \ne 2$. In this chapter we will show that $\ell_1$ has an uncomplemented subspace.

Another way to approach the complemented subspace problem is to demonstrate that $\ell_p$ has a subspace that is not isomorphic to the whole space. Here we meet another question dating back to Banach:

**The homogeneous space problem.** *Let $X$ be a Banach space that is isomorphic to every one of its infinite-dimensional closed subspaces. Is $X$ isomorphic to a Hilbert space?*

This problem was finally solved, again positively, by Komorowski and Tomczak-Jaegermann [175] in 1996 (using an important ingredient by Gowers [114]).

Oddly enough, the $\ell_p$-spaces for $p \ne 2$ are not as regular as one would expect. In fact, for every $p \ne 2$, $\ell_p$ contains a subspace without a basis. For $p > 2$ this was proved by Davie in 1973 [55]; for general $p$ it was obtained by Szankowski [289] a few years later. However, the construction of such subspaces is far from easy and will not be covered in this book. Notice that this provides an example of a separable Banach space without a basis.

One natural idea that comes out of Theorem 2.2.4 is the notion that the $\ell_p$-spaces and $c_0$ are the building blocks from which Banach spaces are constructed; by analogy they might play the role of primes in number theory. This thinking is behind the following definition:

**Definition 2.2.5.** A Banach space $X$ is called *prime* if every complemented infinite-dimensional subspace of $X$ is isomorphic to $X$.

Thus the $\ell_p$-spaces and $c_0$ are prime. Are there other primes? One may immediately ask about $\ell_\infty$, and indeed, this is a (nonseparable) prime space, as was shown by Lindenstrauss in 1967 [194]; we will show this later. The quest for other

prime spaces has proved difficult; some candidates have been found, but in general it is very hard to prove that a particular space is prime. Eventually another prime space was found by Gowers and Maurey [117], but the construction is very involved, and the space is far from being "natural." In fact, the Gowers–Maurey prime space has the property that the only complemented subspaces of infinite dimension are of *finite* codimension. One can say that this space is prime only because it has very few complemented subspaces at all!

## 2.3 The Space $\ell_1$

The space $\ell_1$ has a special role in Banach space theory. In this section we develop some of its elementary properties. We start by proving a universal property of $\ell_1$ with respect to separable spaces due to Banach and Mazur [19] from 1933.

**Theorem 2.3.1.** *If $X$ is a separable Banach space, then there exists a continuous operator $Q : \ell_1 \to X$ from $\ell_1$ onto $X$.*

*Proof.* It suffices to show that $X$ admits of a continuous operator $Q: \ell_1 \to X$ such that $Q(\{\xi \in \ell_1 : \|\xi\|_1 < 1\}) = \{x \in X : \|x\| < 1\}$.

Let $(x_n)_{n=1}^\infty$ be a dense sequence in $B_X$ and define $Q : \ell_1 \to X$ by $Q(\xi) = \sum_{n=1}^\infty \xi(n)x_n$. Notice that $Q$ is well defined: for every $\xi = (\xi(n))_{n=1}^\infty \in \ell_1$ the series $\sum_{n=1}^\infty \xi(n)x_n$ is absolutely convergent in $X$. The map $Q$ is clearly linear and has norm one, since

$$\|Q(\xi)\| = \Big\| \sum_{n=1}^\infty \xi(n)x_n \Big\| \le \sum_{n=1}^\infty |\xi(n)| = \|(\xi(n))\|_1 .$$

The set $Q(B_{\ell_1})$ is dense in $B_X$; hence given $x \in B_X$ and $0 < \epsilon < 1$, there exists $\xi_1 \in B_{\ell_1}$ such that $\|x - Q\xi_1\| < \epsilon$. Next we find $\xi_2' \in B_{\ell_1}$ such that

$$\Big\| \frac{1}{\epsilon}(x - Q\xi_1) - Q\xi_2' \Big\| < \epsilon.$$

If we let $\xi_2 = \epsilon\xi_2'$, we obtain

$$\|x - Q(\xi_1 + \xi_2)\| < \epsilon^2.$$

Iterating, we obtain a sequence $(\xi_n)_{n=1}^\infty$ in $B_{\ell_1}$ satisfying

- $\|\xi_n\|_1 < \epsilon^{n-1}$, and
- $\|x - Q(\xi_1 + \cdots + \xi_n)\| < \epsilon^n$.

Let $\xi = \sum_{n=1}^\infty \xi_n$. Then $\|\xi\|_1 \le (1-\epsilon)^{-1}$ and $Q\xi = x$. Since $\epsilon$ is arbitrary, by scaling we deduce that $Q(\{\xi \in \ell_1 : \|\xi\|_1 < 1\}) = \{x \in X : \|x\| < 1\}$. □

**Corollary 2.3.2.** *If $X$ is a separable Banach space, then $X$ is isometrically isomorphic to a quotient of $\ell_1$.*

*Proof.* Let $Q : \ell_1 \to X$ be the quotient map in the proof of Theorem 2.3.1. Then it follows that $\ell_1/\ker Q$ is isometrically isomorphic to $X$.

□

**Corollary 2.3.3.** *The space $\ell_1$ has an uncomplemented closed subspace.*

*Proof.* Take $X$ a separable Banach space that is not isomorphic to $\ell_1$. Theorem 2.3.1 yields an operator $Q$ from $\ell_1$ onto $X$ whose kernel is a closed subspace of $\ell_1$. If $\ker Q$ were complemented in $\ell_1$, then we would have $\ell_1 = \ker Q \oplus M$ for some closed subspace $M$ of $\ell_1$ and therefore

$$X = \ell_1/\ker Q \approx M.$$

But this can occur only if $X$ is isomorphic to $\ell_1$ by Theorem 2.2.4. □

**Definition 2.3.4.** A Banach space $X$ has the *Schur property* (or $X$ is a *Schur space*) if weak and norm sequential convergence coincide in $X$, i.e., a sequence $(x_n)_{n=1}^\infty$ in $X$ converges to 0 weakly if and only if $(x_n)_{n=1}^\infty$ converges to 0 in norm.

*Example 2.3.5.* Neither the spaces $\ell_p$ for $1 < p < \infty$ nor $c_0$ has the Schur property, since the canonical basis is weakly null but cannot converge to 0 in norm.

The next result was discovered in an equivalent form by Schur in 1920 [281].

**Theorem 2.3.6.** *The space $\ell_1$ has the Schur property.*

*Proof.* Suppose $(x_n)_{n=1}^\infty$ is a weakly null sequence in $\ell_1$ that does not converge to 0 in norm. Using Proposition 2.1.3, we see that $(x_n)_{n=1}^\infty$ contains a subsequence that is basic and equivalent to the canonical basis; this gives a contradiction, because the canonical basis of $\ell_1$ is clearly not weakly null. □

**Theorem 2.3.7.** *Let $X$ be a Banach space with the Schur property. Then a subset $W$ of $X$ is weakly compact if and only if $W$ is norm-compact.*

*Proof.* Suppose $W$ is weakly compact and consider a sequence $(x_n)_{n=1}^\infty$ in $W$. By the Eberlein–Šmulian theorem $W$ is weakly sequentially compact, so $(x_n)_{n=1}^\infty$ has a subsequence $(x_{n_k})_{k=1}^\infty$ that converges weakly to some $x \in W$. Since $X$ has the Schur property, $(x_{n_k})_{k=1}^\infty$ converges to $x$ in norm as well. Therefore $W$ is compact for the norm topology. □

**Corollary 2.3.8.** *If $X$ is a reflexive Banach space with the Schur property, then $X$ is finite-dimensional.*

*Proof.* If a reflexive Banach space $X$ has the Schur property, then its unit ball is norm-compact by Theorem 2.3.7, and so $X$ is finite-dimensional. □

**Definition 2.3.9.** A sequence $(x_n)_{n=1}^\infty$ in a Banach space $X$ is *weakly Cauchy* if $\lim_n x^*(x_n)$ exists for every $x^*$ in $X^*$.

Every weakly Cauchy sequence $(x_n)_{n=1}^{\infty}$ in a Banach space $X$ is norm-bounded by the uniform boundedness principle. If $X$ is reflexive, by Corollary 1.6.4, $(x_n)_{n=1}^{\infty}$ will have a weak cluster point, $x$, and so $(x_n)_{n=1}^{\infty}$ will converge weakly to $x$. If $X$ is nonreflexive, however, there may be sequences that are weakly Cauchy but not weakly convergent.

**Definition 2.3.10.** A Banach space $X$ is said to be *weakly sequentially complete* (wsc) if every weakly Cauchy sequence in $X$ converges weakly.

*Example 2.3.11.* In the space $c_0$ consider the sequence $x_n = e_1 + \cdots + e_n$, where $(e_n)$ is the unit vector basis. Then $(x_n)_{n=1}^{\infty}$ is obviously weakly Cauchy, but it does not converge weakly in $c_0$. The sequence $(x_n)_{n=1}^{\infty}$ converges weak* in the bidual, $\ell_\infty$, to the element $(1, 1, \ldots, 1, \ldots)$. Thus $c_0$ is not weakly sequentially complete.

**Proposition 2.3.12.** *Every Banach space with the Schur property (in particular $\ell_1$) is weakly sequentially complete.*

*Proof.* Suppose $(x_n)_{n=1}^{\infty}$ is weakly Cauchy. Then for every two strictly increasing sequences of integers $(n_k)_{k=1}^{\infty}$, $(m_k)_{k=1}^{\infty}$ the sequence $(x_{m_k} - x_{n_k})_{k=1}^{\infty}$ is weakly null and so $\lim_{k\to\infty} \|x_{m_k} - x_{n_k}\| = 0$. Thus, being norm-Cauchy, $(x_n)_{n=1}^{\infty}$ is norm-convergent and hence weak-convergent. □

## 2.4 Convergence of Series and Operators on $c_0$

In this section we study different properties related to the convergence of reordered series in Banach spaces. The goal is to prove the Orlicz–Pettis theorem, which says that weak subseries convergence is equivalent to unconditional convergence in norm.

**Definition 2.4.1.** Let $(x_n)_{n=1}^{\infty}$ be a sequence in a Banach space $X$. A (formal) series $\sum_{n=1}^{\infty} x_n$ in $X$ is said to be *unconditionally convergent* if $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation $\pi$ of $\mathbb{N}$.

We will see in Chapter 8 that except in finite-dimensional spaces, unconditional convergence of a series $\sum_{n=1}^{\infty} x_n$ in a Banach space $X$ is weaker than *absolute convergence*, i.e., convergence of the scalar series $\sum_{n=1}^{\infty} \|x_n\|$.

**Lemma 2.4.2.** *Given a series $\sum_{n=1}^{\infty} x_n$ in a Banach space $X$, the following are equivalent:*

*(a)* $\sum_{n=1}^{\infty} x_n$ *is unconditionally convergent.*
*(b) The series* $\sum_{k=1}^{\infty} x_{n_k}$ *converges for every increasing sequence* $(n_k)_{k=1}^{\infty}$.
*(c) The series* $\sum_{n=1}^{\infty} \varepsilon_n x_n$ *converges for every choice of signs* $(\varepsilon_n)$.
*(d) For every* $\epsilon > 0$ *there exists* $n$ *such that if* $F$ *is any finite subset of* $\{n+1, n+2, \ldots\}$*, then*

$$\Big\| \sum_{j\in F} x_j \Big\| < \epsilon.$$

*Proof.* We will establish only $(a) \Rightarrow (d)$ and leave the other easier implications to the reader. Suppose that $(d)$ fails. Then there exists $\epsilon > 0$ such that for every $n$ we can find a finite subset $F_n$ of $\{n+1, \ldots\}$ with

$$\Big\| \sum_{j \in F_n} x_j \Big\| \geq \epsilon.$$

We will build a permutation $\pi$ of $\mathbb{N}$ such that $\sum_{n=1}^{\infty} x_{\pi(n)}$ diverges.

First we take $n_1 = 1$ and let $A_1 = F_{n_1}$. Next pick $n_2 = \max A_1$ and let $B_1 = \{n_1+1, \ldots, n_2\} \setminus A_1$. Now we repeat the process and take $A_2 = F_{n_2}$, $n_3 = \max A_2$, and $B_2 = \{n_2+1, \ldots, n_3\} \setminus A_2$. Iterating, we generate a sequence $(n_k)_{k=1}^{\infty}$ and a partition $\{n_k+1, \ldots, n_{k+1}\} = A_k \cup B_k$. Define $\pi$ such that $\pi$ permutes the elements of $\{n_k+1, \ldots, n_{k+1}\}$ in such a way that $A_k$ precedes $B_k$. Then the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ is divergent, because the Cauchy condition fails. □

**Definition 2.4.3.** A (formal) series $\sum_{n=1}^{\infty} x_n$ in a Banach space $X$ is *weakly unconditionally Cauchy* (for short, WUC) or *weakly unconditionally convergent* if $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for every $x^* \in X^*$.

**Proposition 2.4.4.** *Suppose the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally to some $x$ in a Banach space $X$. Then*

*(i)* $\sum_{n=1}^{\infty} x_{\pi(n)} = x$ *for every permutation* $\pi$.
*(ii)* $\sum_{n \in \mathbb{A}} x_n$ *converges unconditionally for every infinite subset* $\mathbb{A}$ *of* $\mathbb{N}$.
*(iii)* $\sum_{n=1}^{\infty} x_n$ *is WUC.*

*Proof.* Parts $(i)$ and $(ii)$ are immediate. For $(iii)$, given $x^* \in X^*$, the scalar series $\sum_{n=1}^{\infty} x^*(x_{\pi(n)})$ converges for every permutation $\pi$. It is a classical theorem of Riemann that for scalar sequences the series $\sum_{n=1}^{\infty} a_n$ converges unconditionally if and only if it converges absolutely, i.e., $\sum_{n=1}^{\infty} |a_n| < \infty$. Thus we have $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$. □

Let us notice that the name "weakly unconditionally convergent" series can be misleading, because such series need not be weakly convergent; we will therefore use the term weakly unconditionally Cauchy or more usually its abbreviation (WUC).

*Example 2.4.5.* The series $\sum_{n=1}^{\infty} e_n$ in $c_0$, where $(e_n)_{n=1}^{\infty}$ is the canonical basis of the space, is WUC but fails to converge weakly (and so it cannot converge unconditionally). In fact, this is in a certain sense the only counterexample, as we shall see.

In Proposition 2.4.7 we shall prove that WUC series are in a very natural correspondence with bounded operators on $c_0$. Let us first see a lemma.

**Lemma 2.4.6.** *Let $\sum_{n=1}^{\infty} x_n$ be a formal series in a Banach space $X$. The following are equivalent:*

*(i)* $\sum_{n=1}^{\infty} x_n$ *is WUC.*

*(ii) There exists* $C > 0$ *such that for all* $(\xi(n)) \in c_{00}$ *we have*

$$\Big\| \sum_{n=1}^{\infty} \xi(n) x_n \Big\| \le C \max_n |\xi(n)|.$$

*(iii) There exists* $C' > 0$ *such that for every finite subset* $F$ *of* $\mathbb{N}$ *and all* $\varepsilon_n = \pm 1$*,*

$$\Big\| \sum_{n \in F} \varepsilon_n x_n \Big\| \le C'.$$

*Proof.* $(i) \Rightarrow (ii)$. Put

$$S = \Big\{ \sum_{n=1}^{\infty} \xi(n) x_n \in X : \ \xi = (\xi(n)) \in c_{00}, \ \|\xi\|_\infty \le 1 \Big\}.$$

The WUC property implies that $S$ is weakly bounded. Therefore, it is norm-bounded by the uniform boundedness principle.

$(ii)$ implies $(iii)$ is obvious. For $(iii) \Rightarrow (i)$, given $x^* \in X^*$, let $\varepsilon_n = \operatorname{sgn} x^*(x_n)$. Then for each integer $N$ we have

$$\sum_{n=1}^{N} |x^*(x_n)| = \Big| x^* \Big( \sum_{n=1}^{N} \varepsilon_n x_n \Big) \Big| \le C \|x^*\|,$$

and therefore the series $\sum_{n=1}^{\infty} |x^*(x_n)|$ converges. □

**Proposition 2.4.7.** *Let* $\sum_{n=1}^{\infty} x_n$ *be a series in a Banach space* $X$*. Then* $\sum_{n=1}^{\infty} x_n$ *is WUC if and only if there is a bounded operator* $T\colon c_0 \to X$ *with* $Te_n = x_n$*.*

*Proof.* If $\sum_{n=1}^{\infty} x_n$ is WUC, then the operator $T\colon c_{00} \to X$ defined by $T\xi = \sum_{n=1}^{\infty} \xi(n) x_n$ is bounded for the $c_0$-norm by Lemma 2.4.6. By density, $T$ extends to a bounded operator $T : c_0 \to X$.

For the converse, let $T : c_0 \to X$ be a bounded operator with $Te_n = x_n$ for all $n$. For each $x^* \in X^*$ we have

$$\sum_{n=1}^{\infty} |x^*(x_n)| = \sum_{n=1}^{\infty} |x^*(Te_n)| = \sum_{n=1}^{\infty} |T^*(x^*)(e_n)|,$$

which is finite, since $\sum_{n=1}^{\infty} e_n$ is WUC. □

**Proposition 2.4.8.** *Let* $\sum_{n=1}^{\infty} x_n$ *be a WUC series in a Banach space* $X$*. Then* $\sum_{n=1}^{\infty} x_n$ *converges unconditionally in* $X$ *if and only if the operator* $T\colon c_0 \to X$ *such that* $Te_n = x_n$ *is compact.*

*Proof.* Suppose that $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent. We will show that $\lim_{n\to\infty} \|T - TS_n\| = 0$, where $(S_n)_{n=1}^{\infty}$ are the partial sum projections associated to the canonical basis $(e_n)_{n=1}^{\infty}$ of $c_0$. Thus, being a uniform limit of finite-rank operators, $T$ will be compact.

Given $\epsilon > 0$, we use Lemma 2.4.2 to find $n = n(\epsilon)$ such that if $F$ is a finite subset of $\{n+1, n+2, \ldots\}$, then $\|\sum_{j\in F} x_j\| \le \epsilon/2$. For every $x^* \in X^*$ with $\|x^*\| \le 1$ we have

$$\sum_{\{j\in F\,:\, x^*(x_j)\ge 0\}} x^*(x_j) \le \frac{\epsilon}{2},$$

and therefore

$$\sum_{j\in F} |x^*(x_j)| \le \epsilon.$$

Hence if $\xi \in c_{00}$ with $\|\xi\|_\infty \le 1$, it follows that $|x^*(T - TS_m)\xi| \le \epsilon$ for $m \ge n$ and $x^* \in X^*$. By density we conclude that $\|T - TS_m\| \le \epsilon$.

Assume, conversely, that $T$ is compact. Let us consider

$$T^{**} : c_0^{**} = \ell_\infty \longrightarrow X \subset X^{**}.$$

The restriction of $T^{**}$ to $B_{\ell_\infty}$ is weak$^*$-to-norm continuous, because on a norm compact set, the weak$^*$ topology agrees with the norm topology. Since $\sum_{n=1}^{\infty} e_{\pi(n)}$ converges weak$^*$ in $\ell_\infty$ for every permutation $\pi$, $\sum_{n=1}^{\infty} x_n$ also converges unconditionally in $X$. □

Note that the above argument also implies the following stability property of unconditionally convergent series with respect to multiplication by bounded sequences. The proof is left as an exercise.

**Proposition 2.4.9.** *A series $\sum_{n=1}^{\infty} x_n$ in a Banach space $X$ is unconditionally convergent if and only if $\sum_{n=1}^{\infty} t_n x_n$ converges (unconditionally) for all $(t_n)_{n=1}^{\infty} \in \ell_\infty$.*

The next theorem and its consequences are essentially due to Bessaga and Pełczyński in their 1958 paper [24] and represent some of the earliest applications of the basic sequence methods.

**Theorem 2.4.10.** *Suppose $T : c_0 \to X$ is a bounded operator. The following conditions on $T$ are equivalent:*

*(i) $T$ is compact,*
*(ii) $T$ is weakly compact,*
*(iii) $T$ is strictly singular.*

*Proof.* $(i) \Rightarrow (ii)$ is obvious. For $(ii) \Rightarrow (iii)$, let us suppose that $T$ fails to be strictly singular. Then there exists an infinite-dimensional subspace $Y$ of $c_0$ such that $T|_Y$ is an isomorphism onto its range. If $T$ is weakly compact, this forces $Y$ to be reflexive, contradicting Proposition 2.2.2.

We now consider $(iii) \Rightarrow (i)$. Assume that $T$ fails to be compact. Then, by Proposition 2.4.8, $\sum_{n=1}^{\infty} Te_n$ does not converge unconditionally, so by Lemma 2.4.2, there exist $\epsilon > 0$ and a sequence of disjoint finite subsets of integers $(F_n)_{n=1}^{\infty}$ such that $\| \sum_{k \in F_n} Te_k \| \geq \epsilon$ for every $n$. Let $x_n = \sum_{k \in F_n} Te_k$. The sequence $(x_n)_{n=1}^{\infty}$ is weakly null in $X$, since $\sum_{k \in F_n} e_k$ is weakly null in $c_0$. Using Proposition 1.3.10, we can, by passing to a subsequence of $(x_n)_{n=1}^{\infty}$, assume it is basic in $X$ with basis constant $K$, say. Then for $\xi = (\xi(n))_{n=1}^{\infty} \in c_{00}$,

$$\Big\| \sum_{n=1}^{\infty} \xi(n) x_n \Big\| = \Big\| T\Big( \sum_{n=1}^{\infty} \xi(n) \sum_{k \in F_n} e_k \Big) \Big\| \leq \|T\| \max_{n \in \mathbb{N}} |\xi(n)|.$$

On the other hand,

$$\max_{n \in \mathbb{N}} |\xi(n)| \leq 2K \Big\| \sum_{n=1}^{\infty} \xi(n) x_n \Big\|.$$

Thus $(x_n)_{n=1}^{\infty}$ is equivalent to the canonical $c_0$ basis and hence to $(\sum_{k \in F_n} e_k)_{n=1}^{\infty}$. We conclude that $T$ cannot be strictly singular. □

From now on, whenever we say that a Banach space $X$ *contains a copy of a Banach space* $Y$, we mean that $X$ contains a closed subspace $E$ that is isomorphic to $Y$. Using Theorem 2.4.10, we obtain a very nice characterization of spaces that contain a copy of $c_0$.

**Theorem 2.4.11.** *In order that every WUC series in a Banach space $X$ be unconditionally convergent, it is necessary and sufficient that $X$ contain no copy of $c_0$.*

*Proof.* Suppose that $X$ contains no copy of $c_0$ and that $\sum_{n=1}^{\infty} x_n$ is a WUC series in $X$. By Proposition 2.4.7 there exists a bounded operator $T \colon c_0 \to X$ such that $Te_n = x_n$ for all $n$. The operator $T$ must be strictly singular, since every infinite-dimensional subspace of $c_0$ contains a copy of $c_0$ (Proposition 2.2.1), so $T$ is compact by Theorem 2.4.10. Hence the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally by Proposition 2.4.8. The converse follows trivially from Example 2.4.5. □

*Remark 2.4.12.* This theorem of Bessaga and Pełczyński is a prototype for exclusion theorems that say that if we can exclude a certain subspace from a Banach space, then it will have a particular property. It had considerable influence in suggesting that such theorems might be true. In Chapter 11 we will see a similar and much more difficult result for Banach spaces not containing $\ell_1$ (due to Rosenthal [273]) that when combined with the Bessaga–Pełczyński theorem gives a very

elegant pair of bookends in Banach space theory. It is also worth noting that the hypothesis that a Banach space fails to contain $c_0$ becomes ubiquitous in the theory precisely because of Theorem 2.4.11.

We have seen that a series $\sum_{n=1}^{\infty} x_n$ in a Banach space $X$ converges unconditionally in norm if and only if each subseries $\sum_{k=1}^{\infty} x_{n_k}$ does. In particular, every subseries of an unconditionally convergent series is weakly convergent. The Orlicz–Pettis theorem establishes that the converse is true as well. First we see an auxiliary result.

**Lemma 2.4.13.** *Let $m_0$ be the set of all sequences of scalars assuming only finitely many different values. Then $m_0$ is dense in $\ell_\infty$.*

*Proof.* Let $a = (a_n)_{n=1}^{\infty}$ be a sequence of scalars with $\|a\|_\infty \leq 1$. For every $\epsilon > 0$ pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then the sequence $b = (b_n)_{n=1}^{\infty} \in m_0$ given by

$$b_n = (\operatorname{sgn} a_n)\frac{j}{N} \quad \text{if} \quad \frac{j}{N} \leq |a_n| \leq \frac{j+1}{N}, \quad j = 1, \ldots, N,$$

satisfies $\|a - b\|_\infty \leq \frac{1}{N} < \epsilon$. □

**Theorem 2.4.14 (The Orlicz–Pettis Theorem).** *Suppose $\sum_{n=1}^{\infty} x_n$ is a series in a Banach space $X$ for which every subseries $\sum_{k=1}^{\infty} x_{n_k}$ converges weakly. Then $\sum_{n=1}^{\infty} x_n$ converges unconditionally in norm.*

*Proof.* The hypothesis easily yields that $\sum_{n=1}^{\infty} x_n$ is a WUC series, so by Proposition 2.4.7, there exists a bounded operator $T : c_0 \to X$ with $Te_n = x_n$ for all $n$. We will show that $T$ is compact.

Let us look at $T^{**} : \ell_\infty \to X^{**}$. For every $A \subset \mathbb{N}$ let us denote by $\chi_A = (\chi_A(k))_{k=1}^{\infty}$ the element of $\ell_\infty$ such that $\chi_A(k) = 1$ if $k \in A$ and 0 otherwise. By hypothesis $\sum_{n \in A} x_n$ converges weakly in $X$, and it follows that $T^{**}(\chi_A) \in X$. The linear span of all such $\chi_A$ consists of the space $m_0$ of scalar sequences taking only finitely many different values, which by Lemma 2.4.13 is dense in $\ell_\infty$. Hence $T^{**}$ maps $\ell_\infty$ into $X$. This means that $T$ is a weakly compact operator. Now Theorem 2.4.10 implies that $T$ is a compact operator, and Proposition 2.4.8 completes the proof. □

Now, as a corollary, we can give a reciprocal of Proposition 2.4.4 (*iii*).

**Corollary 2.4.15.** *If a Banach space $X$ is weakly sequentially complete, then every WUC series in $X$ is unconditionally convergent.*

*Proof.* If $\sum_{n=1}^{\infty} x_n$ is WUC, then $\sum_{n=1}^{\infty} x^*(x_n)$ is absolutely convergent for every $x^* \in X^*$, which is equivalent to saying that $\sum_{k=1}^{\infty} x^*(x_{n_k})$ converges for each subseries $\sum_{k=1}^{\infty} x_{n_k}$ and each $x^* \in X^*$. Hence $\sum_{k=1}^{\infty} x_{n_k}$ is weakly Cauchy and therefore weakly convergent by hypothesis. We deduce that $\sum_{n=1}^{\infty} x_n$ converges unconditionally in norm by the Orlicz–Pettis theorem. □

The Orlicz–Pettis theorem predates basic sequence techniques. The result is stated without proof in the *Remarques* of Banach's book [18], where it is properly attributed to Orlicz. Indeed, after Orlicz had proved in 1929 [232] that in a weakly sequentially complete Banach space every WUC series is unconditionally convergent, he realized that by replacing the WUC assumption with weak subseries convergence his proof showed what is nowadays known as the Orlicz–Pettis theorem. Pettis needed the result in his study of vector measures and wrote an independent proof in [250], where it is quoted as *Orlicz Lemma*. It is also interesting to observe that both proofs are basically the same, using what today would be called the Schur property of $\ell_1$. The result became known later on as the Orlicz–Pettis theorem because of its importance in the theory of vector measures. If $\Sigma$ is a $\sigma$-algebra of sets and $\mu : \Sigma \to X$ is a map such that for every $x^* \in X^*$ the set function $x^* \circ \mu$ is a (countably additive) measure, then the Orlicz–Pettis theorem implies that $\mu$ is countably additive in the norm topology. Thus weakly countably additive set functions are norm countably additive. This is an attractive theorem, and as a result, it has been proved, re-proved, and generalized many times since then. It is not clear that there is much left to say on this subject! We will suggest some generalizations in the problems.

## 2.5 Complementability of $c_0$

Let us discuss the following extension problem. Suppose that $X$ and $Y$ are Banach spaces and that $E$ is a subspace of $X$. Let $T : E \to Y$ be a bounded operator. Can we extend $T$ to a bounded operator $\tilde{T} : X \to Y$? If we consider the special case that $Y = E$ and $T$ is the identity map on $E$, we are asking simply whether $E$ is the range of a projection on $X$, i.e., whether $E$ is complemented in $X$.

The Hahn–Banach theorem asserts that if $Y$ has dimension one, then such an extension is possible with preservation of norm. However, in general such an extension is not possible, and we have discussed the fact that there are noncomplemented subspaces in almost all Banach spaces. For instance, we have seen that $\ell_1$ must have an uncomplemented subspace, but the construction of this subspace as the kernel of a certain quotient map means that it is rather difficult to see exactly what it is. In this section we will study a very natural example. Let us formalize the notion of an injective Banach space.

**Definition 2.5.1.** A Banach space $Y$ is called *injective* if whenever $X$ is a Banach space, $E$ is a closed subspace of $X$, and $T : E \to Y$ is a bounded operator then there is a bounded linear operator $\tilde{T} : X \to Y$ that is an extension of $T$. A Banach space $Y$ is called *isometrically injective* if $\tilde{T}$ can be additionally chosen to have $\|\tilde{T}\| = \|T\|$.

We will defer our discussion of injective spaces to later and restrict ourselves to one almost trivial observation:

**Proposition 2.5.2.** *The space $\ell_\infty$ is an isometrically injective space. Hence, if a Banach space $X$ has a subspace $E$ isomorphic to $\ell_\infty$, then $E$ is necessarily complemented in $X$.*

*Proof.* Suppose $E$ is a subspace of $X$ and $T : E \to \ell_\infty$ is bounded. Then $Te = (e_n^*(e))_{n=1}^\infty$ for some sequence $(e_n^*)_{n=1}^\infty$ in $E^*$; clearly $\|T\| = \sup_n \|e_n^*\|$. By the Hahn–Banach theorem we choose extensions $x_n^* \in X^*$ with $\|x_n^*\| = \|e_n^*\|$ for each $n$. By letting $\tilde{T}x = (x_n^*(x))_{n=1}^\infty$, we are done. □

The space $c_0$ is a subspace of $\ell_\infty$ (its bidual), and it is easy to see that $c_0$ is injective if and only if it is complemented in $\ell_\infty$. Must a Banach space be complemented in its bidual? Certainly this is true for every space that is the dual of another space, since for every Banach space $X$, the space $X^*$ is always complemented in its bidual, $X^{***}$. To see this, consider the natural embedding $j : X \to X^{**}$. Then $j^* : X^{***} \to X^*$ is a norm-one operator. Denote by $J$ the canonical injection of $X^*$ into $X^{***}$. We claim that $j^*J$ is the identity $I_{X^*}$ on $X^*$. Indeed, suppose $x^* \in X^*$ and that $x \in X$. Then $\langle x, j^*J(x^*)\rangle = \langle jx, Jx^*\rangle = \langle x, x^*\rangle$. Thus $j^*$ is a norm-one projection of $X^{***}$ onto $X^*$. If $X$ is isomorphic (but not necessarily isometric) to a dual space, we leave for the reader the details to check that $X$ will still be complemented in its bidual. So we may also ask whether $c_0$ is isomorphic to a dual space.

As we will see next, $c_0$ is *not* complemented in $\ell_\infty$. This was proved essentially by Phillips [252] in 1940, although it was first formally observed by Sobczyk [286] the following year. Phillips in fact proved the result for the subspace $c$ of convergent sequences. The proof we give is due to Whitley [302] and requires a simple lemma:

**Lemma 2.5.3.** *Every countably infinite set $\mathbb{S}$ has an uncountable family of infinite subsets $\{\mathbb{A}_i\}_{i\in\mathcal{I}}$ such that any two members of the family have finite intersection.*

*Proof.* The proof is very simple but rather difficult to spot! Without loss of generality we can identify $\mathbb{S}$ with the set of the rational numbers $\mathbb{Q}$. For each irrational number $\theta$, take a sequence of rational numbers $(q_n)_{n=1}^\infty$ converging to $\theta$. Then the sets of the form $\mathbb{A}_\theta = \{(q_n)_{n=1}^\infty : q_n \to \theta\}$ satisfy the lemma. □

If $\mathbb{A}$ is any subset of $\mathbb{N}$, we denote by $\ell_\infty(\mathbb{A})$ the subspace of $\ell_\infty$ given by

$$\ell_\infty(\mathbb{A}) = \big\{\xi = (\xi(k))_{k=1}^\infty \in \ell_\infty : \xi(k) = 0 \text{ if } k \notin \mathbb{A}\big\}.$$

**Theorem 2.5.4.** *Let $T : \ell_\infty \to \ell_\infty$ be a bounded operator such that $T\xi = 0$ for all $\xi \in c_0$. Then there is an infinite subset $\mathbb{A}$ of $\mathbb{N}$ such that $T\xi = 0$ for every $\xi \in \ell_\infty(\mathbb{A})$.*

*Proof.* We use the family $(\mathbb{A}_i)_{i\in\mathcal{I}}$ of infinite subsets of $\mathbb{N}$ given by Lemma 2.5.3. Suppose that for every such set we can find $\xi_i \in \ell_\infty(\mathbb{A}_i)$ with $T\xi_i \neq 0$. We can assume by normalization that $\|\xi_i\|_\infty = 1$ for every $i \in \mathcal{I}$. There must exist $n \in \mathbb{N}$ such that the set $\mathcal{I}_n = \{i \in \mathcal{I} : T\xi_i(n) \neq 0\}$ is uncountable. Similarly, there exists $k \in \mathbb{N}$ such that the set $\mathcal{I}_{n,k} = \{i : |T\xi_i(n)| \geq k^{-1}\}$ is also uncountable. For each $i \in \mathcal{I}_{n,k}$ choose $\alpha_i$ with $|\alpha_i| = 1$ and $\alpha_i T\xi_i(n) = |T\xi_i(n)|$.

Let $\mathbb{F}$ be a finite subset of $\mathcal{I}_{n,k}$. Consider $y = \sum_{i\in\mathbb{F}} \alpha_i \xi_i$. Since the intersection of the supports of any two distinct $\xi_i$ is finite, we can write $y = u + v$, where $\|u\|_\infty \le 1$ and $v$ has finite support. Thus

$$\|Ty\|_\infty = \|Tu\|_\infty \le \|T\|,$$

and so

$$e_n^*(Ty) = \sum_{i\in\mathbb{F}} |T\xi_i(n)| \le \|T\|.$$

It follows that if $|\mathbb{F}| = m$, we have $mk^{-1} \le \|T\|$, i.e., $m \le k\|T\|$. Since this holds for every finite subset of $\mathcal{I}_{n,k}$, we have shown that $\mathcal{I}_{n,k}$ is in fact finite, which is a contradiction. □

**Theorem 2.5.5 (Phillips and Sobczyk [252, 286]).** *There is no bounded projection from $\ell_\infty$ onto $c_0$.*

*Proof.* If $P$ is such a projection, we can apply Theorem 2.5.4 to $T = I - P$, with $I$ the identity operator on $\ell_\infty$, and then it is clear that $P\xi = \xi$ for all $\xi \in \ell_\infty(\mathbb{A})$ for some infinite set $\mathbb{A}$, which gives a contradiction. □

**Corollary 2.5.6.** *$c_0$ is not isomorphic to a dual space.*

*Proof.* If $c_0$ were isomorphic to a dual space, then by the comments that follow the proof of Proposition 2.5.2, $c_0$ should be complemented in $c_0^{**}$, which would lead to a contradiction with Theorem 2.5.5. □

Several comments are in order here. Theorem 2.5.4 proves more than is needed for the Phillips–Sobczyk theorem. It shows that there is no bounded one-to-one operator from the quotient space $\ell_\infty/c_0$ into $\ell_\infty$; in other words, the points of $\ell_\infty/c_0$ cannot be separated by countably many bounded linear functionals. (Of course, if $E$ is a complemented subspace of a Banach space $X$, then $X/E$ must be isomorphic to a subspace of $X$ that is complementary to $E$.)

Now we are also in a position to note that $c_0$ is not an injective space. Actually there are no separable injective spaces, but we will see this later, when we discuss the structure of $\ell_\infty$ in more detail. For the moment let us note the dual statement of Theorem 2.3.1.

**Theorem 2.5.7.** *Every separable Banach space embeds isometrically in $\ell_\infty$.*

*Proof.* Let $(x_n)_{n=1}^\infty$ be a dense sequence in $X$. For each integer $n$ pick $x_n^* \in X^*$ such that $\|x_n^*\| = 1$ and $x_n^*(x_n) = \|x_n\|$. The sequence $(x_n^*)_{n=1}^\infty$ is norming in $X$. Therefore the operator $T: X \to \ell_\infty$ defined for each $x$ in $X$ by $T(x) = (x_n^*(x))_{n=1}^\infty$ provides the desired embedding. □

Thus $X$ separable can be injective only if it is isomorphic to a complemented subspace of $\ell_\infty$. Therefore classifying the complemented subspaces of $\ell_\infty$ becomes important; we will see in Chapter 5 the (already mentioned) theorem of Lindenstrauss [194] that $\ell_\infty$ is a prime space, and this will answer our question.

In the meantime, we turn to Sobczyk's main result in his 1941 paper, which gives some partial answers to these questions. The proof we present here is due to Veech [300].

**Theorem 2.5.8 (Sobczyk [286]).** *Let $X$ be a separable Banach space. If $E$ is a closed subspace of $X$ and $T\colon E \longrightarrow c_0$ is a bounded operator, then there exists an operator $\tilde{T}\colon X \longrightarrow c_0$ such that $\tilde{T}|_E = T$ and $\|\tilde{T}\| \leq 2\,\|T\|$.*

*Proof.* Without loss of generality we can assume that $\|T\| = 1$. It is immediate to realize that the operator $T$ must be of the form

$$Tx = (f_n^*(x))_{n=1}^\infty, \qquad x \in E,$$

for some $(f_n^*)_{n=1}^\infty \subset E^*$. Moreover, $\|f_n^*\| \leq 1$ for all $n$, and $(f_n^*)_{n=1}^\infty$ converges to 0 in the weak$^*$ topology of $E^*$. By the Hahn–Banach theorem, for each $n \in \mathbb{N}$ there exists $\varphi_n^* \in X^*$, $\|\varphi_n^*\| \leq 1$, such that $\varphi_n^*|_E = f_n^*$.

The fact that $X$ is separable implies that $(B_{X^*}, w^*)$ is metrizable (Lemma 1.4.1). Let $\rho$ be the metric on $B_{X^*}$ that induces the weak$^*$ topology on $B_{X^*}$. We claim that $\lim_{n\to\infty} \rho(\varphi_n^*, B_{X^*} \cap E^\perp) = 0$. If this were not the case, there would be some $\epsilon > 0$ and a subsequence $(\varphi_{n_k}^*)$ of $(\varphi_n^*)$ such that $\rho(\varphi_{n_k}^*, B_{X^*} \cap E^\perp) \geq \epsilon$ for every $k$. Let $(\varphi_{n_{k_j}}^*)$ be a subsequence of $(\varphi_{n_k}^*)$ such that $\varphi_{n_{k_j}}^* \xrightarrow{w^*} \varphi^*$. Then $\varphi^* \in E^\perp \cap B_{X^*}$, since for each $e \in E$ we have

$$\varphi^*(e) = \lim_j \varphi_{n_{k_j}}^*(e) = \lim_j f_{n_{k_j}}^*(e) = 0.$$

Hence

$$\rho(\varphi_{n_{k_j}}^*, \varphi^*) \geq \epsilon \ \text{ for all } j. \tag{2.1}$$

On the other hand,

$$\lim_{j\to\infty} \rho(\varphi_{n_{k_j}}^*, B_{X^*} \cap E^\perp) = \rho(\varphi^*, B_{X^*} \cap E^\perp) = 0, \tag{2.2}$$

since the function $\rho(\,\cdot\,, B_{X^*} \cap E^\perp)$ is weak$^*$ continuous on $B_{X^*}$. Clearly we cannot have (2.1) and (2.2) at the same time, so our claim holds.

Recall that $E^\perp$ is weak$^*$ closed; hence $B_{X^*} \cap E^\perp$ is weak$^*$ compact. Therefore for each $n$ we can pick $v_n^* \in B_{X^*} \cap E^\perp$ such that

$$\rho(\varphi_n^*, v_n^*) = \rho(\varphi_n^*, B_{X^*} \cap E^\perp).$$

Let $x_n^* = \varphi_n^* - v_n^*$ and define the operator $\tilde{T}$ on $X$ by $\tilde{T}(x) = (x_n^*(x))_{n=1}^\infty$. Notice that $\tilde{T}(x) \in c_0$, because $x_n^* \xrightarrow{w^*} 0$. Moreover, for each $x \in X$ we have

$$\|\tilde{T}(x)\| = \sup_n |x_n^*(x)| = \sup_n(|\varphi_n^*(x) - v_n^*(x)|) \leq \sup_n(\|\varphi_n^*\| + \|v_n^*\|)\,\|x\| \leq 2\,\|x\|\,,$$

and so $\|\tilde{T}\| \leq 2$. □

**Corollary 2.5.9.** *If $E$ is a closed subspace of a separable Banach space $X$ and $E$ is isomorphic to $c_0$, then there is a projection $P$ from $X$ onto $E$.*

*Proof.* Suppose that $T : E \to c_0$ is an isomorphism and let $\tilde{T} : X \to c_0$ be the extension of $T$ given by the preceding theorem. Then $P = T^{-1}\tilde{T}$ is a projection from $X$ onto $E$. (Note that since $\|\tilde{T}\| \leq 2\|T\|$, if $E$ is *isometric* to $c_0$, then $\|P\| \leq 2$.) □

*Remark 2.5.10.* It follows that if a separable Banach space $X$ contains a copy of $c_0$, then $X$ is not injective.

We finish this chapter by observing that in light of Theorem 2.5.8, it is natural to define a Banach space $Y$ to be *separably injective* if whenever $X$ is a separable Banach space, $E$ is a closed subspace of $X$, and $T : E \to Y$ is a bounded operator, then $T$ can be extended to an operator $\tilde{T} : X \to Y$. It was for a long time conjectured that $c_0$ is the only separable and separably injective space. This was solved in 1977 [310] by Zippin who showed that indeed, $c_0$ is, up to isomorphism, the only separable space that is separably injective.

We also note that the constant 2 in Theorem 2.5.8 is the best possible (see Problem 2.7).

## Problems

**2.1.** Let $T : X \to Y$ be an operator between the Banach spaces $X$, $Y$.

(a) Show that if $T$ is strictly singular, then in every infinite-dimensional subspace $E$ of $X$ there is a normalized basic sequence $(x_n)$ with $\|Tx_n\| < 2^{-n}\|x_n\|$ for all $n$.
(b) Deduce that $T$ is strictly singular if and only if every infinite-dimensional closed subspace $E$ contains a further infinite-dimensional closed subspace $F$ such that the restriction of $T$ to $F$ is compact.

**2.2.** Show that the sum of two strictly singular operators is strictly singular. Show also that if $T_n : X \to Y$ are strictly singular and $\|T_n - T\| \to 0$, then $T$ is strictly singular.

**2.3.** Show that the set of all strictly singular operators on a Banach space $X$ forms a closed two-sided ideal in the algebra $\mathcal{B}(X)$ of all bounded linear operators from $X$ to $X$.

**2.4.** Show that if $1 < p < \infty$ and $T : \ell_p \to \ell_p$ is not compact, then there is a complemented subspace $E$ of $\ell_p$ such that $T$ is an isomorphism of $E$ onto a complemented subspace $T(E)$. Deduce that the Banach algebra $\mathcal{L}(\ell_p)$ contains exactly one proper closed two-sided ideal (the ideal of compact operators). Note that every strictly singular operator is compact in these spaces.

**2.5.** Show that $\mathcal{B}(\ell_p \oplus \ell_r)$ for $p \neq r$ contains at least two nontrivial closed two-sided ideals.

**2.6.** Suppose $X$ is a Banach space whose dual is separable. Suppose that $\sum_{n=1}^{\infty} x_n^*$ is a formal series in $X^*$ that has the property that every subseries $\sum_{k=1}^{\infty} x_{n_k}^*$ converges in the weak* topology. Show that $\sum_{n=1}^{\infty} x_n^*$ converges in norm. [*Hint*: Every $x^{**} \in X^{**}$ is the limit of a weak* converging sequence from $X$.]

**2.7.** Let $c$ be the subspace of $\ell_\infty$ of converging sequences. Show that for every bounded projection $P$ of $c$ onto $c_0$ we have $\|P\| \geq 2$. This proves that 2 is the best possible constant in Sobczyk's theorem (Theorem 2.5.8).

**2.8.** In this exercise we will focus on the special properties of $\ell_1$ as a target space for operators and show its *projectivity*.

(a) Suppose $T: X \to \ell_1$ is an operator from a Banach space $X$ *onto* $\ell_1$. Show that then $X$ contains a complemented subspace isomorphic to $\ell_1$.
(b) Prove that if $Y$ is a separable infinite-dimensional Banach space with the property that whenever $T: X \to Y$ is a bounded surjective operator then $Y$ is isomorphic to a complemented subspace of $X$, then $Y$ is isomorphic to $\ell_1$.

**2.9.** Let $X$ be a Banach space.

(a) Show that for every $x^{**} \in X^{**}$ and every finite-dimensional subspace $E$ of $X^*$ there exists $x \in X$ such that

$$\|x\| < (1 + \epsilon)\|x^{**}\|,$$

and

$$x^*(x) = x^{**}(x^*), \qquad x^* \in E.$$

(b) Use part (a) to deduce the following result of Bessaga and Pełczyński [24]: If $X^*$ contains a subspace isomorphic to $c_0$, then $X$ contains a complemented subspace isomorphic to $\ell_1$, and hence $X^*$ contains a subspace isomorphic to $\ell_\infty$. In particular, no separable dual space can contain an isomorphic copy of $c_0$. [This may also be used in Problem 2.6.]

**2.10.** For an arbitrary set $\Gamma$ we define $c_0(\Gamma)$ as the space of functions $\xi: \Gamma \to \mathbb{R}$ such that for each $\epsilon > 0$ the set $\{\gamma : |\xi(\gamma)| > \epsilon\}$ is finite. When normed by $\|\xi\| = \max_{\gamma \in \Gamma} |\xi(\gamma)|$, the space $c_0(\Gamma)$ becomes a Banach space.

(a) Show that $c_0(\Gamma)^*$ can be identified with $\ell_1(\Gamma)$, the space of functions $\eta : \Gamma \to \mathbb{R}$ such that $\eta \in c_0(\Gamma)$ and $\|\eta\| = \sum_{\gamma\in\Gamma} |\eta(\gamma)| < \infty$.
(b) Show that $\ell_1(\Gamma)^* = \ell_\infty(\Gamma)$.
(c) Show, using the methods of Lemma 2.5.3 and Theorem 2.5.4, that $c_0(\mathbb{R})$ is isomorphic to a subspace of $\ell_\infty/c_0$.

**2.11.** Let $\Gamma$ be an infinite set and let $\mathcal{P}\Gamma$ denote its power set $\mathcal{P}\Gamma = \{A : A \subset \Gamma\}$.

(a) Show that $\ell_1(\mathcal{P}\Gamma)$ is isometric to a subspace of $\ell_\infty(\Gamma)$. [*Hint*: For each $\gamma \in \Gamma$ define $\varphi_\gamma \in \ell_\infty(\mathcal{P}\Gamma)$ by $\varphi_\gamma = 1$ when $\gamma \in A$ and $-1$ when $\gamma \notin A$.]
(b) Show that if $\ell_1(\Gamma)$ is a quotient of a subspace of $X$, then $\ell_1(\Gamma)$ embeds into $X$ (compare with Problem 2.8).
(c) Deduce that if $\ell_1(\Gamma)$ embeds into $X$, then $\ell_1(\mathcal{P}\Gamma)$ embeds into $X^*$.
(d) Deduce that $\ell_1^{**}$ contains an isometric copy of $\ell_1(\mathcal{P}\mathbb{R})$.

# Chapter 3
# Special Types of Bases

Knowing whether a separable Banach space has a Schauder basis and identifying one that allows one to compute easily the norm of its elements is important. However, this knowledge becomes very limited if we are interested in using bases as a tool to delve deeper into the geometry of the space. In this chapter we look a bit more carefully at special classes of bases. In particular we will consider the notions of shrinking, boundedly complete, and unconditional basis, which was already hinted at in Section 2.4. Much of this chapter is based on classical work of James in the early 1950s. James techniques illustrate the use of bases with additional features to obtain crucial structural information on a space.

## 3.1 Unconditional Bases

Unconditional bases are the most useful and extensively studied special bases because of the good structural properties of the spaces they span. Unconditional bases seem to have first appeared in 1948 [165] in work of Karlin, who proved that $\mathcal{C}[0,1]$ fails to have an unconditional basis. We will prove this fact later on in the chapter. In the older literature the term *absolute basis* is often used in place of unconditional basis, but this usage has largely disappeared.

**Definition 3.1.1.** A basis $(u_n)_{n=1}^{\infty}$ of a Banach space $X$ is *unconditional* if for each $x \in X$ the series $\sum_{n=1}^{\infty} u_n^*(x)u_n$ converges unconditionally.

Obviously, $(u_n)_{n=1}^{\infty}$ is an unconditional basis of $X$ if and only if $(u_{\pi(n)})_{n=1}^{\infty}$ is a basis of $X$ for every permutation $\pi : \mathbb{N} \to \mathbb{N}$.



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*Example 3.1.2.* The standard unit vector basis $(e_n)_{n=1}^\infty$ is an unconditional basis of $c_0$ and $\ell_p$ for $1 \le p < \infty$. An example of a basis that is *conditional* (i.e., not unconditional) is the *summing basis* of $c_0$, defined as

$$f_n = e_1 + \cdots + e_n, \quad n \in \mathbb{N}.$$

To see that $(f_n)_{n=1}^\infty$ is a basis for $c_0$ we prove that for each $\xi = (\xi(n))_{n=1}^\infty \in c_0$ we have $\xi = \sum_{n=1}^\infty f_n^*(\xi) f_n$, where $f_n^* = e_n^* - e_{n+1}^*$ are the biorthogonal functionals of $(f_n)_{n=1}^\infty$. Given $N \in \mathbb{N}$,

$$\begin{aligned}
\sum_{n=1}^N f_n^*(\xi) f_n &= \sum_{n=1}^N \big(e_n^*(\xi) - e_{n+1}^*(\xi)\big) f_n \\
&= \sum_{n=1}^N (\xi(n) - \xi(n+1)) f_n \\
&= \sum_{n=1}^N \xi(n) f_n - \sum_{n=2}^{N+1} \xi(n) f_{n-1} \\
&= \sum_{n=1}^N \xi(n)(f_n - f_{n-1}) - \xi(N+1) f_N \\
&= \Big(\sum_{n=1}^N \xi(n) e_n\Big) - \xi(N+1) f_N,
\end{aligned}$$

where we have used the convention that $f_0 = 0$. Therefore,

$$\begin{aligned}
\Big\| \xi - \sum_{n=1}^N f_n^*(\xi) f_n \Big\|_\infty &= \Big\| \sum_{N+1}^\infty \xi(n) e_n + \xi(N+1) f_N \Big\|_\infty \\
&\le \Big\| \sum_{N+1}^\infty \xi_n e_n \Big\|_\infty + |\xi(N+1)| \, \|f_N\|_\infty \overset{N\to\infty}{\to} 0,
\end{aligned}$$

and $(f_n)_{n=1}^\infty$ is a basis.

Now we will identify the set $S$ of coefficients $(\alpha_n)_{n=1}^\infty$ such that the series $\sum_{n=1}^\infty \alpha_n f_n$ converges. In fact, we have that $(\alpha_n)_{n=1}^\infty \in S$ if and only if there exists $\xi = (\xi(n))_{n=1}^\infty \in c_0$ such that $\alpha_n = \xi(n) - \xi(n+1)$ for all $n$. Then, clearly, unless the series $\sum_{n=1}^\infty \alpha_n$ converges absolutely, the convergence of $\sum_{n=1}^\infty \alpha_n f_n$ in $c_0$ is not equivalent to the convergence of $\sum_{n=1}^\infty \epsilon_n \alpha_n f_n$ for all choices of signs $(\epsilon_n)_{n=1}^\infty$. Hence $(f_n)_{n=1}^\infty$ cannot be unconditional.

**Proposition 3.1.3.** *A basis $(u_n)_{n=1}^\infty$ of a Banach space $X$ is unconditional if and only if there is a constant $K \ge 1$ such that for all $N \in \mathbb{N}$,*

$$\Big\| \sum_{n=1}^{N} a_n u_n \Big\| \le K \Big\| \sum_{n=1}^{N} b_n u_n \Big\|, \tag{3.1}$$

*whenever* $a_1, \ldots, a_N$, $b_1, \ldots, b_N$ *are scalars satisfying* $|a_n| \le |b_n|$ *for* $n = 1, \ldots, N$.

*Proof.* Assume $(u_n)_{n=1}^\infty$ is unconditional. If $\sum_{n=1}^\infty a_n u_n$ converges, then so does $\sum_{n=1}^\infty t_n a_n u_n$ for all $(t_n)_{n=1}^\infty \in \ell_\infty$ by Proposition 2.4.9. By the Banach–Steinhaus theorem, the linear map $T_{(t_n)}\colon X \to X$ given by $\sum_{n=1}^\infty a_n u_n \mapsto \sum_{n=1}^\infty t_n a_n u_n$ is continuous. Now the uniform boundedness principle yields $K$ such that equation (3.1) holds.

Conversely, let us take a convergent series $\sum_{n=1}^\infty a_n u_n$ in $X$. We are going to prove that the subseries $\sum_{k=1}^\infty a_{n_k} u_{n_k}$ is convergent for every increasing sequence of integers $(n_k)_{k=1}^\infty$ and appeal to Lemma 2.4.2 to deduce that it is unconditionally convergent. Given $\epsilon > 0$; there is $N = N(\epsilon) \in \mathbb{N}$ such that if $m_2 > m_1 \ge N$, then

$$\Big\| \sum_{n=m_1+1}^{m_2} a_n u_n \Big\| < \frac{\epsilon}{K}.$$

By hypothesis, if $N \le n_k < \cdots < n_{k+l}$, we have

$$\Big\| \sum_{j=k+1}^{k+l} a_{n_j} u_{n_j} \Big\| \le K \Big\| \sum_{j=n_k+1}^{n_{k+l}} a_j u_j \Big\| < \epsilon,$$

and so $\sum_{k=1}^\infty a_{n_k} u_{n_k}$ is Cauchy. □

**Definition 3.1.4.** Let $(u_n)_{n=1}^\infty$ be an unconditional basis of a Banach space $X$. The *unconditional basis constant* $\mathsf{K}_\mathsf{u}$ *of* $(u_n)_{n=1}^\infty$ is the least constant $K$ such that equation (3.1) holds. We then say that $(u_n)_{n=1}^\infty$ is $K$*-unconditional* whenever $K \ge \mathsf{K}_\mathsf{u}$.

Suppose $(u_n)_{n=1}^\infty$ is an unconditional basis for a Banach space $X$. For each sequence of scalars $(\alpha_n)$ with $|\alpha_n| = 1$, let $T_{(\alpha_n)} : X \to X$ be the isomorphism defined by $T_{(\alpha_n)}(\sum_{n=1}^\infty a_n u_n) = \sum_{n=1}^\infty \alpha_n a_n u_n$. Then

$$\mathsf{K}_\mathsf{u} = \sup \Big\{ \|T_{(\alpha_n)}\| \colon (\alpha_n) \text{ scalars}, \ \ |\alpha_n| = 1 \text{ for all } n \Big\}.$$

Let $(u_n)_{n=1}^\infty$ be an unconditional basis of $X$. For every $A \subseteq \mathbb{N}$ there is a linear projection $P_A$ from $X$ onto $[u_k : k \in A]$ defined for each $x = \sum_{k=1}^\infty u_k^*(x) u_k$ in $X$ by

$$P_A(x) = \sum_{k \in A} u_k^*(x) u_k.$$

The members of the set $\{P_A \colon A \subseteq \mathbb{N}\}$ are the natural projections associated to the unconditional basis $(u_n)_{n=1}^\infty$ .

**Proposition 3.1.5.** *Let $(u_n)_{n=1}^{\infty}$ be a basis of a Banach space $X$. The following are equivalent:*

*(i)* *The basis $(u_n)_{n=1}^{\infty}$ is unconditional.*
*(ii)* *The map $P_A$ is well defined for every $A \subseteq \mathbb{N}$.*
*(iii)* *The map $P_A$ is well defined for every $A \subseteq \mathbb{N}$ and $\sup_A \|P_A\| < \infty$.*
*(iv)* $\sup\{\|P_F(x)\| \colon F \subseteq \mathbb{N},\ F \textit{ finite}\} < \infty$.
*(v)* *The map $P_B$ is well defined for every cofinite subset $B$ of $\mathbb{N}$ and*

$$\sup\{\|P_B(x)\| \colon B \text{ cofinite subset of } \mathbb{N}\} < \infty.$$

*Moreover, if any of the above statements holds, then the suprema in (iii), (iv), and (v) coincide.*

*Proof.* The implication $(i) \Rightarrow (ii)$ is a consequence of Proposition 3.1.3.

$(ii) \Rightarrow (iii)$ follows readily from the uniform boundedness principle.

$(iii) \Rightarrow (iv)$ and $(iii) \Rightarrow (v)$ are trivial.

$(iv) \Rightarrow (iii)$ Let $A$ be any subset (finite or infinite) of $\mathbb{N}$. For every $x \in X$ with finite support in $(u_n)_{n=1}^{\infty}$ let $S = \operatorname{supp}(x)$. Since $P_A(x) = P_{A\cap S}(x)$,

$$\|P_A(x)\| = \|P_{A\cap S}(x)\| \le \|P_{A\cap S}\|\|x\| \le \sup\{\|P_F\| \colon F \text{ finite}\}\|x\|.$$

By density $P_A$ extends to a bounded operator from $X$ to $X$.

To close the cycle of equivalences we will show that $(v)$ implies $(iii)$. Let $A$ be any subset of $\mathbb{N}$. Take $x$ finitely supported in $(u_n)_{n=1}^{\infty}$ and let $S = \operatorname{supp}(x)$. Since $P_A(x) = P_{A\cup S^c}(x)$,

$$\|P_A(x)\| = \|P_{A\cup S^c}\|\|x\| \le \sup\{\|P_B\| \colon B \subseteq \mathbb{N} \text{ cofinite}\}\|x\|.$$

As before, by density $P_A$ extends to a bounded operator from $X$ to $X$. □

**Definition 3.1.6.** If $(u_n)_{n=1}^{\infty}$ is an unconditional basis of $X$, the number

$$\mathsf{K}_{\mathsf{su}} = \sup\{\|P_A\| : A \subseteq \mathbb{N}\}$$

is called the *suppression-unconditional constant* of the basis.

Let us observe that in general, we have

$$1 \le \mathsf{K}_{\mathsf{su}} \le \mathsf{K}_{\mathsf{u}} \le 2\mathsf{K}_{\mathsf{su}}.$$

## 3.2 Bases and Duality: Boundedly Complete and Shrinking Bases

Suppose $(e_n)_{n=1}^\infty$ is a basis for a Banach space $X$ and that $(e_n^*)_{n=1}^\infty$ is the sequence of its biorthogonal functionals. One of our goals in this section is to establish necessary and sufficient conditions for $(e_n^*)_{n=1}^\infty$ to be a basis for $X^*$. This is not always the case, since it is necessary that $X^*$ be separable. In fact, there are Banach spaces $X$ with a basis and with a separable dual such that $X^*$ does not have the approximation property; hence $X^*$ cannot have a basis [195]. As it happens, though, $(e_n^*)_{n=1}^\infty$ is always a basic sequence in $X^*$.

**Proposition 3.2.1.** *Let $(e_n)_{n=1}^\infty$ be a basis for a Banach space $X$ with basis constant $\mathsf{K_b}$ and biorthogonal functionals $(e_n^*)_{n=1}^\infty$. Then $(e_n^*)_{n=1}^\infty$ is a basis for the subspace $Z = [e_n^*]$ with basis constant not greater than $\mathsf{K_b}$. More specifically:*

*(a) The coordinate functionals $(e_n^{**})_{n=1}^\infty$ associated to $(e_n^*)_{n=1}^\infty$ are given by $e_n^{**} = j(e_n)|_Z$ for every $n$, where $j$ is the canonical embedding of $X$ into its second dual $X^{**}$.*
*(b) The partial sum projections associated to the basic sequence $(e_n^*)_{n=1}^\infty$ are the restrictions $(S_N^*|_Z)_{N=1}^\infty$ of the adjoint operators of the partial sum projections associated to $(e_n)_{n=1}^\infty$. Moreover,*

$$Z = \left\{ x^* \in X^* \colon \lim_{N\to\infty} S_N^*(x^*) = x^* \right\}.$$

*Proof.* The functionals $(e_n^{**})_{n=1}^\infty$ satisfy $e_n^{**}(e_k^*) = 1$ if $n = k$ and 0 otherwise. Hence if suffices to show that the operators $T_N \colon Z \to Z$ defined by

$$T_N(x^*) = \sum_{k=1}^N e_k^{**}(x^*) e_k^*, \quad x^* \in Z,$$

satisfy $\|T_N\| \le \mathsf{K_b}$ and appeal to the partial converse of the Banach–Steinhaus theorem (see E.14 in the appendix).

Let $(S_N^*)_{N=1}^\infty$ be the sequence of adjoint operators of the partial sum projections associated with $(e_n)_{n=1}^\infty$. Notice that for $x \in X$ and $x^* \in X^*$,

$$S_N^*(x^*)(x) = x^*(S_N(x)) = \sum_{k=1}^N e_k^*(x) x^*(e_k) = \sum_{k=1}^N e_k^*(x) j(e_k)(x^*).$$

Hence $S_N^* = \sum_{k=1}^N j(e_k)\, e_k^*$, and so $T_N = S_N^*|_Z$. Thus,

$$\sup_N \|T_N\| \le \sup_N \|S_N^*\| = \sup_N \|S_N\| = \mathsf{K_b}.$$

□

*Example 3.2.2.* The natural identification of $\ell_1^*$ with $\ell_\infty$ shows that $(e_n^*)_{n=1}^\infty$ is a basis for a subspace of $\ell_1^*$ isometrically isomorphic to $c_0$.

Roughly speaking, the next proposition says that if $(e_n)_{n=1}^\infty$ is a basis for a Banach space $X$, then the subspace $Z = [e_n^*]$ of $X^*$ is reasonably big, in the sense that it is $1/\mathsf{K}_\mathsf{b}$-norming for $X$ (see Definition 1.4.3).

**Proposition 3.2.3.** *Let $(e_n)_{n=1}^\infty$ be a basis for a Banach space $X$ with basis constant $\mathsf{K}_\mathsf{b}$ and let $Z = [e_n^*]$. The norm on $X$ defined by*

$$\|x\|_Z = \sup\left\{|h(x)| \colon h \in Z, \|h\| \leq 1\right\}$$

*satisfies*

$$\mathsf{K}_\mathsf{b}^{-1}\, \|x\| \leq \|x\|_Z \leq \|x\|\,, \qquad x \in X. \tag{3.2}$$

*Thus:*

*(i) The subspace $Z$ of $X^*$ is $1/\mathsf{K}_\mathsf{b}$-norming for $X$.*
*(ii) The map $x \mapsto j(x)|_Z$ defines an isomorphic embedding of $X$ into $Z^*$. This embedding is isometric if $(e_n)_{n=1}^\infty$ is monotone.*

*Proof.* Let $x \in X$. Since $Z \subseteq X^*$,

$$\|x\|_Z \leq \sup\{|x^*(x)| : x^* \in X^*, \|x^*\| \leq 1\} = \|x\|\,.$$

For the reverse inequality pick $x^* \in S_{X^*}$ such that $x^*(x) = \|x\|$. For each $N$,

$$\frac{|(S_N^* x^*)x|}{\mathsf{K}_\mathsf{b}} \leq \frac{|(S_N^* x^*)x|}{\left\|S_N^* x^*\right\|} \leq \sup\left\{|h(x)| \colon h \in Z, \|h\| \leq 1\right\} = \|x\|_Z\,.$$

We let $N \to \infty$ and use that if $\|S_N(x) - x\| \to 0$, then $|S_N^* x^*(x)| = |x^*(S_N x)| \to \|x\|$. □

As a consequence of the isomorphic embedding of $X$ into $Z^*$ and Proposition 3.2.1, we obtain an interesting reflexivity property for basic sequences.

**Corollary 3.2.4.** *Let $(e_n)_{n=1}^\infty$ be a basis for a Banach space $X$. The basic sequence $(e_n^{**})_{n=1}^\infty$ is equivalent to $(e_n)_{n=1}^\infty$ (isometrically equivalent if $(e_n)_{n=1}^\infty$ is monotone).*

Although $(e_n^*)_{n=1}^\infty$ may not be a basis for $X^*$, one could go further and, based on the following proposition, say that $(e_n^*)_{n=1}^\infty$ is a *weak* basis* for $X^*$.

**Proposition 3.2.5.** *Let $(e_n)_{n=1}^\infty$ be a basis for a Banach space $X$ with biorthogonal functionals $(e_n^*)_{n=1}^\infty$. Then for every $x^* \in X^*$ there is a unique sequence of scalars $(a_n)_{n=1}^\infty$ such that*

$$x^* = \sum_{n=1}^{\infty} a_n e_n^*,$$

*the convergence of the series being in the weak$^*$ topology of $X^*$. More precisely,* $x^* = \text{weak}^* - \lim_{N\to\infty} \sum_{n=1}^{N} x^*(e_n)e_n^*$.

*Proof.* For every $x \in X$,

$$|(x^* - S_N^*(x^*))(x)| = |(x^*(x - S_N(x))| \le \|x^*\| \|x - S_N(x)\| \overset{N\to\infty}{\longrightarrow} 0.$$

□

From Proposition 3.2.1 (a) we infer that $(e_n^*)_{n=1}^{\infty}$ will be a basis for $X^*$ (in the regular sense) if and only if $[e_n^*] = X^*$. Our next result provides a useful test for this, but first let us see a trivial case.

**Proposition 3.2.6.** *If $(e_n)_{n=1}^{\infty}$ is a basis for a reflexive Banach space X, then $(e_n^*)_{n=1}^{\infty}$ is a basis for $X^*$.*

*Proof.* Proposition 3.2.5 yields that the linear span of $(e_n^*)_{n=1}^{\infty}$ is weak$^*$ dense in $X^*$. Since $X$ is reflexive, the weak and the weak$^*$ topologies of $X^*$ coincide, so that the linear span of $(e_n^*)_{n=1}^{\infty}$ is actually weakly dense in $X^*$. An application of Mazur's theorem (see Appendix G.1) now yields that the linear span of $(e_n^*)_{n=1}^{\infty}$ is norm dense in $X^*$, which put together with Proposition 3.2.1 (a) gives that $(e_n^*)_{n=1}^{\infty}$ is a basis for $X^*$. □

*Example 3.2.7.* Suppose $1 < p < \infty$. If $(e_n)_{n=1}^{\infty}$ is a basis for $\ell_p$ [respectively, $L_p[0, 1]$], then $(e_n^*)_{n=1}^{\infty}$ is a basis for $\ell_q$ [respectively, $L_q[0, 1]$], where $1/p + 1/q = 1$.

**Proposition 3.2.8.** *Suppose $(e_n)_{n=1}^{\infty}$ is a basis for a Banach space X. The coordinate functionals $(e_n^*)_{n=1}^{\infty}$ are a basis for $X^*$ if and only if*

$$\lim_{N\to\infty} \|x^*\|_N = 0 \qquad \textit{for every } x^* \in X^*, \tag{3.3}$$

*where $\|x^*\|_N$ is the norm of $x^*$ restricted to the (tail) space $[e_n]_{n>N}$, i.e.,*

$$\|x^*\|_N = \sup\Big\{|x^*(y)| : y \in [e_n]_{n>N}, \|y\| \le 1\Big\}.$$

**Definition 3.2.9.** A basis $(e_n)_{n=1}^{\infty}$ fulfilling property (3.3) is said to be *shrinking*.

*Proof of Proposition 3.2.8.* Suppose $(e_n^*)_{n=1}^{\infty}$ is a basis for $X^*$. Every $x^* \in X^*$ can be decomposed as $(x^* - S_N^* x^*) + S_N^* x^*$ for each $N$. Then

$$\|x^*\|_N \le \big\|\big(x^* - S_N^* x^*\big)|_{[e_n]_{n>N}}\big\| + \underbrace{\big\|S_N^* x^*|_{[e_n]_{n>N}}\big\|}_{\text{this term is } 0} \le \big\|x^* - S_N^* x^*\big\| \overset{N\to\infty}{\longrightarrow} 0.$$

Assume now that (3.3) holds. Let $x^*$ be an element in $X^*$. Since for every $x \in X$, $(I_X - S_N)(x)$ is in the subspace $[e_n]_{n>N}$, we have

$$\begin{aligned}|(x^* - S_N^* x^*)(x)| &= |x^*(I_X - S_N)(x)| \\ &\leq \left\| x^*|_{[e_n]_{n\geq N+1}} \right\| \|I_X - S_N\| \|x\| \\ &\leq (\mathsf{K_b} + 1) \left\| x^*|_{[e_n]_{n\geq N+1}} \right\| \|x\| .\end{aligned}$$

Hence

$$\|x^* - S_N^* x^*\| \leq (\mathsf{K_b} + 1) \, \|x^*|_{[e_n]_{n\geq N+1}}\| \overset{N\to\infty}{\longrightarrow} 0.$$

Thus $X^* = [e_n^*]$ and we are done. □

*Example 3.2.10.* The unit vector basis $(e_n)_{n=1}^\infty$ of $c_0$ is shrinking, since $c_0^* = \ell_1$ and given $x^* = (a_n)_{n=1}^\infty \in \ell_1$, we have

$$\|x^*\|_N = \sum_{n=N+1}^{\infty} |a_n| \overset{N\to\infty}{\longrightarrow} 0.$$

In general, just because a space has a shrinking basis does not mean that every basis for that space is shrinking too. The summing basis $(f_n)_{n=1}^\infty$ of $c_0$, for instance, is not shrinking. To see this, you can take the coordinate functional $e_1^*$ corresponding to the first vector of the canonical basis of $c_0$ and simply observe that $e_1^*(f_n) = 1$ for all $n$, so condition (3.3) cannot hold.

Next we give one further characterization of shrinking bases in light of the selection principles that will prove useful in Section 3.3.

**Proposition 3.2.11.** *A basis $(e_n)_{n=1}^\infty$ of a Banach space $X$ is shrinking if and only if every bounded block basic sequence of $(e_n)_{n=1}^\infty$ is weakly null.*

*Proof.* Assume $(e_n)_{n=1}^\infty$ is not shrinking. Then $Z \neq X^*$; hence there is $x^*$ in $X^* \setminus [e_n^*]$, $\|x^*\| = 1$, such that by Proposition 3.2.5 the series $\sum_{n=1}^\infty x^*(e_n)e_n^*$ converges to $x^*$ in the weak* topology of $X^*$ but does not converge in the norm topology of $X^*$. Using the Cauchy condition, we can find two sequences of positive integers $(p_n)_{n=1}^\infty$, $(q_n)_{n=1}^\infty$ and $\delta > 0$ such that $p_1 \leq q_1 < p_2 \leq q_2 < p_3 \leq q_3 < \cdots$ and $\| \sum_{n=p_k}^{q_k} x^*(e_n)e_n^* \| > \delta$ for all $k \in \mathbb{N}$. Thus for each $k$ there exists $x_k \in X$, $\|x_k\| = 1$, for which $\sum_{n=p_k}^{q_k} x^*(e_n)e_n^*(x_k) > \delta$. Put

$$y_k = \sum_{n=p_k}^{q_k} e_n^*(x_k)e_n, \quad k = 1, 2, \ldots.$$

The sequence $(y_k)_{k=1}^\infty$ is a block basic sequence of $(e_n)_{n=1}^\infty$ that is not weakly null, since $x^*(y_k) > \delta$ for all $k$.

The converse implication follows readily from Proposition 3.2.8. □

A companion notion to that of shrinking bases that was introduced by James as a tool in the study of the structure of Banach spaces is that of *boundedly complete* bases. This property had been used by Dunford and Morse in 1936 [73] to guarantee the existence of derivatives of Lipschitz maps on the real line taking values in Banach spaces (see Section 5.5).

**Definition 3.2.12.** Let $X$ be a Banach space. A basis $(e_n)_{n=1}^{\infty}$ for $X$ is *boundedly complete* if whenever $(a_n)_{n=1}^{\infty}$ is a sequence of scalars such that

$$\sup_N \left\| \sum_{n=1}^{N} a_n e_n \right\| < \infty,$$

then the series $\sum_{n=1}^{\infty} a_n e_n$ converges.

*Example 3.2.13. (a)* The canonical basis of $\ell_p$ for $1 \leq p < \infty$ is boundedly complete.

*(b)* The natural basis $(e_n)_{n=1}^{\infty}$ of $c_0$ is not boundedly complete. Indeed, the series $\sum_{n=1}^{\infty} e_n$ is not convergent in $c_0$ despite the fact that

$$\sup_N \left\| \sum_{n=1}^{N} e_n \right\|_\infty = \sup_N \big\| \underbrace{(1, 1, \dots, 1}_{N}, 0, 0, \dots) \big\|_\infty = 1.$$

(c) The summing basis $(f_n)_{n=1}^{\infty}$ of $c_0$ is not boundedly complete, since

$$\sup_N \left\| \sum_{n=1}^{N} (-1)^n f_n \right\|_\infty = 1,$$

but the series $\sum_{n=1}^{\infty} (-1)^n f_n$ does not converge.

The next two theorems will show that boundedly complete bases and shrinking bases are in duality. Before stating these results and proving them, though, we shall see a lemma about boundedly complete bases that is of some interest by itself and that isolates a property that we need.

**Lemma 3.2.14.** *Suppose $(e_n)_{n=1}^{\infty}$ is a boundedly complete basis for a Banach space $X$ with biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$. Then for each $x^{**} \in X^{**}$ we have*

$$\lim_{N\to\infty} \sum_{n=1}^{N} x^{**}(e_n^*) e_n \in X.$$

*Proof.* Note that for each $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} x^{**}(e_n^*)e_n = S_N^{**}(x^{**}),$$

where $S_N^{**}$ is the double adjoint of $S_N$. Hence

$$\left\| \sum_{n=1}^{N} x^{**}(e_n^*)e_n \right\| = \left\| S_N^{**}(x^{**}) \right\| \le \sup_N \left\| S_N^{**} \right\| \|x^{**}\| = \mathsf{K}_\mathsf{b} \|x^{**}\| .$$

The fact that $(e_n)_{n=1}^{\infty}$ is boundedly complete implies that $\lim_N \sum_{n=1}^{N} x^{**}(e_n^*)e_n \in X$. □

**Theorem 3.2.15.** *Let $(e_n)_{n=1}^{\infty}$ be a basis for a Banach space $X$ with biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$. The following are equivalent:*

*(i) $(e_n)_{n=1}^{\infty}$ is a boundedly complete basis for $X$.*
*(ii) $(e_n^*)_{n=1}^{\infty}$ is a shrinking basis for $Z = [e_n^*]$, i.e., the sequence of its biorthogonal functionals $(j(e_n)|_Z)_{n=1}^{\infty}$ is a basis for $Z^*$.*
*(iii) The map $x \mapsto j(x)|_Z$ defines an isomorphism from $X$ onto $Z^*$ that is isometric if $\mathsf{K}_\mathsf{b} = 1$.*

*Proof.* $(i) \Rightarrow (iii)$ Using Proposition 3.2.3, we need only show that the map is onto. Given $h^* \in Z^*$, there exists $x^{**} \in X^{**}$ such that $x^{**}|_Z = h^*$. By Lemma 3.2.14 the series $\sum_{n=1}^{\infty} x^{**}(e_n^*)e_n$ converges to some $x \in X$. Now $j(x)|_Z = h^*$, since for each $k \in \mathbb{N}$ we have

$$j(x)|_Z(e_k^*) = e_k^*(x) = x^{**}(e_k^*) = h^*(e_k^*).$$

$(iii) \Rightarrow (ii)$ Assume that $x \mapsto j(x)|_Z$ is an isomorphism from $X$ onto $Z^*$. Then $(j(e_n)|_Z)_{n=1}^{\infty}$ is a basis for $Z^*$ and it is also the sequence of coordinate functionals for $(e_n^*)_{n=1}^{\infty}$. This means that $(e_n^*)_{n=1}^{\infty}$ is a shrinking basis for $Z$.

$(ii) \Rightarrow (i)$ Let $(a_n)_{n=1}^{\infty}$ be a sequence of scalars for which

$$\sup_N \left\| \sum_{n=1}^{N} a_n e_n \right\| < \infty. \tag{3.4}$$

Since $(\sum_{n=1}^{N} a_n j(e_n))_{N=1}^{\infty}$ is bounded in $X^{**}$, the Banach–Alaoglu theorem yields the existence of a weak$^*$ cluster point $x^{**} \in X^{**}$ of that sequence. Let $h^* = x^{**}|_Z$. Since

$$\lim_N \sum_{n=1}^{N} a_n j(e_n)(e_k^*) = \lim_N \sum_{n=1}^{N} a_n e_k^*(e_n) = a_k,$$

we have $h^*(e_k^*) = x^{**}(e_k^*) = a_k$ for all $k$. Using the hypothesis and Proposition 3.2.1, we obtain

$$h^* = \sum_{n=1}^{\infty} e_n^{***}(h^*)e_n^{**} = \sum_{n=1}^{\infty} h^*(e_n^*)e_n^{**} = \sum_{n=1}^{\infty} a_n j(e_n)|_Z.$$

In particular, the above series converges in $Z^*$. Since by Proposition 3.2.3, the mapping $x \mapsto j(x)|_Z$ is an isomorphic embedding, the series $\sum_{n=1}^{\infty} a_n e_n$ converges in $X$. □

**Corollary 3.2.16.** *The space $c_0$ has no boundedly complete basis.*

*Proof.* The result follows from Theorem 3.2.15, taking into account that $c_0$ is not isomorphic to a dual space (Corollary 2.5.6). □

**Theorem 3.2.17.** *Let $(e_n)_{n=1}^{\infty}$ be a basis for a Banach space $X$ with biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$. The following are equivalent:*

*(i)* $(e_n)_{n=1}^{\infty}$ *is a shrinking basis for $X$.*
*(ii)* $(e_n^*)_{n=1}^{\infty}$ *is a boundedly complete basis for $Z = [e_n^*]$.*
*(iii)* $Z = X^*$.

*Proof.* Just apply Theorem 3.2.15 to the basis $(e_n^*)_{n=1}^{\infty}$ of $Z$ and take into account Corollary 3.2.4. □

In 1948, Karlin [165] raised the following question: If the dual $X^*$ of a Banach space $X$ has a basis, does $X$ itself have a basis? Johnson, Rosenthal, and Zippin gave in 1971 an affirmative answer to this question [142], thus solving one of the fundamental problems in basis theory. They showed, in fact, that such an $X$ has a shrinking basis. From their result it follows that if $X$ has a basis and $X^*$ is separable and satisfies the approximation property, then $X^*$ has a basis. A related, but easier, question to handle is this: If $X$ has a basis $(e_n)_{n=1}^{\infty}$, which property on $(e_n)_{n=1}^{\infty}$ will tell us whether $X$ is the dual of another Banach space with a basis? Theorem 3.2.15 gives us the answer: if $(e_n)_{n=1}^{\infty}$ is boundedly complete, then $X$ is isomorphic to the dual space $Z^* = [e_n^*]^*$.

We may not know much more about $X^*$ other than that it has a boundedly complete basis if all we know about $X$ is that it has a shrinking basis. However, it is possible to give a very useful description of $X^{**}$ as the space of all sequences of scalars $(a_n)_{n=1}^{\infty}$ for which $\sup_N \|\sum_{n=1}^{N} a_n e_n\| < \infty$. The original purpose of the following theorem was to lay the foundation for the James space $\mathcal{J}$, which we will cover in Section 3.4. Its proof uses a combination of ideas essentially contained in the proof of Theorem 3.2.15, and so we leave it as an exercise for the reader.

**Theorem 3.2.18.** *Suppose $(e_n)_{n=1}^{\infty}$ is a shrinking basis for a Banach space $X$ with biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$. The mapping*

$$x^{**} \mapsto (x^{**}(e_n^*))_{n=1}^{\infty}$$

*defines an isomorphism of $X^{**}$ with the space of all sequences of scalars $(a_n)_{n=1}^{\infty}$ such that $\sup_N \|\sum_{n=1}^N a_n e_n\| < \infty$. In particular,*

$$\|x^{**}\| \approx \sup_N \Big\| \sum_{n=1}^N x^{**}(e_n^*) e_n \Big\|,$$

*and if $(e_n)_{n=1}^{\infty}$ is monotone, then*

$$\|x^{**}\| = \lim_{N\to\infty} \Big\| \sum_{n=1}^N x^{**}(e_n^*) e_n \Big\|.$$

*The canonical image of $X$ inside $X^{**}$ corresponds to all sequences of scalars $(a_n)_{n=1}^{\infty}$ for which $(\sum_{n=1}^N a_n e_n)_{N=1}^{\infty}$ not only is bounded but converges in norm.*

Now we come to the main result of the section, which is due to James [126].

**Theorem 3.2.19 (James [126]).** *Let $X$ be a Banach space. If $X$ has a basis $(e_n)_{n=1}^{\infty}$ then $X$ is reflexive if and only if $(e_n)_{n=1}^{\infty}$ is both boundedly complete and shrinking.*

*Proof.* Assume that $X$ is reflexive and that $(e_n)_{n=1}^{\infty}$ is a basis for $X$. Then $X^* = Z$. If not, using the Hahn–Banach theorem, one could find $0 \neq x^{**} \in X^{**}$ such that $x^{**}(h) = 0$ for all $h \in Z$. By reflexivity there is $0 \neq x = \sum_{n=1}^{\infty} e_n^*(x) e_n \in X$ such that $x = x^{**}$. In particular, we would have $0 = x^{**}(e_n^*) = e_n^*(x)$ for all $n$, which would imply $x = 0$. Thus $(e_n)_{n=1}^{\infty}$ is shrinking. Now consider $(e_n^*)_{n=1}^{\infty}$ as a basis for the reflexive Banach space $X^* = Z$. The above shows that $(e_n^*)_{n=1}^{\infty}$ is shrinking; hence by Theorem 3.2.15, $(e_n)_{n=1}^{\infty}$ is boundedly complete.

Conversely, $(e_n)_{n=1}^{\infty}$ shrinking implies $Z = X^*$, and since $(e_n)_{n=1}^{\infty}$ is boundedly complete as well, the map $X \to Z^*$ in Theorem 3.2.15 (*iii*) is now the canonical embedding of $X$ *onto* $X^{**}$. □

This theorem gives a criterion for reflexivity that is enormously useful, particularly in the construction of examples. Notice that the facts that the canonical basis of $\ell_1$ fails to be shrinking and that the canonical basis of $c_0$ fails to be boundedly complete are explained now in the nonreflexivity of these spaces.

During the 1960s it was very fashionable to study the structure of Banach spaces by understanding the properties of their bases. Of course, this viewpoint was somewhat undermined when Enflo showed that not every separable Banach space has a basis [88]. One of the high points of this theory was the theorem of Zippin [309] that a Banach space with a basis is reflexive if and only if *every* basis is boundedly complete or if and only if *every* basis is shrinking. Thus, every nonreflexive Banach space that has a basis must have at least one non boundedly complete basis and at least one nonshrinking basis.

## 3.3 Nonreflexive Spaces with Unconditional Bases

Now let us consider the boundedly complete and shrinking unconditional bases. Again we follow the classic paper of James [126].

**Theorem 3.3.1.** *Let $X$ be a Banach space with unconditional basis $(u_n)_{n=1}^{\infty}$. The following are equivalent:*

*(i)* $(u_n)_{n=1}^{\infty}$ *fails to be shrinking.*
*(ii)* *$X$ contains a complemented subspace isomorphic to $\ell_1$.*
*(iii)* *There exists a complemented block basic sequence $(y_n)_{n=1}^{\infty}$ with respect to $(u_n)_{n=1}^{\infty}$ that is equivalent to the canonical basis of $\ell_1$.*
*(iv)* *$X$ contains a subspace isomorphic to $\ell_1$.*

*Proof.* The implications $(iii) \Rightarrow (ii) \Rightarrow (iv)$ are obvious.

$(iv) \Rightarrow (i)$ is also immediate, because if $X$ contains $\ell_1$, then $X^*$ cannot be separable, and so $(u_n)_{n=1}^{\infty}$ is not shrinking.

$(i) \Rightarrow (iii)$ If $(u_n)_{n=1}^{\infty}$ is not shrinking, by Proposition 3.2.11 we can find a bounded block basic sequence $(y_k)_{k=1}^{\infty}$ of $(u_n)_{n=1}^{\infty}$, $\delta > 0$, and $x^* \in X^*$ with $\|x^*\| = 1$, such that $x^*(y_k) > \delta$ for all $k$. Let $\mathsf{C} = \sup_k \|y_k\|$.

For every $(a_k)_{k=1}^{\infty} \in c_{00}$,

$$\Big\| \sum_{k=1}^{\infty} a_k y_k \Big\| \geq \Big| \sum_{k=1}^{\infty} x^*(y_k) a_k \Big|.$$

By picking $\epsilon_k = \operatorname{sgn} a_k$ for each $k$ we obtain

$$\Big\| \sum_{k=1}^{\infty} \epsilon_k a_k y_k \Big\| \geq \sum_{k=1}^{\infty} |x^*(y_k) a_k| \geq \delta \sum_{k=1}^{\infty} |a_k|.$$

Being a block basis of $(u_n)_{n=1}^{\infty}$, $(y_k)_{k=1}^{\infty}$ is an unconditional basic sequence with unconditional basis constant not greater than the unconditional basis constant $\mathsf{K}_{\mathsf{u}}$ of $(u_n)_{n=1}^{\infty}$. Therefore,

$$\Big\| \sum_{k=1}^{\infty} a_k y_k \Big\| \geq \delta \mathsf{K}_{\mathsf{u}}^{-1} \sum_{k=1}^{\infty} |a_k|.$$

On the other hand, since $(y_k)_{k=1}^{\infty}$ is bounded, the triangle law yields an upper $\ell_1$-estimate for $\| \sum_{k=1}^{\infty} a_k y_k \|$, and hence $(y_k)_{k=1}^{\infty}$ is equivalent to the standard $\ell_1$-basis. It remains to define a linear projection from $X$ onto $[y_k]$. For each $k$ put

$$y_k^* = \frac{1}{x^*(y_k)} \sum_{n=p_k}^{q_k} x^*(u_n) u_n^*.$$

Clearly, the sequence $(y_k^*)_{k=1}^\infty$ is orthogonal to $(y_k)_{k=1}^\infty$ and $\|y_k^*\| \le \delta^{-1}\mathsf{K}_\mathsf{u}$. For every $N \in \mathbb{N}$ let us consider the projection from $X$ onto $[y_k]_{1\le k\le N}$ defined as

$$P_N(x) = \sum_{k=1}^N y_k^*(x) y_k.$$

The sequence $(P_N)_{N=1}^\infty$ is bounded: given any $x \in X$, if we pick $\epsilon_k = \operatorname{sgn} y_k^*(x)$, we have

$$\begin{aligned}
\|P_N(x)\| &\le \mathsf{c}\sum_{k=1}^N |y_k^*(x)| \\
&= \mathsf{c}\sum_{k=1}^N \epsilon_k y_k^*(x) \\
&= \mathsf{c}\sum_{k=1}^N \sum_{n=p_k}^{q_k} \frac{\epsilon_k}{x^*(y_k)} x^*(u_n) u_n^*(x) \\
&= \mathsf{c}x^*\Big(\sum_{k=1}^N \sum_{n=p_k}^{q_k} \frac{\epsilon_k}{x^*(y_k)} u_n^*(x) u_n\Big) \\
&\le \mathsf{c}\mathsf{K}_\mathsf{u} \max_k \left|\frac{1}{x^*(y_k)}\right| \|x\| \\
&\le \mathsf{c}\mathsf{K}_\mathsf{u}\delta^{-1} \|x\| .
\end{aligned}$$

Since $\lim_{N\to\infty} P_N(x)$ exists for each $x$, by the Banach–Steinhaus theorem the operator

$$P : X \to [y_k], \qquad x \mapsto P(x) = \sum_{k=1}^\infty y_k^*(x) y_k,$$

is bounded by $\mathsf{c}\mathsf{K}_\mathsf{u}\delta^{-1}$ and is obviously the desired projection. □

**Theorem 3.3.2.** *Let $X$ be a Banach space with unconditional basis $(u_n)_{n=1}^\infty$. The following are equivalent:*

*(i) The basis $(u_n)_{n=1}^\infty$ fails to be boundedly complete.*
*(ii) $X$ contains a complemented subspace isomorphic to $c_0$.*
*(iii) There exists a complemented block basic sequence $(y_n)_{n=1}^\infty$ with respect to $(u_n)_{n=1}^\infty$ equivalent to the canonical basis of $c_0$.*
*(iv) $X$ contains a subspace isomorphic to $c_0$.*

*Proof.* Note that $(ii)$ and $(iv)$ are equivalent, since $c_0$ is separably injective (Sobczyk's theorem, Theorem 2.5.8).

$(i) \Rightarrow (iii)$ If $(u_n)_{n=1}^{\infty}$ is not boundedly complete, there exists a sequence of scalars $(a_n)_{n=1}^{\infty}$ such that $\sup_N \|\sum_{n=1}^{N} a_n u_n\| < \infty$ but $\sum_{n=1}^{\infty} a_n u_n$ does not converge in $X$.

Given any $x^* \in X^*$, pick $\epsilon_n = \operatorname{sgn} x^*(u_n)$. By the unconditionality of the basis, for $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} |a_n||x^*(u_n)| = \sum_{n=1}^{N} \epsilon_n a_n x^*(u_n) \leq \mathsf{K_u} \|x^*\| \left\| \sum_{n=1}^{N} a_n u_n \right\|.$$

Thus $\sum_{n=1}^{\infty} |x^*(a_n u_n)|$ converges for all $x^* \in X^*$. That is, $\sum_{n=1}^{\infty} a_n u_n$ is a WUC series in $X$ that is not unconditionally convergent. Proposition 2.4.7 yields $T : c_0 \to X$ such that $T(e_n) = a_n u_n$ for all $n$, where $(e_n)_{n=1}^{\infty}$ denotes the standard unit vector basis of $c_0$. Furthermore, by Proposition 2.4.8, the operator $T$ cannot be compact. Using Theorem 2.4.10, we can extract a block basic sequence $(x_k)_{k=1}^{\infty}$ with respect to the canonical basis of $c_0$ such that $T|_{[x_k]}$ is an isomorphism onto its range. Then $y_k := T(x_k)$ defines a block basic sequence in $X$ with respect to the basis $(u_n)_{n=1}^{\infty}$ such that $[y_k]$ is isomorphic to $c_0$. Corollary 2.5.9 implies that $[y_k]$ is complemented in $X$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ Suppose that $(ii)$ holds and that $(u_n)_{n=1}^{\infty}$ is boundedly complete. Then, by Theorem 3.2.15, $X$ is a dual space and so there is a bounded projection of $X^{**}$ onto $X$ (see the discussion after Proposition 2.5.2). Hence there is a projection of $X^{**}$ onto a subspace $E$ of $X$ isomorphic to $c_0$. However, if $E$ is a subspace of $X$, then $E^{**}$ embeds as a subspace of $X^{**}$ (it can be identified with $E^{\perp\perp}$, which is also the weak$^*$ closure of $E$). Hence there is a projection of $E^{**}$ onto $E$. This contradicts Theorem 2.5.5. □

**Corollary 3.3.3.** *Suppose that $X$ is a Banach space with an unconditional basis. Then:*

*(a) Either all unconditional bases of $X$ are shrinking or none is.*
*(b) Either all unconditional bases of $X$ are boundedly complete or none is.*

The following theorem is again due to James [126], except that the last statement was proved earlier, using different techniques, by Karlin [165].

**Theorem 3.3.4.** *Suppose that $X$ is a Banach space with an unconditional basis. If $X$ is not reflexive, then either $c_0$ is complemented in $X$, or $\ell_1$ is complemented in $X$ (or both). In either case, $X^{**}$ is nonseparable.*

*Proof.* The first statement of the theorem follows immediately from Theorem 3.2.19, Theorem 3.3.1, and Theorem 3.3.2. Now, for the latter statement, if $c_0$ were complemented in $X$, then $X^{**}$ would contain a (complemented) copy $\ell_\infty$. If $\ell_1$ were complemented in $X$, then $X^*$ would be nonseparable, since it would contain a (complemented) copy of $\ell_\infty$. In either case, $X^{**}$ is nonseparable. □

## 3.4 The James Space $\mathcal{J}$

Continuing with the classic paper of James [126], we come to his construction of one of the most important examples in Banach space theory. This space, nowadays known as the James space, is, in fact, quite a natural space, consisting of sequences of bounded 2-variation. The James space will provide an example of a Banach space with a basis but with no unconditional basis; it also answered several other open questions at the time. For example, it was not known whether a Banach space $X$ was necessarily reflexive if its bidual was separable. The James space $\mathcal{J}$ is separable and has codimension one in $\mathcal{J}^{**}$, and so gives a counterexample. Later, James [127] went further and modified the definition of the norm to make $\mathcal{J}$ isometric to $\mathcal{J}^{**}$, thus showing that a Banach space can be isometrically isomorphic to its bidual yet fail to be reflexive!

Let us define $\tilde{\mathcal{J}}$ to be the space of all sequences $\xi = (\xi(n))_{n=1}^{\infty}$ of real numbers with finite square variation; that is, $\xi \in \tilde{\mathcal{J}}$ if and only if there is a constant $M$ such that for every choice of integers $(p_j)_{j=0}^{n}$ with $1 \le p_0 < p_1 < \cdots < p_n$ we have

$$\sum_{j=1}^{n} (\xi(p_j) - \xi(p_{j-1}))^2 \le M^2.$$

It is easy to verify that if $\xi \in \tilde{\mathcal{J}}$, then $\lim_{n\to\infty} \xi(n)$ exists. We then define $\mathcal{J}$ as the subspace of $\tilde{\mathcal{J}}$ of all $\xi$ such that $\lim_{n\to\infty} \xi(n) = 0$.

**Definition 3.4.1.** The *James space* $\mathcal{J}$ is the (real) Banach space of all sequences $\xi = (\xi(n))_{n=1}^{\infty} \in \tilde{\mathcal{J}}$ such that $\lim_{n\to\infty} \xi(n) = 0$, endowed with the norm

$$\|\xi\|_{\mathcal{J}} = \frac{1}{\sqrt{2}} \sup \left\{ \left( (\xi(p_n) - \xi(p_0))^2 + \sum_{k=1}^{n} (\xi(p_k) - \xi(p_{k-1}))^2 \right)^{1/2} \right\},$$

where the supremum is taken over all $n \in \mathbb{N}$ and all choices of integers $(p_j)_{j=0}^{n}$ with $1 \le p_0 < p_1 < \cdots < p_n$.

The definition of the norm in the James space is not quite natural; clearly, the norm is equivalent to the alternative norm given by the formula

$$\|\xi\|_0 = \sup \left\{ \left( \sum_{k=1}^{n} (\xi(p_k) - \xi(p_{k-1}))^2 \right)^{1/2} \right\},$$

where again, the supremum is taken over all sequences of integers $(p_j)_{j=0}^{n}$ with $1 \le p_0 < p_1 < \cdots < p_n$. In fact,

$$\frac{1}{\sqrt{2}} \|\xi\|_0 \le \|\xi\|_{\mathcal{J}} \le \sqrt{2} \|\xi\|_0, \qquad \xi \in \mathcal{J}.$$

Notice that $\|e_n\|_{\mathcal{J}} = 1$ for all $n$, but $\|e_n\|_0 = \sqrt{2}$ for $n \geq 2$. We also note that $\|\cdot\|_{\mathcal{J}}$ can be canonically extended to $\tilde{\mathcal{J}}$ by

$$\|\xi\|_{\mathcal{J}} = \frac{1}{\sqrt{2}} \sup \left\{ \left( (\xi(p_n) - \xi(p_0))^2 + \sum_{k=1}^{n} (\xi(p_k) - \xi(p_{k-1}))^2 \right)^{1/2} \right\},$$

but this defines only a seminorm on $\tilde{\mathcal{J}}$ vanishing on all constant sequences.

**Proposition 3.4.2.** *The sequence $(e_n)_{n=1}^{\infty}$ of standard unit vectors is a monotone basis for $\mathcal{J}$ in both norms $\|\cdot\|_{\mathcal{J}}$ and $\|\cdot\|_0$.*

*Proof.* We will leave for the reader the verification that $(e_n)_{n=1}^{\infty}$ is a monotone basic sequence in both norms. To prove that it is a basis, we need only consider the norm $\|\cdot\|_0$.

Suppose $\xi \in \mathcal{J}$. For each $N$ let

$$\xi_N = \xi - \sum_{j=1}^{N} \xi(j) e_j.$$

Given $\epsilon > 0$, pick $1 \leq p_0 < p_1 < \cdots < p_n$ for which $\sum_{j=1}^{n} (\xi(p_j) - \xi(p_{j-1}))^2 > \|\xi\|_0^2 - \epsilon^2$. In order to estimate the norm of $\xi_N$ when $N > p_n$, it is enough to consider positive integers $q_0 \leq q_1 < q_2 < \cdots < q_m$, where $N \leq q_0$. Then for the partition $1 \leq p_0 < p_1 < \cdots < p_n < q_0 < q_2 < \cdots < q_m$ we have

$$\begin{aligned}
\|\xi\|_0^2 &\geq \sum_{j=1}^{n} (\xi(p_j) - \xi(p_{j-1}))^2 + (\xi(q_0) - \xi(p_n))^2 + \sum_{j=1}^{m} (\xi(q_j) - \xi(q_{j-1}))^2 \\
&\geq \sum_{j=1}^{n} (\xi(p_j) - \xi(p_{j-1}))^2 + \sum_{j=1}^{m} (\xi(q_j) - \xi(q_{j-1}))^2.
\end{aligned}$$

Hence

$$\sum_{j=1}^{m} (\xi(q_j) - \xi(q_{j-1}))^2 \leq \epsilon^2.$$

Thus $\|\xi_N\|_0 < \epsilon$ for $N > p_n$. □

**Proposition 3.4.3.** *Let $(\eta_k)_{k=1}^{\infty}$ be a normalized block basic sequence with respect to $(e_n)_{n=1}^{\infty}$ in $(\mathcal{J}, \|\cdot\|_0)$. Then for every sequence of scalars $(\lambda_k)_{k=1}^{n}$,*

$$\left\| \sum_{k=1}^{n} \lambda_k \eta_k \right\|_0 \leq \sqrt{5} \left( \sum_{k=1}^{n} \lambda_k^2 \right)^{1/2}.$$

*Proof.* For each $k$ let

$$\eta_k = \sum_{j=q_{k-1}+1}^{q_k} \eta_k(j)e_j,$$

where $0 = q_0 < q_1 < \cdots$, and put

$$\xi = \sum_{k=1}^{n} \lambda_k \eta_k.$$

Suppose $1 \le p_0 < p_1 < \cdots < p_m$. Fix $i \le n$. Let $A_i$ be the set of $k$ such that $q_{i-1} < p_{k-1} < p_k \le q_i$. If $k \in A_i$, then

$$\xi(p_k) - \xi(p_{k-1}) = \lambda_i(\eta_i(p_k) - \eta_i(p_{k-1})).$$

Hence

$$\sum_{k \in A_i} (\xi(p_k) - \xi(p_{k-1}))^2 \le \lambda_i^2.$$

If $A = \cup_i A_i$, we thus have

$$\sum_{k \in A} (\xi(p_k) - \xi(p_{k-1}))^2 \le \sum_{i=1}^{n} \lambda_i^2.$$

Let $B$ be the set of $1 \le k \le m$ with $k \notin A$. For each such $k$ there exist $i = i(k), j = j(k)$ such that $q_{i-1} < p_{k-1} \le q_i$ and $q_{j-1} < p_k \le q_j$. Then,

$$\begin{aligned}(\xi(p_k) - \xi(p_{k-1}))^2 &= (\lambda_j \eta_j(p_k) - \lambda_i \eta_i(p_{k-1}))^2 \\ &\le 2(\lambda_j^2 \eta_j(p_k)^2 + \lambda_i^2 \eta_j(p_{k-1})^2) \\ &\le 2(\lambda_j^2 + \lambda_i^2).\end{aligned}$$

Thus,

$$\sum_{k=1}^{m} (\xi(p_k) - \xi(p_{k-1}))^2 \le \sum_{i=1}^{n} \lambda_i^2 + 2\sum_{k \in B} \lambda_{i(k)}^2 + 2\sum_{k \in B} \lambda_{j(k)}^2.$$

Since the $i(k)$'s and similarly the $j(k)$'s are distinct for $k \in B$, it follows that

$$\sum_{k=1}^{m} (\xi(p_k) - \xi(p_{k-1}))^2 \le 5\sum_{i=1}^{n} \lambda_i^2,$$

and this completes the proof. ☐

**Proposition 3.4.4.** *The sequence $(e_n)_{n=1}^{\infty}$ is a shrinking basis for $\mathcal{J}$ (for both norms $\|\cdot\|_{\mathcal{J}}$ and $\|\cdot\|_0$).*

*Proof.* We will prove that every bounded block basic sequence of $(e_n)$ is weakly null, and then we will appeal to Proposition 3.2.11. Let $(\eta_k)_{k=1}^{\infty}$ be a normalized block basic sequence in $(\mathcal{J}, \|\cdot\|_0)$. Using Proposition 3.4.3, we see that the operator $S : \ell_2 \to [\eta_k] \subset \mathcal{J}$ defined for each $\lambda = (\lambda_k)_{k=1}^{\infty} \in \ell_2$ by

$$S(\lambda) = \sum_{k=1}^{\infty} \lambda_k \eta_k$$

is bounded. The norm-continuity of $S$ implies that $S$ is weak-to-weak continuous. Since the sequence of the unit vector basis of $\ell_2$ is weakly null, it follows that their images, the block basic sequence $(\eta_k)_{k=1}^{\infty}$, must converge to 0 weakly as well. □

*Remark 3.4.5.* Notice that the standard unit vector basis of $\mathcal{J}$ is not boundedly complete, since

$$\Big\| \sum_{n=1}^{N} e_n \Big\|_{\mathcal{J}} = \|(1, 1, \ldots, 1, 0, \ldots)\|_0 = 1$$

for all $N$, but the series $\sum_{n=1}^{\infty} e_n$ does not converge in $\mathcal{J}$.

Since $(e_n)_{n=1}^{\infty}$ is shrinking, we can identify each $x^{**} \in \mathcal{J}^{**}$ with the sequence given by $\xi(n) = x^{**}(e_n^*)$ for every $n$ (see Theorem 3.2.18). Under this identification $\mathcal{J}^{**}$ becomes the space of sequences $\xi$ such that

$$\|\xi\|_{\mathcal{J}^{**}} = \sup_n \|(\xi(1), \ldots, \xi(n), 0, \ldots)\|_{\mathcal{J}} < \infty.$$

Note that we now specialize to the use of the norm $\|\cdot\|_{\mathcal{J}}$ on $\mathcal{J}$. That $\|\cdot\|_{\mathcal{J}^{**}}$ is the bidual norm on $\mathcal{J}^{**}$ follows easily from the fact that the basis $(e_n)_{n=1}^{\infty}$ is monotone. It is clear from the definition that $\mathcal{J}^{**}$ coincides with $\tilde{\mathcal{J}}$, i.e., the space of sequences of bounded square variation.

We have already noticed that the canonical extension of $\|\cdot\|_{\mathcal{J}}$ to $\tilde{\mathcal{J}} = \mathcal{J}^{**}$ is only a seminorm. In fact, the relationship between $\|\cdot\|_{\mathcal{J}^{**}}$ and $\|\cdot\|_{\mathcal{J}}$ is

$$\|\xi\|_{\mathcal{J}^{**}} = \max(\|\xi\|_{\mathcal{J}}, \|\xi\|_1),$$

where

$$\|\xi\|_1 = \frac{1}{\sqrt{2}} \sup \left\{ \Big( \xi(p_n)^2 + \xi(p_0)^2 + \sum_{k=1}^{n} (\xi(p_k) - \xi(p_{k-1}))^2 \Big)^{1/2} \right\},$$

and, as usual, the supremum is taken over all $n \in \mathbb{N}$ and all choices of integers $(p_j)_{j=0}^{n}$ with $1 \le p_0 < p_1 < \cdots < p_n$.

**Theorem 3.4.6.** *The space $\mathcal{J}$ is a subspace of codimension* 1 *in $\mathcal{J}^{**}$, and $\mathcal{J}^{**}$ is isometric to $\mathcal{J}$.*

*Proof.* Clearly, $\mathcal{J} = \{\xi \in \mathcal{J}^{**} : \lim_{n\to\infty} \xi(n) = 0\}$ has codimension one in its bidual. To prove the fact that it is isometric to its bidual we observe that

$$\|\xi\|_{\mathcal{J}^{**}} = \|(0, \xi(1), \xi(2), \dots)\|_{\mathcal{J}}, \qquad \xi \in \mathcal{J}^{**}.$$

Let

$$L(\xi) = \lim_{n\to\infty} \xi(n), \qquad \xi \in \mathcal{J}^{**}.$$

We define

$$S(\xi) = (-L(\xi), \xi(1) - L(\xi), \xi(2) - L(\xi), \dots).$$

Then $S$ maps $\mathcal{J}^{**}$ onto $\mathcal{J}$ and is one-to-one. Since $\|\cdot\|_{\mathcal{J}}$ is a seminorm on $\mathcal{J}^{**}$ vanishing on constants,

$$\|S(\xi)\|_{\mathcal{J}} = \|(0, \xi(1), \dots)\|_{\mathcal{J}} = \|\xi\|_{\mathcal{J}^{**}}.$$

Thus $S$ is an isometry. □

**Corollary 3.4.7.** *$\mathcal{J}$ does not have an unconditional basis.*

*Proof.* The result follows immediately from the separability of $\mathcal{J}^{**}$, Theorem 3.3.4 and Theorem 3.4.6. □

After the appearance of James's example, the term *quasi-reflexive* was often used for Banach spaces $X$ such that $X^{**}/X$ is finite-dimensional.

The ideas of the James construction have been repeatedly revisited to produce more sophisticated examples of similar type. For example, Lindenstrauss [195] showed that for every separable Banach space $X$ there is a Banach space $\mathcal{Z}$ with a shrinking basis such that $\mathcal{Z}^{**}/\mathcal{Z}$ is isomorphic to $X$ (see Section 15.1).

## 3.5 A Litmus Test for Having Unconditional Bases

We now want to go a little further and show that $\mathcal{J}$ cannot even be isomorphic to a subspace of a Banach space with an unconditional basis. We therefore need to identify a property of subspaces of spaces with unconditional bases that we can test. For this we use Pełczyński's property (u) introduced in 1958 [240].

**Definition 3.5.1.** A Banach space $X$ *has property (u)* if whenever $(x_n)_{n=1}^\infty$ is a weakly Cauchy sequence in $X$, there is a WUC series $\sum_{k=1}^\infty u_k$ in $X$ such that

$$x_n - \sum_{k=1}^n u_k \to 0 \text{ weakly}.$$

**Proposition 3.5.2.** *If a Banach space $X$ has property (u), then every closed subspace $Y$ of $X$ has property (u).*

*Proof.* Let $(y_s)_{s=1}^\infty$ be a weakly Cauchy sequence in a closed subspace $Y$ of $X$. Since $X$ has property (u), there is a WUC series $\sum_{i=1}^\infty u_i$ in $X$ such that the sequence $(y_s - \sum_{i=1}^s u_i)_{s=1}^\infty$ converges to 0 weakly. By Mazur's theorem there is a sequence of convex combinations of members of $(y_s - \sum_{i=1}^s u_i)_{s=1}^\infty$ that converges to 0 in norm. Using the Cauchy condition, we obtain integers $(p_k)_{k=1}^\infty$, $0 = p_0 < p_1 < p_2 < \cdots$, and convex combinations $(\sum_{j=p_{k-1}+1}^{p_k} \lambda_j (y_j - \sum_{i=1}^j u_i))_{k=1}^\infty$ such that

$$\Big\| \sum_{j=p_{k-1}+1}^{p_k} \lambda_j (y_j - \sum_{i=1}^j u_i) \Big\| \le 2^{-k} \quad \text{for all } k.$$

Put $z_0 = 0$, and for each integer $k \ge 1$ let

$$z_k = \sum_{j=p_{k-1}+1}^{p_k} \lambda_j y_j \in Y.$$

Then for every $x^* \in X^*$, $\|x^*\| = 1$, we have

$$\begin{aligned} &|x^*(z_k - z_{k-1})| \\ &\quad \le \frac{1}{2^k} + \frac{1}{2^{k-1}} + \left| x^* \left( \sum_{j=p_{k-1}+1}^{p_k} \lambda_j \sum_{i=p_{k-2}+1}^{j} u_i - \sum_{j=p_{k-2}+1}^{p_{k-1}} \lambda_j \sum_{i=p_{k-2}+1}^{j} u_i \right) \right|. \end{aligned}$$

Thus,

$$|x^*(z_k - z_{k-1})| \le 3 \cdot 2^{-k} + 2 \sum_{j=p_{k-2}+1}^{p_k} |x^*(u_j)|,$$

which implies

$$\sum_{k=1}^\infty |x^*(z_k - z_{k-1})| \le \frac{3}{2} + 4 \sum_{j=1}^\infty |x^*(u_j)| < \infty.$$

Therefore $\sum_{k=1}^\infty (z_k - z_{k-1})$ is a WUC series in $Y$. Now one easily checks that the sequence

$$\Big( y_n - \sum_{k=1}^n (z_k - z_{k-1}) \Big)_{n=1}^\infty = (y_n - z_n)_{n=1}^\infty$$

converges weakly to 0. □

**Proposition 3.5.3 (Pełczyński [240]).** *If a Banach space $X$ has an unconditional basis, then $X$ has property (u).*

*Proof.* Let $(u_n)_{n=1}^\infty$ be a $\mathsf{K_u}$-unconditional basis of $X$ with biorthogonal functionals $(u_n^*)_{n=1}^\infty$. If $(x_n)_{n=1}^\infty$ is a weakly Cauchy sequence in $X$, then for each $k$ the scalar sequence $(u_k^*(x_n))_{n=1}^\infty$ converges, say to $\alpha_k$. Hence the sequence $(\sum_{k=1}^N t_k u_k^*(x_n)u_k)_{n=1}^\infty$ converges weakly to $\sum_{k=1}^N t_k\alpha_k u_k$ for each $N$ and all scalars $(t_k)$. Therefore,

$$\Big\|\sum_{k=1}^N \epsilon_k\alpha_k u_k\Big\| \le \mathsf{K_u}\sup_n \|x_n\|$$

for all $N$ and every sequence of signs $(\epsilon_k)$. Being weakly Cauchy, $(x_n)_{n=1}^\infty$ is norm-bounded; thus $\sum_{k=1}^\infty \alpha_k u_k$ is a WUC series. Put

$$y_n = x_n - \sum_{k=1}^n \alpha_k u_k.$$

The sequence $(y_n)_{n=1}^\infty$ is weakly Cauchy. Also, $\lim_{n\to\infty} u_s^*(y_n) = 0$ for all $s\in\mathbb{N}$. We claim that $(y_n)_{n=1}^\infty$ converges weakly to 0. If not, there is $x^*\in X^*$ such that $\lim_{n\to\infty} x^*(y_n) = 1$. Using the Bessaga–Pełczyński selection principle (Proposition 1.3.10), we can extract a subsequence $(y_{n_j})_{j=1}^\infty$ of $(y_n)_{n=1}^\infty$ and find a block basic sequence $(z_j)_{j=1}^\infty$ of $(u_n)_{n=1}^\infty$ such that $(z_j)_{j=1}^\infty$ is equivalent to $(y_{n_j})_{j=1}^\infty$ and $\|y_{n_j} - z_j\| \to 0$. We deduce that $x^*(z_j)\to 1$, since

$$|x^*(z_j) - 1| \le |x^*(z_j - y_{n_j})| + |x^*(y_{n_j}) - 1| \le \|x^*\|\,\underbrace{\|z_j - y_{n_j}\|}_{\text{this tends to 0}} + \underbrace{|x^*(y_{n_j}) - 1|}_{\text{this tends to 0}}.$$

Without loss of generality we can assume that $|x^*(z_j)| > 1/2$ for all $j$. Given $(a_j)_{j=1}^\infty \in c_{00}$, by letting $\epsilon_j = \operatorname{sgn} a_j x^*(z_j)$ we have

$$\sum_{j=1}^\infty |a_j||x^*(z_j)| = \Big|\sum_{j=1}^\infty \epsilon_j a_j x^*(z_j)\Big| = \Big|x^*\Big(\sum_{j=1}^\infty \epsilon_j a_j z_j\Big)\Big| \le \|x^*\|\,\mathsf{K_u}\Big\|\sum_{j=1}^\infty a_j z_j\Big\|.$$

Hence

$$\Big\|\sum_{j=1}^\infty a_j z_j\Big\| \ge \frac{1}{2\mathsf{K_u}\,\|x^*\|}\sum_{j=1}^\infty |a_j|.$$

On the other hand, we obtain an upper $\ell_1$-estimate for $\|\sum_{j=1}^\infty a_j z_j\|$ using the boundedness of the sequence $(z_j)_{j=1}^\infty$ and the triangle inequality. We conclude that $(z_j)_{j=1}^\infty$ is equivalent to the standard $\ell_1$-basis. This is a contradiction because $(z_j)_{j=1}^\infty$ is weakly Cauchy, whereas the canonical basis of $\ell_1$ is not. Therefore our claim holds and this finishes the proof. □

**Proposition 3.5.4.** *(i) The James space $\mathcal{J}$ does not have property (u), and so $\mathcal{J}$ cannot be embedded in any Banach space with an unconditional basis.*

*(ii)* ***(Karlin [165])*** *The space $\mathcal{C}[0,1]$ does not have an unconditional basis, and cannot be embedded in a space with unconditional basis.*

*Proof. (i)* Assume that $\mathcal{J}$ has property (u). Since the sequence defined for each $n$ by $s_n = \sum_{k=1}^n e_k$ is weakly Cauchy in $\mathcal{J}$, there exists a WUC series in $\mathcal{J}$, $\sum_{k=1}^\infty u_k$, such that the sequence $(\sum_{k=1}^n e_k - \sum_{k=1}^n u_k)_{n=1}^\infty$ converges weakly to 0. One easily notices that the series $\sum_{k=1}^\infty u_k$ cannot be unconditionally convergent in $\mathcal{J}$, because that would force the sequence $(s_n)_{n=1}^\infty$ to converge weakly to the same limit when $(s_n)_{n=1}^\infty$ is not weakly convergent in $\mathcal{J}$ (it does converge weakly, though, to $(1,1,1,\dots,1,\dots) \in \tilde{\mathcal{J}}$). Therefore using Theorem 2.4.11, $c_0$ embeds in $\mathcal{J}$, which implies that $\ell_\infty$ embeds in $\mathcal{J}^{**}$, contradicting the separability of $\mathcal{J}^{**}$.

That $\mathcal{J}$ does not embed into any space with unconditional basis follows immediately from Proposition 3.5.2 and Proposition 3.5.3.

*(ii)* This follows from $(i)$ because $\mathcal{J}$ embeds isometrically into $\mathcal{C}[0,1]$ by the Banach–Mazur theorem (Theorem 1.4.4). □

Thus we have seen that having an unconditional basis is very special, and one cannot rely on the existence of such bases in most spaces. It is, however, true that most of the spaces that are useful in harmonic analysis or partial differential equations such as the spaces $L_p$ for $1 < p < \infty$ do have unconditional bases (which we will see in Chapter 6). We will see also that $L_1$ fails to have an unconditional basis. It is perhaps reasonable to argue that the reason the spaces $L_p$ for $1 < p < \infty$ seem to be more useful for applications in these areas is precisely because they admit unconditional bases!

From the point of view of abstract Banach space theory, in this context it was natural to ask the following question:

**The unconditional basic sequence problem.** *Does every Banach space contain at least an unconditional basic sequence?*

This problem was regarded as perhaps the single most important problem in the area after the solution of the approximation problem by Enflo in 1973. Eventually a counterexample was found by Gowers and Maurey in 1993 [116]. The construction is extremely involved but has led to a variety of other applications, some of which we have already met (see, e.g., [114, 117, 175]).

## Problems

**3.1.** Let $(u_n)_{n=1}^\infty$ be a $\mathsf{K}_\mathsf{u}$-unconditional basis in a Banach space $X$.

(a) Show that if $(y_n)_{n=1}^\infty$ is a block basic sequence of $(u_n)_{n=1}^\infty$, then $(y_n)_{n=1}^\infty$ is an unconditional basic sequence in $X$ with unconditional constant $\le \mathsf{K}_\mathsf{u}$.

(b) Show that the sequence of biorthogonal functionals $(u_n^*)_{n=1}^\infty$ of $(u_n)_{n=1}^\infty$ is an unconditional basic sequence in $X^*$ with unconditional constant $\le \mathsf{K}_\mathsf{u}$.

**3.2.** Let $(u_n)_{n=1}^{\infty}$ be an unconditional basis for a Banach space $X$ with suppression-unconditional constant $\mathsf{K}_{\mathsf{su}}$. Prove that for all $N$, whenever $a_1, \dots, a_N$, $b_1, \dots, b_N$ are scalars such that $|a_n| \leq |b_n|$ for all $1 \leq n \leq N$ and $a_n b_n > 0$, we have

$$\Big\| \sum_{n=1}^{N} a_n u_n \Big\| \leq \mathsf{K}_{\mathsf{su}} \Big\| \sum_{n=1}^{N} b_n u_n \Big\|.$$

That is, the suppression-unconditional constant can replace the unconditional constant in equation (3.1) when the signs of the coefficients in the linear combinations of the basis coincide.

**3.3.** Do the proof of Theorem 3.2.18.

**3.4.** Show that the sequence $(e_n)_{n=1}^{\infty}$ of standard unit vectors is a monotone basic sequence for $\mathcal{J}$ in both norms $\|\cdot\|_{\mathcal{J}}$ and $\|\cdot\|_0$ (see Proposition 3.4.2).

**3.5 (Orlicz sequence spaces).** An *Orlicz function* is a continuous convex function $F : [0, \infty) \to [0, \infty)$ with $F(0) = 0$ and $F(x) > 0$ for $x > 0$. Let us assume that for suitable $1 < q < \infty$ we have that $F(x)/x^q$ is a decreasing function (caution: this is a mild additional assumption; see [203] for the full picture). The corresponding *Orlicz sequence space* $\ell_F$ is the space of (real) sequences $(\xi(n))_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} F(|\xi(n)|) < \infty.$$

(a) Prove that $\ell_F$ is a linear space that becomes a Banach space under the norm

$$\|\xi\|_{\ell_F} = \inf\{\lambda > 0 : \sum_{n=1}^{\infty} F(\lambda^{-1}|\xi(n)|) \leq 1\}.$$

(b) Show that the canonical basis $(e_n)_{n=1}^{\infty}$ is an unconditional basis for $\ell_F$.
(c) Show that the canonical bases of $\ell_F$ and $\ell_G$ are equivalent if and only if there is a constant $C$ such that

$$F(x)/C \leq G(x) \leq CF(x), \qquad 0 \leq x \leq 1.$$

**3.6.** (Continuation of the previous problem)

(a) By considering the behavior of block basic sequences, show that $\ell_F$ contains no subspace isomorphic to $c_0$.
(b) Now assume additionally that there exists $1 < p < \infty$ such that $F(x)/x^p$ is an increasing function. Show that $\ell_F$ is reflexive.

**3.7.** Let $X$ be a subspace of a space with unconditional basis. Show that if $X$ contains no copy of $c_0$ or $\ell_1$ then $X$ is reflexive.

**3.8.** Let $X$ be a Banach space with property (u) and separable dual. Suppose $Y$ is a Banach space containing no copy of $c_0$. Show that every bounded operator $T : X \to Y$ is weakly compact.

**3.9.** Let $X$ be a Banach space.

(a) Show that if $X$ contains a non boundedly complete basic sequence, then $X$ contains a basic sequence $(x_n)_{n=1}^{\infty}$ with $\inf_n \|x_n\| > 0$ and $\sup_n \|\sum_{i=1}^n x_i\| < \infty$.
(b) (Continuation of (a)) Show that $y_n = \sum_{i=1}^n x_i$ is also a basic sequence.
(c) Show that if $X$ contains a nonshrinking basic sequence, then $X$ contains a basic sequence $(x_n)_{n=1}^{\infty}$ such that $\sup_n \|x_n\| < \infty$ but for some $x^* \in X^*$ we have $x^*(x_n) = 1$ for all $n$.
(d) (Continuation of (c)) Show that if $y_1 = x_1$ and $y_n = x_n - x_{n-1}$ for $n \geq 2$, then $(y_n)_{n=1}^{\infty}$ is also a basic sequence. [We remind the reader of Problem 1.3.]

**3.10.** Let $X$ be a Banach space. Show that the following conditions are equivalent:

(i) Every basic sequence in $X$ is shrinking.
(ii) Every basic sequence in $X$ is boundedly complete.
(iii) $X$ is reflexive.

This result is due to Singer [283]; later Zippin [309] improved the result to replace *basic sequence* by *basis* when $X$ is known to have a basis (see Problem 9.7).

**3.11.** Let $(e_n)_{n=1}^{\infty}$ be the canonical basis of the James space $\mathcal{J}$. Show that the sequence defined by $f_n = e_1 + \cdots + e_n$ is a boundedly complete basis and that the regular norm on $\mathcal{J}$ is equivalent to the norm given by

$$\Big|\Big|\Big| \sum_{j=1}^{\infty} a_j f_j \Big|\Big|\Big| = \sup \left\{ \Big( \sum_{j=1}^{n} \Big( \sum_{i=p_{j-1}+1}^{p_j} a_i \Big)^2 \Big)^{1/2} \right\},$$

where the supremum is taken over all $n$ and all integers $(p_j)_{j=0}^{n}$ with $0 = p_0 < p_1 < \cdots < p_n$.

# Chapter 4
# Banach Spaces of Continuous Functions

We are now going to shift our attention from sequence spaces to spaces of functions, and we start in this chapter by considering spaces of type $\mathcal{C}(K)$.

If $K$ is a compact Hausdorff space, $\mathcal{C}(K)$ will denote the space of all real-valued continuous functions on $K$.

Then $\mathcal{C}(K)$ is a Banach space with the norm

$$\|f\|_{\mathcal{C}(K)} = \|f\|_\infty = \max_{s\in K} |f(s)|.$$

It can be argued that the space $\mathcal{C}[0,1]$ was the first Banach space studied in Fredholm's 1903 paper [96]. Indeed, prior to the development of Lebesgue measure, the spaces of continuous functions were the only readily available Banach spaces!

We will begin by establishing some well known classical facts. We include an optional section on characterization of real $\mathcal{C}(K)$-spaces. Then we turn to the classification of isometrically injective spaces. Continuing in the spirit of considering the isomorphic theory of Banach spaces, we will also be interested in classifying $\mathcal{C}(K)$-spaces at least for $K$ metrizable. This will give us the opportunity to use some of the techniques we have already developed in Chapters 2 and 3.

The highlight of the chapter is a celebrated result of Miljutin from 1966 that states that if $K$ and $L$ are uncountable compact metric spaces, then $\mathcal{C}(K)$ and $\mathcal{C}(L)$ are isomorphic as Banach spaces. This is a very elegant application of some of the ideas developed in the previous chapters. However, we will not use this result later, so the impatient reader can safely skip it.



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## 4.1 Basic Facts About the Spaces $\mathcal{C}(K)$

Most of the material in this section is classical. For convenience we will always consider spaces of real-valued functions, although the extension of the main results to complex-valued functions is not difficult.

Let us start by recalling some of the basic facts about spaces of continuous functions. The first is the classical Riesz representation theorem.

**Theorem 4.1.1 (Riesz Representation Theorem).** *If $K$ is a compact Hausdorff space, then $\mathcal{C}(K)^*$ is isometrically isomorphic to the space $\mathcal{M}(K)$ of all finite regular signed Borel measures on $K$ with the norm $\|\mu\| = |\mu|(K)$. The duality is given by*

$$\langle f, \mu \rangle = \int_K f \, d\mu.$$

*If, in addition, $K$ is metrizable, then every Borel measure is regular, and so $\mathcal{M}(K)$ coincides with the space of all finite Borel measures.*

**Theorem 4.1.2 (The Stone–Weierstrass Theorem).** *Suppose that $K$ is a compact Hausdorff space.*

*(a) (Real case) Let $\mathcal{A}$ be a subalgebra of $\mathcal{C}(K)$ (i.e., $\mathcal{A}$ is a linear subspace of $\mathcal{C}(K)$ and sums, products, and scalar multiples of functions from $\mathcal{A}$ are in $\mathcal{A}$) containing constants. If $\mathcal{A}$ separates points of $K$ (i.e., for every $s_1, s_2 \in K$ with $s_1 \neq s_2$ there is some $f \in \mathcal{A}$ such that $f(s_1) \neq f(s_2)$), then $\overline{\mathcal{A}} = \mathcal{C}(K)$.*

*(b) (Complex case) Let $\mathcal{A}$ be a subalgebra of $\mathcal{C}_{\mathbb{C}}(K)$ containing constants. If $\mathcal{A}$ is self-adjoint (i.e., $f \in \mathcal{A}$ implies $\overline{f} \in \mathcal{A}$) and separates points of $K$, then $\overline{\mathcal{A}} = \mathcal{C}_{\mathbb{C}}(K)$.*

**Theorem 4.1.3.** *If $K$ is compact Hausdorff, then the space $\mathcal{C}(K)$ is separable if and only if $K$ is metrizable.*

*Proof.* There is a natural embedding $s \mapsto \delta_s$ (the point mass at $s$) of $K$ into $\mathcal{M}(K)$. This is a homeomorphism for the weak* topology of $\mathcal{M}(K)$. By Lemma 1.4.1 (*i*), this shows that $K$ is metrizable if $\mathcal{C}(K)$ is separable. For the converse, let us begin by observing that if $K$ is a metrizable compact Hausdorff space, then in particular, it is separable. Let $d$ be a metric inducing the topology and let $(s_n)_{n=1}^{\infty}$ be a dense countable subset of $K$. For $n = 1, 2, \dots$, let $d_n : K \to \mathbb{R}$ be the (continuous) function defined for each $s \in K$ by $d_n(s) = d(s, s_n)$. The algebra $A$ generated in $\mathcal{C}(K)$ by the countable set $D = \{1, d_1, d_2, \dots\}$ (here 1 denotes the constantly one function) is dense in $\mathcal{C}(K)$ by the Stone–Weierstrass theorem. The set of all polynomials of several variables in the functions from $D$ with rational coefficients is a countable dense set in $A$; hence it is dense in $\mathcal{C}(K)$, so $\mathcal{C}(K)$ is separable. □

Let us recall that a *separation* of a topological space $X$ is a pair $U$, $V$ of disjoint open subsets of $X$ whose union is $X$. Then, the space $X$ is said to be *connected* if there does not exist a separation of $X$, i.e., if the only subsets of $X$ that are both

open and closed in $X$ (or *clopen*) are the empty set and $X$ itself. On the other hand, a space is *totally disconnected* if its only connected subsets are one-point sets. This is equivalent to saying that each point in $X$ has a base of neighborhoods consisting of sets that are both open and closed in $X$. The Cantor set $\Delta = \{0,1\}^{\mathbb{N}}$ is an example of a totally disconnected compact metric space. We will need the following elementary fact:

**Proposition 4.1.4.** *If $K$ is a totally disconnected compact Hausdorff space, then the collection of simple continuous functions (i.e., functions $f$ of the form $f = \sum_{j=1}^{n} a_j \chi_{U_j}$, where $U_1, \dots, U_n$ are disjoint clopen sets) is dense in $\mathcal{C}(K)$.*

*Proof.* This is an easy deduction from the Stone–Weierstrass theorem since the simple functions form a subalgebra of $\mathcal{C}(K)$. □

We conclude the section with another basic theorem from the classical theory, the Banach–Stone theorem, whose proof is proposed as an exercise (see Problem 4.2).

**Theorem 4.1.5 (Banach–Stone).** *Suppose $K$ and $L$ are two compact Hausdorff spaces such that $\mathcal{C}(K)$ and $\mathcal{C}(L)$ are isometrically isomorphic Banach spaces. Then $K$ and $L$ are homeomorphic.*

The Banach–Stone theorem appears for $K, L$ metrizable in Banach's 1932 book [18]. In full generality it was proved by M.H. Stone in 1937. In fact, general topology was in its infancy in that period, and Banach was constrained by the imperfect state of development of nonmetrizable topology; thus, for example, Alaoglu's theorem on the weak* compactness of the dual unit ball was not obtained till 1941, because it required Tychonoff's theorem.

## 4.2 An Intrinsic Characterization of Real $\mathcal{C}(K)$-Spaces

In this section we give purely algebraic characterizations of real Banach algebras that are isometrically isomorphic to (real) spaces $\mathcal{C}(K)$ of continuous functions on compact Hausdorff spaces $K$.

One needs to know that certain spaces such as $\ell_\infty$ and $L_\infty[0,1]$ are $\mathcal{C}(K)$-spaces in disguise. The standard derivation of such facts requires considering the complex versions of these spaces as commutative $C^*$-algebras (or $B^*$-algebras) and invoking the standard representation of such algebras as $\mathcal{C}(K)$-spaces via the Gelfand transform [52, pp. 242ff]. Readers familiar with this approach can skip the section, which is presented to remain within the category of real spaces. Our approach will allow us to avoid complex analysis and the general methods of Banach algebra that depend heavily on the use of complex scalars.

In $\mathcal{C}(K)$ the two algebraic operations, addition and multiplication, are related by the formula

$$2|f(s)g(s)| \le f^2(s) + g^2(s), \quad \forall\, s \in K,$$

so that by taking the supremum on $s \in K$ on both sides, we obtain

$$\|2fg\|_{\mathcal{C}(K)} \leq \|f^2 + g^2\|_{\mathcal{C}(K)}, \quad \forall f, g \in \mathcal{C}(K).$$

As it turns out, this inequality characterizes the commutative real Banach algebras with unit that are $\mathcal{C}(K)$-spaces [5, 7], and our goal will be to prove the following:

**Theorem 4.2.1.** *Let $\mathcal{A}$ be a commutative real Banach algebra with an identity $e$ such that $\|e\| = 1$. Then $\mathcal{A}$ is isometrically isomorphic to the algebra $\mathcal{C}(K)$ for some compact Hausdorff space $K$ if and only if*

$$\|2ab\| \leq \|a^2 + b^2\|, \qquad \forall\, a, b \in \mathcal{A}. \tag{4.1}$$

In 1947, Arens [14] gave the first-known criterion for checking whether a commutative real Banach algebra with unit *is* a $\mathcal{C}(K)$-space:

**Theorem 4.2.2 (Arens).** *Let $\mathcal{A}$ be a commutative real Banach algebra with an identity $e$ such that $\|e\| = 1$. Then $\mathcal{A}$ is isometrically isomorphic to the algebra $\mathcal{C}(K)$ for some compact Hausdorff space $K$ if and only if*

$$\|a\|^2 \leq \|a^2 + b^2\| \qquad \forall\, a, b \in \mathcal{A}. \tag{4.2}$$

Unlike the proof of Theorem 4.2.2, which uses the complexification technique, the proof of Theorem 4.2.1 will rely exclusively on *intrinsic* methods, i.e., it will not require any complex function theory, and instead, the proof remains inside the structure of real Banach algebras.

Of course, conditions (4.2) and (4.1) must be with hindsight equivalent, since they characterize the same space. Let us show this directly with the aid of a third additional condition.

**Lemma 4.2.3.** *Let $\mathcal{A}$ be a commutative real Banach algebra with an identity $e$ of norm $\|e\| = 1$. The following conditions are equivalent:*

*(i)* $\|2ab\| \leq \|a^2 + b^2\| \quad \forall\, a, b \in \mathcal{A}$.
*(ii)* $\|a\|^2 \leq \|a^2 + b^2\| \quad \forall\, a, b \in \mathcal{A}$.
*(iii)* $\|a^2 - b^2\| \leq \|a^2 + b^2\| \quad \forall\, a, b \in \mathcal{A}$.

*Proof.* $(i) \Rightarrow (ii)$ If we put $y = e$ in $(i)$, we get

$$2\|x\| \leq \|x^2 + e\| \leq \|x^2\| + 1.$$

Thus for all $x$ in $\mathcal{A}$ of norm $\|x\| = 1$, we have $2 \leq \|x^2\| + 1$, i.e., $\|x^2\| \geq 1$. Therefore,

$$1 \leq \left\| \frac{x^2}{\|x\|^2} \right\|, \quad \forall\, x \in \mathcal{A}, x \neq 0,$$

which yields

$$\|x\|^2 \leq \|x^2\|, \quad \forall\, x \in \mathcal{A}.$$

Hence, given any $a, b \in \mathcal{A}$,

$$\begin{aligned}\|a\|^2 = \|a^2\| &= \frac{1}{2}\|(a^2 - b^2) + (b^2 + a^2)\| \\ &\leq \frac{1}{2}(\|a^2 - b^2\| + \|a^2 + b^2\|) \\ &\leq \|a^2 + b^2\|.\end{aligned}$$

$(ii) \Rightarrow (iii)$ If we assume $(ii)$ and if $a$ and $b$ belong to $\mathcal{A}$, then

$$\|a^2 - b^2\|^2 \leq \|(a^2 - b^2)^2 + 4a^2b^2\| = \|(a^2 + b^2)^2\| \leq \|a^2 + b^2\|^2.$$

$(iii) \Rightarrow (i)$ Given $x$ and $y$ in $\mathcal{A}$, we pick $a, b$ in $\mathcal{A}$ such that $x = a - b$ and $y = a + b$ (i.e., $a = \frac{x+y}{2}, b = \frac{y-x}{2}$). Then,

$$\begin{aligned}\|2xy\| &= 2\|a^2 - b^2\| \\ &\leq 2\|a^2 + b^2\| \\ &= 2\Big\|\Big(\frac{x+y}{2}\Big)^2 + \Big(\frac{y-x}{2}\Big)^2\Big\| \\ &= \|x^2 + y^2\|.\end{aligned}$$

□

Let us next introduce the ingredients we will need in the proof of Theorem 4.2.1.

**Definition 4.2.4.** Suppose $\mathcal{A}$ is a commutative real Banach algebra with identity $e$ such that $\|e\| = 1$. The *state space* of $\mathcal{A}$ is the set

$$\mathcal{S} = \{\varphi \in \mathcal{A}^* : \|\varphi\| = \varphi(e) = 1\},$$

where $\mathcal{A}^*$ denotes the dual space of $\mathcal{A}$. An element of $\mathcal{S}$ is called a *state*.

The set of states $\mathcal{S}$ of a commutative real Banach algebra $\mathcal{A}$ with identity is nonempty by the Hahn–Banach theorem.

We recall that the weak* topology on $\mathcal{A}^*$ is the topology of pointwise convergence. Then $\mathcal{S}$ is compact in the weak* topology. This follows easily, since by Alaoglu's theorem, the closed unit ball $B_{\mathcal{A}^*}$ of $\mathcal{A}^*$ is weak* compact, and the relations in the definition of $\mathcal{S}$ define a weak* closed subset of $B_{\mathcal{A}^*}$. Now we just take into account the fact that the weak* topology is Hausdorff, so a weak* closed subset of a weak* compact set is also weak* compact.

We use $\mathcal{A}_+$ to denote the norm-closure of the set of squares in $\mathcal{A}$, that is,

$$\mathcal{A}_+ = \overline{\{a^2 : a \in \mathcal{A}\}}.$$

The two properties of $\mathcal{A}_+$ stated in the next lemma are trivially verified.

**Lemma 4.2.5.** *The following statements are true for a commutative real Banach algebra $\mathcal{A}$:*

*(i) If $x, y \in \mathcal{A}_+$, then $xy \in \mathcal{A}_+$.*
*(ii) If $x \in \mathcal{A}_+$ and $\lambda \geq 0$, then $\lambda x \in \mathcal{A}_+$.*

Part (i) of the next proposition is the well known *square root lemma* from Banach algebra theory (see, for example, [236, Theorem 3.4.5, p. 361]).

**Proposition 4.2.6.** *A commutative real Banach algebra $\mathcal{A}$ with an identity $e$ of norm 1 has the following properties:*

*(i) If $x \in \mathcal{A}$ is such that $\|x\| \leq 1$, then $e + x \in \mathcal{A}_+$.*
*(ii) $\mathcal{A} = \mathcal{A}_+ - \mathcal{A}_+$.*

*Proof.* Let $x$ in $\mathcal{A}$ have $\|x\| < 1$. By writing $(1+t)^{1/2}$ in its binomial series, valid for scalars $t$ with $|t| < 1$, we see that the series $\sum_{n=0}^{\infty} \binom{1/2}{n} x^n$ is absolutely convergent, therefore convergent to some $y$ in $\mathcal{A}$. By expanding $(1 + t)^{1/2}(1 + t)^{1/2}$ for a real variable $t$ when $|t| < 1$, it becomes clear that

$$\sum_{m+n=k} \binom{1/2}{m} \binom{1/2}{n} = \begin{cases} 1 & \text{if } k = 0 \text{ or } 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

We deduce that $y^2 = e + x$. Since $\mathcal{A}_+$ is closed, we see that $e + x \in \mathcal{A}_+$ if $\|x\| \leq 1$. This establishes $(i)$.

Part $(ii)$ follows immediately (with the use of Lemma 4.2.5), since when $\|x\| \leq 1$, we can write

$$x = \tfrac{1}{2}(e + x) - \tfrac{1}{2}(e - x).$$

□

We now turn to the proof of Theorem 4.2.1. Let us first note two simple deductions from the hypothesis. The equivalence between (4.1) and condition $(iii)$ in Lemma 4.2.3 gives

$$\|x - y\| \leq \|x + y\|, \quad x, y \in \mathcal{A}_+. \tag{4.3}$$

Since $\|x\| \leq \frac{1}{2}(\|x - y\| + \|x + y\|)$, equation (4.3) yields

$$\|x\| \leq \|x + y\|, \quad x, y \in \mathcal{A}_+. \tag{4.4}$$

Before completing the proof, we prove two preparatory lemmas.

**Lemma 4.2.7.** *Suppose that $\mathcal{A}$ satisfies (4.4). Then $\varphi(x) \geq 0$ whenever $\varphi \in \mathcal{S}$ and $x \in \mathcal{A}_+$.*

*Proof.* Take $x \in \mathcal{A}_+$ with $\|x\| = 1$. By Proposition 4.2.6, $e - x \in \mathcal{A}_+$, and by (4.4),

$$\|e - x\| \le \|(e - x) + x\| = 1.$$

Hence for $\varphi \in \mathcal{S}$ we have

$$1 = \|\varphi\| \ge \varphi(e - x) = 1 - \varphi(x),$$

whence $\varphi(x) \ge 0$. □

We recall that a point $x$ in a nonempty convex subset $S$ of a vector space is an *extreme point of* $S$ if whenever $x = \lambda x_1 + (1 - \lambda)x_2$ with $0 < \lambda < 1$, then $x = x_1 = x_2$. We use $\partial_e S$ to denote the set of extreme points of $S$. We will use the Krein–Milman theorem, which in our context states that if $S$ is a weak* compact convex subset of the dual of a Banach space, then $\partial_e S$ is nonempty and $S$ is the weak* closure of the convex hull of $\partial_e S$ (see Appendix F).

**Lemma 4.2.8.** *Suppose that $\mathcal{A}$ satisfies* (4.4). *Let $K$ be the set of all multiplicative states of $\mathcal{A}$, i.e.,*

$$K = \{\varphi \in \mathcal{S} : \varphi(xy) = \varphi(x)\varphi(y) \ \textit{for all} \ \ x, y \in \mathcal{A}\}.$$

*Then $K$ is a compact Hausdorff space in the weak* topology of $\mathcal{A}^*$ that contains the set $\partial_e \mathcal{S}$ of extreme points of $\mathcal{S}$ (and in particular, $K$ is nonempty).*

*Proof.* It is trivial to show that $K$ is a closed subset of the closed unit ball of $\mathcal{A}^*$. This ensures that $K$ is compact for the weak* topology.

Since $\mathcal{S}$ is convex and compact in the weak* topology of $\mathcal{A}^*$, the Krein–Milman theorem guarantees that $\partial_e \mathcal{S}$ is nonempty. Suppose that $\varphi$ lies in $\partial_e \mathcal{S}$. We claim that $\varphi$ is in $K$. Since $\mathcal{A} = \mathcal{A}_+ - \mathcal{A}_+$, it suffices to show that $\varphi(xy) = \varphi(x)\varphi(y)$ whenever $x \in \mathcal{A}_+$ and $y \in \mathcal{A}$.

Consider $x$ in $\mathcal{A}_+$ with $\|x\| \le 1$ and $y$ in $\mathcal{A}$ with $\|y\| \le 1$. By Proposition 4.2.6, $e \pm y \in \mathcal{A}_+$. Therefore, by Lemma 4.2.7,

$$\varphi(x(e \pm y)) \ge 0,$$

which implies that

$$|\varphi(xy)| \le \varphi(x). \tag{4.5}$$

Similarly, $e - x \in \mathcal{A}_+$ by Proposition 4.2.6, so

$$|\varphi((e - x)y)| \le 1 - \varphi(x). \tag{4.6}$$

Notice that both equations (4.5) and (4.6) hold, in fact, for arbitrary $y$ in $\mathcal{A}$.

If $\varphi(x) = 0$, inequality (4.5) yields $\varphi(xy) = \varphi(x)\varphi(y)$. Similarly, if $\varphi(x) = 1$, then using (4.6), it is immediate that $\varphi(xy) = \varphi(x)\varphi(y)$.

If $0 < \varphi(x) < 1$, we define $\psi_1$ and $\psi_2$ on $\mathcal{A}$ by

$$\psi_1(y) = \varphi(x)^{-1}\varphi(xy)$$

and

$$\psi_2(y) = (1 - \varphi(x))^{-1}\varphi((e - x)y).$$

Using (4.5) and (4.6), we see that $\psi_1$ and $\psi_2$ are states. Now we can write

$$\varphi = \varphi(x)\psi_1 + (1 - \varphi(x))\psi_2.$$

By the fact that $\varphi$ is an extreme point of $\mathcal{S}$ we must have $\psi_1 = \varphi$, and therefore,

$$\varphi(xy) = \varphi(x)\varphi(y) \quad \forall\ x \in \mathcal{A}_+ \text{ and } y \in \mathcal{A}.$$

□

*Completion of the Proof of Theorem 4.2.1.* Suppose $\mathcal{A}$ satisfies the condition (4.1). Let $J : \mathcal{A} \to \mathcal{C}(K)$ be the natural map given by

$$Jx(\varphi) = \varphi(x), \quad x \in \mathcal{A}, \varphi \in K,$$

where $K$ is the set of all multiplicative states of $\mathcal{A}$. Clearly, $J$ is an algebra homomorphism, $J(e) = 1$ and $\|J\| = 1$. In order to prove that $J$ is an isometry, we need to establish the following:

**Claim.** *Suppose $x \in \mathcal{A}$ is such that $\|Jx\|_{\mathcal{C}(K)} \leq 1$. Then for every $\epsilon > 0$ there exists $t_\varepsilon > 0$ such that*

$$\|e - t_\varepsilon(1 + \epsilon)e - t_\varepsilon x\| < 1.$$

If the claim fails to be true, there is an $x$ in $\mathcal{A}$ with $\|Jx\|_{\mathcal{C}(K)} \leq 1$ such that for some $\epsilon > 0$ we have

$$\|e - t(1 + \epsilon)e - tx\| \geq 1, \quad \forall\ t \geq 0.$$

By the Hahn–Banach theorem (invoked to separate $\{e - t(1 + \epsilon)e - tx :\ t \geq 0\}$ from the open unit ball) we can find a linear functional $\varphi$ with $\|\varphi\| = 1$ and

$$\varphi(e - t(1 + \epsilon)e - tx) \geq 1, \quad \forall\ t \geq 0.$$

In particular, $\varphi$ lies in $\mathcal{S}$ and $\varphi((1 + \epsilon)e + x) \leq 0$. Hence $|\varphi(x)| \geq 1 + \epsilon$. But now, using the Krein–Milman theorem and Lemma 4.2.8, we deduce that there exists $\psi$ in $K$ with $|\psi(x)| \geq 1 + \epsilon$. Thus $\|Jx\|_{\mathcal{C}(K)} > 1$, a contradiction.

Combining the claim with Proposition 4.2.6 (i), we see that $\|Jx\|_{\mathcal{C}(K)} \leq 1$ implies that $(1+\epsilon)e + x \in \mathcal{A}_+$ for all $\epsilon > 0$, so $e + x \in \mathcal{A}_+$. Applying the same reasoning to $-x$, we have $e - x \in \mathcal{A}_+$. Hence, by (4.3), we obtain

$$\|x\| = \tfrac{1}{2}\|(e+x) - (e-x)\| \leq \tfrac{1}{2}\|(e+x) + (e-x)\| = 1.$$

Thus $J$ is an isometry. Finally, $J$ maps $\mathcal{A}$ onto $\mathcal{C}(K)$ by the Stone–Weierstrass theorem. □

*Example 4.2.9.* If we consider $\mathcal{A} = \ell_\infty$ (with the multiplication of two sequences defined coordinatewise), Theorem 4.2.1 yields that $\mathcal{A} = \mathcal{C}(K)$ (isometrically) for some compact Hausdorff space $K$. This set $K$ is usually denoted by $\beta\mathbb{N}$. We also note that if $(\Omega, \Sigma, \mu)$ is any $\sigma$-finite measure space, then $L_\infty(\Omega, \mu)$ is again a $\mathcal{C}(K)$-space. In each case the isomorphism *preserves order* (i.e., nonnegative functions are mapped to nonnegative functions), since squares are mapped to squares.

*Remark 4.2.10.* Condition (4.1) may appear to be innocuous to the reader, but there are well known commutative real algebras with identity where (4.1) fails. We illustrate this with a few examples:

(a) Every complex Banach space $\mathcal{C}_{\mathbb{C}}(K)$ of continuous functions on a compact Hausdorff space $K$ is in particular a commutative real Banach algebra. One readily sees that condition (4.1) fails by taking, for instance, $a$ to be the constant function 1 and $b$ the constant function $i$.

(b) The real algebra $\mathcal{C}^{(1)}[0,1]$ of continuously differentiable real-valued functions on $[0,1]$ with the norm

$$\|f\| = \max_{0\leq t\leq 1} |f(t)| + \max_{0\leq t\leq 1} |f'(t)|$$

is a commutative Banach algebra with unit that seems similar to $\mathcal{C}[0,1]$ but fails to obey (4.1). Take, for instance, $a = e^x$ and $b = e^{-x}$.

(c) Let $\ell_1(\mathbb{Z}_+)$ be the space of all formal power series $\sum_{n=0}^\infty a_n t^n$ (with real coefficients) with $(a_n)_{n=0}^\infty \in \ell_1$ and with the norm

$$\Big\| \sum_{n=0}^\infty a_n t^n \Big\| = \sum_{n=0}^\infty |a_n|.$$

To see that condition (4.1) fails in $\ell_1(\mathbb{Z}_+)$ take, for instance, $a = 1 - 2t^2$ and $b = 2t + t^2$.

*Remark 4.2.11.* Observe that our proof of Theorem 4.2.1 required the full force of hypothesis (4.1) only at the very last step. Prior to that we used only the weaker hypothesis

$$\|a^2\| \leq \|a^2 + b^2\|, \quad \forall\, a, b \in \mathcal{A}. \tag{4.7}$$

Condition (4.7) implies (4.4), which was used in Lemmas 4.2.7 and 4.2.8. However, this hypothesis allows us to deduce only that $\mathcal{A}$ is 2-isomorphic to $\mathcal{C}(K)$, i.e.,

$$\frac{1}{2}\|x\| \le \|Jx\|_{\mathcal{C}(K)} \le \|x\|, \quad \forall\, x \in \mathcal{A},$$

so that $\|J\| = 1$ and $\|J^{-1}\| \le 2$. That this is best possible is clear from the norm on $\mathcal{C}(K)$ given by

$$|||f||| = \|f_+\|_{\mathcal{C}(K)} + \|f_-\|_{\mathcal{C}(K)},$$

where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Under this norm $\mathcal{C}(K)$ is a commutative real Banach algebra satisfying inequality (4.7) but not inequality (4.1).

## 4.3 Isometrically Injective Spaces

We now turn to the problem of classifying isometrically injective spaces, originally introduced in Chapter 2 (Section 2.5). There we saw that $\ell_\infty$, which we identify with $\mathcal{C}(\beta\mathbb{N})$, is isometrically injective but that $c_0$ is not an (isomorphically) injective space (although it is separably injective). Let us recall that $\beta\mathbb{N}$ is the Stone–Čech compactification of $\mathbb{N}$ endowed with the discrete topology, i.e., $\beta\mathbb{N}$ is the unique compact Hausdorff space containing $\mathbb{N}$ as a dense subspace such that every bounded continuous function on $\mathbb{N}$ extends to a continuous function on $\beta\mathbb{N}$.

The complete classification of isometrically injective spaces was achieved in the early 1950s by the combined efforts of Nachbin [225], Goodner [110], and Kelley [169]. The basic approach developed by Nachbin and Goodner was to abstract the essential ingredient of the Hahn–Banach theorem, which is the order-completeness (i.e., the least upper bound axiom) of the real numbers.

**Definition 4.3.1.** We say that the space $\mathcal{C}(K)$ is *order-complete* if whenever $A, B$ are nonempty subsets of $\mathcal{C}(K)$ with $f \le g$ for all $f \in A$ and $g \in B$, then there exists $h \in \mathcal{C}(K)$ such that $f \le h \le g$ whenever $f \in A$ and $g \in B$.

*Remark 4.3.2. (a)* If $\mathcal{C}(K)$ is order-complete, then every subset $A$ of $\mathcal{C}(K)$ that has an upper bound has also a least upper bound, which we denote by $\sup A$. Indeed, let $B$ be the set of all upper bounds of $A$ and apply the preceding definition. The (uniquely determined) function $h$ must be the least upper bound. It is important to stress that $h$ is a continuous function and may not coincide with the pointwise supremum $\tilde{h}(s) = \sup_{f\in A} f(s)$, which need not be a continuous function. Similar statements may be made about greatest lower bounds (i.e., infima).

*(b)* The previous definition can easily be extended to any space with a suitable order structure such as $\ell_\infty$ or $L_\infty$. It is clear that $\ell_\infty$ is order-complete for its natural order and therefore $\mathcal{C}(\beta\mathbb{N})$ is also order-complete. To compute the supremum of $A$ in $\ell_\infty$ one does indeed take the pointwise supremum, but the corresponding supremum in $\mathcal{C}(\beta\mathbb{N})$ is not necessarily a pointwise supremum.

We will say that a map $V : F \to \mathcal{C}(K)$, where $F$ is a linear subspace of a Banach space $X$, is *sublinear* if

(i) $V(\alpha x) = \alpha V(x)$ for all $\alpha \geq 0$ and $x \in F$, and
(ii) $V(x + y) \leq V(x) + V(y)$ for all $x, y \in F$.

A sublinear map $V : X \to \mathcal{C}(K)$ is *minimal* provided there is no sublinear map $U : X \to \mathcal{C}(K)$ such that $U(x) \leq V(x)$ for all $x \in X$ and $U \neq V$.

**Lemma 4.3.3.** *Let $X$ be a Banach space and $F$ a linear subspace of $X$. Suppose $V : X \to \mathcal{C}(K)$ and $W : F \to \mathcal{C}(K)$ are sublinear maps such that $W(y) + V(-y) \geq 0$ for all $y \in F$. If $\mathcal{C}(K)$ is order-complete, then the map $V \wedge W : X \to \mathcal{C}(K)$ given by*

$$V \wedge W(x) = \inf\{V(x - y) + W(y) : \; y \in F\}$$

*is well defined and sublinear.*

*Proof.* For each fixed $x \in X$ we have

$$V(x - y) + W(y) \geq V(-y) - V(-x) + W(y) \geq -V(-x)$$

for all $y \in F$. That is, $-V(-x)$ is a lower bound of the set $\{V(x-y)+W(y) : \; y \in F\}$. Thus, by the order-completeness of $\mathcal{C}(K)$, we can define a map $V \wedge W : F \to \mathcal{C}(K)$ by

$$V \wedge W(x) = \inf\{V(x - y) + W(y) : \; y \in F\}.$$

It is a straightforward verification to check that $V \wedge W$ is sublinear. □

**Lemma 4.3.4.** *Let $V : X \to \mathcal{C}(K)$ be a sublinear map. If $\mathcal{C}(K)$ is order-complete, then there is a minimal sublinear map $W : X \to \mathcal{C}(K)$ with $W(x) \leq V(x)$ for all $x \in X$.*

*Proof.* Put

$$\mathcal{S} = \big\{U : X \to \mathcal{C}(K) \; : \; U \text{ is sublinear and } U(x) \leq V(x) \text{ for all } x \in X\big\}.$$

The set $\mathcal{S}$ is nonempty ($V \in \mathcal{S}$) and partially ordered. Let $\Psi = (U_i)_{i\in I}$ be a chain (i.e., a totally ordered subset) in $\mathcal{S}$. Note that for each $i \in I$ we have $0 = U_i(x + (-x)) \leq U_i(x) + U_i(-x)$ for all $x \in X$; hence

$$U_i(x) \geq -U_i(-x) \geq -V(-x).$$

Thus, for each $x \in X$, the set $\{U_i(x) : i \in I\} \subset \mathcal{C}(K)$ has a lower bound. By the order-completeness of $\mathcal{C}(K)$, the map

$$U_\Psi(x) = \inf_{i\in I} U_i(x)$$

is well defined on $X$ and sublinear. To see this, since $\Psi$ is a totally ordered set, given $i \neq j \in I$, without loss of generality we can assume that $U_i \leq U_j$. Then, for all $x, y \in X$ we have

$$U_\Psi(x+y) \leq U_i(x+y) \leq U_j(x) + U_i(y);$$

therefore, $U_\Psi(x+y) - U_j(x) \leq U_\Psi(y)$, which yields $U_\Psi(x+y) - U_\Psi(y) \leq U_\Psi(x)$. Moreover, $U_\Psi(x) \leq V(x)$ for all $x \in X$. That is, $U_\Psi \in \mathcal{S}$ is a lower bound for the chain $(U_i)_{i\in I}$. Using Zorn's lemma, we deduce the existence of a minimal element $W$ in $\mathcal{S}$. □

**Lemma 4.3.5.** *Suppose that $\mathcal{C}(K)$ is order-complete and let $V : X \to \mathcal{C}(K)$ be a sublinear map. If $V$ is minimal, then $V$ is linear.*

*Proof.* Given an element $x \in X$, let us call $F$ its linear span, $F = \langle x \rangle$. Then, $W(\lambda x) = -\lambda V(-x)$ defines a linear map from $F$ to $\mathcal{C}(K)$. Clearly, $W(\lambda x) \geq -V(-\lambda x)$ for every real $\lambda$. Using Lemma 4.3.3, we can define on $X$ the sublinear map

$$V \wedge W(x) = \inf_{\lambda\in\mathbb{R}} \left\{V(x-\lambda x) + W(\lambda x)\right\}.$$

By the minimality of $V$, $V \wedge W = V$ on $X$. Therefore $V \leq W$ on $F$, which implies that $V(x) \leq -V(-x)$. On the other hand, $V(x) \geq -V(-x)$ by the sublinearity of $V$, so $V(-x) = -V(x)$. Since this holds for all $x \in X$, it is clear that $V$ is linear. □

**Theorem 4.3.6 (Goodner and Nachbin [110, 225]).** *Let $K$ be a compact Hausdorff space. Then $\mathcal{C}(K)$ is isometrically injective if and only if $\mathcal{C}(K)$ is order-complete.*

*Proof.* Assume, first, that $\mathcal{C}(K)$ is order-complete. Let $E$ be a subspace of a Banach space $X$ and let $S : E \to \mathcal{C}(K)$ be a linear operator with $\|S\| = 1$. That is, for each $x \in E$ we have

$$-\|x\| \leq (Sx)(k) \leq \|x\| \quad \text{for all } k \in K,$$

which, if we let 1 denote the constant function 1 on $K$, is equivalent to writing

$$-\|x\| \cdot 1 \leq S(x) \leq \|x\| \cdot 1. \tag{4.8}$$

Thus, if we consider the sublinear map from $X$ to $\mathcal{C}(K)$ given by $V_0(x) = \|x\| \cdot 1$, equation (4.8) tells us that $S(x) \geq -V_0(-x)$ for all $x \in E$, and so we can define on $X$ the sublinear map $V = V_0 \wedge S$ as in Lemma 4.3.3:

$$V(x) = \inf\left\{V_0(x-y) + S(y) \ : y \in E\right\}.$$

By Lemma 4.3.4 there exists $T : X \to \mathcal{C}(K)$, a minimal sublinear map satisfying $T \leq V$. Lemma 4.3.5 yields that $T$ is linear.

On $E$, we have $T(x) \leq S(x)$ and $T(-x) \leq S(-x)$. Therefore, $T|_E = S$. Finally, $T(x) \leq \|x\| \cdot 1$ and $T(-x) \leq \|x\| \cdot 1$ for all $x \in X$, which implies that $\|T\| \leq 1$. Thus, we have successfully extended $S$ from $E$ to $X$.

Suppose, conversely, that $\mathcal{C}(K)$ is isometrically injective. Then there is a norm-one projection $P$ from $\ell_\infty(K)$ onto $\mathcal{C}(K)$, where $\ell_\infty(K)$ denotes the space of all bounded functions on $K$. Suppose that $A, B$ are two nonempty subsets of $\mathcal{C}(K)$ such that $f \in A$ and $g \in B$ implies $f \leq g$. For each $s \in K$, put $a(s) = \sup_{f \in A} f(s)$. Obviously, $a \in \ell_\infty(K)$. Let $h = P(a)$. We will prove that $f \leq h \leq g$ for all $f \in A$ and all $g \in B$.

Since $P(1) = 1$ and $P$ has norm one, it follows that for each $b \in \ell_\infty(K)$ with $b > 0$ we have

$$\|P(1 - \lambda b)\| \leq 1 \text{ for } \quad 0 \leq \lambda \leq 2/\|b\|.$$

We deduce that $P$ is a positive map, that is, $Pb \geq 0$ whenever $b \in \ell_\infty(K)$ and $b \geq 0$. Thus, if $f \in A$, then $f \leq a$, and therefore, $f \leq h$. Analogously, if $g \in B$ we have $g \geq a$, and so $g \geq h$. Hence, $\mathcal{C}(K)$ is order-complete. □

The spaces $K$ such that $\mathcal{C}(K)$ is order-complete are characterized by the property that the closure of every open set remains open; such spaces are called *extremally disconnected*. We refer the reader to the problems for more information.

The natural question arises whether only $\mathcal{C}(K)$-spaces can be isometrically injective. Both Nachbin and Goodner showed that an isometrically injective Banach space $X$ is (isometrically isomorphic to) a $\mathcal{C}(K)$-space, provided the unit ball of $X$ has at least one extreme point. The key here is that the constant function 1 is always an extreme point on the unit ball in $\mathcal{C}(K)$, and they needed to find an element in the space $X$ to play this role. However, two years later, in 1952, Kelley completed the argument and proved the following definitive result:

**Theorem 4.3.7 (Kelley [169]).** *A Banach space $X$ is isometrically injective if and only if it is isometrically isomorphic to an order-complete $\mathcal{C}(K)$-space.*

*Proof.* We need only show the forward implication. For that, we are going to identify $X$ (via an isometric isomorphism) with a suitable $\mathcal{C}(K)$-space which, by the isometric injectivity of $X$, will, by an appeal to Theorem 4.3.6, be order-complete.

The trick is to "find" $K$ as a subset of the dual unit ball $B_{X^*}$. Consider the set $\partial_e B_{X^*}$ of extreme points of $B_{X^*}$ with the weak* topology. There is a maximal open subset, $U$, of $\partial_e B_{X^*}$ subject to the property that $U \cap (-U) = \emptyset$. This is an easy consequence of Zorn's lemma again, since every chain of such open sets has an upper bound, namely, their union. Let $K$ be the weak* closure of $U$ in $B_{X^*}$. Then $K$ is, of course, compact and Hausdorff for the weak* topology.

Let us observe that $K \cap \partial_e B_{X^*}$ cannot meet $-U$, since $\partial_e B_{X^*} \setminus (-U)$ is relatively weak* closed in $\partial_e B_{X^*}$. Then, $K \cap (-U) = \emptyset$.

We claim that $\partial_e B_{X^*} \subset (K \cup (-K))$. Indeed, suppose that there exists $x^* \in \partial_e B_{X^*} \setminus (K \cup (-K))$. Then there is an absolutely convex weak* open neighborhood,

$V$, of 0 such that $x^* \notin V$ and $(x^* + V) \cap (K \cup (-K)) = \emptyset$. Let $U_1 = U \cup \big((x^* + V) \cap \partial_e B_{X^*}\big)$. Then $U_1$ strictly contains $U$, since $x^* \in U_1$. Suppose $y^* \in U_1 \cap (-U_1)$. Then either $y^* \notin U$ or $-y^* \notin U$; thus replacing $y^*$ by $-y^*$ if necessary, we can assume $y^* \notin U$. Then $y^* \in x^* + V$; this implies that $y^* \notin K \cup (-K)$, and so $y^* \notin -U$. Hence $y^* \in -x^* - V$, and so $0 \in 2x^* + 2V$ or $x^* \in V$, yielding a contradiction. Thus $U_1 \cap (-U_1) = \emptyset$, which contradicts the maximality of $U$.

By the Krein–Milman theorem, $B_{X^*}$ must be the weak* closed convex hull of $K \cup (-K)$, and in particular, if $x \in X$, we have

$$\|x\| = \sup_{x^* \in B_{X^*}} |x^*(x)| = \max_{x^* \in K} |x^*(x)|.$$

Thus, the map $J$ that assigns to each $x \in X$ the function $\hat{x} \in \mathcal{C}(K)$ given by $\hat{x}(x^*) = x^*(x)$, $x^* \in K$, is an isometry. We can therefore use the isometric injectivity of $X$ (extending the map $J^{-1} : J(X) \to X$) to define an operator $T : \mathcal{C}(K) \to X$ such that $T(\hat{x}) = x$ for all $x \in X$ with $\|T\| = 1$.

Let us consider the adjoint map $T^* : X^* \to \mathcal{M}(K)$. If $u^* \in U$, then $T^* u^* = \mu \in \mathcal{M}(K)$ with $\|\mu\| \le 1$. Let $V$ be any weak* open neighborhood of $u^*$ relative to $K$ and put $K_0 = K \setminus V$. We can define $v^* \in X^*$ by

$$v^*(x) = \int_V x^*(x)\, d\mu(x^*), \qquad x \in X,$$

and $w^* \in X^*$ by

$$w^*(x) = \int_{K_0} x^*(x)\, d\mu(x^*), \qquad x \in X.$$

Then $\|v^*\| \le |\mu|(V)$ and $\|w^*\| \le |\mu|(K_0)$. But

$$\int_K x^*(x)\, d\mu = \langle \hat{x}, T^*(u^*) \rangle = \langle x, u^* \rangle;$$

hence $v^* + w^* = u^*$. Since $\|u^*\| = 1 \ge \|\mu\|$, we must have $|\mu|(V) + |\mu|(K_0) = 1$. Thus, $\|v^*\| + \|w^*\| = 1$, and so the fact that $u^*$ is an extreme point implies that $v^* = \|v^*\| u^*$ and $w^* = \|w^*\| u^*$.

Suppose $|\mu|(K_0) = \|w^*\| = \alpha > 0$. Then,

$$u^*(x) = \alpha^{-1} \int_{K_0} x^*(x)\, d\mu(x^*), \qquad x \in X,$$

and in particular,

$$|u^*(x)| \le \max_{x^* \in K_0} |x^*(x)|, \qquad x \in X.$$

This implies that $u^*$ is in the weak* closed convex hull, $C$, of $K_0 \cup (-K_0)$. But $u^*$ must be an extreme point in $C$ also, so by Milman's theorem it must belong to the weak* closed set $K_0 \cup (-K_0)$. Since $u^* \notin K_0$, we have that $u^* \in (-K_0)$, i.e., $-u^* \in K_0$. Thus, $K_0$ meets $-U$, so $K$ meets $-U$, which is a contradiction to our previous remarks.

Hence $|\mu|(K_0) = \|w^*\| = 0$, and so $|\mu(V)| = 1$ for every weak* open neighborhood $V$ of $u^*$. By the regularity of $\mu$ we must have that $\mu = \pm\delta_{u^*}$ ($\delta_{u^*}$ is the point mass at $u^*$). Thus $\mu = \delta_{u^*}$ for $u^* \in U$. Since $T^*$ is weak* continuous, we infer that $T^*(x^*) = \delta_{x^*}$ for all $x^* \in K$. We are done, because if $f \in \mathcal{C}(K)$, then

$$\langle Tf, x^* \rangle = f(x^*),$$

so $J$ is onto $\mathcal{C}(K)$. This shows that $X$ is a $\mathcal{C}(K)$-space. □

At this point we have only one example in which $\mathcal{C}(K)$ is order-complete, namely, $\ell_\infty$ (although, of course, $\ell_\infty(\mathcal{I})$ for any index set $\mathcal{I}$ will also work). There are, however, less trivial examples, as the next proposition shows.

**Proposition 4.3.8.**

*(i) If $\mathcal{C}(K)$ is (isometrically isomorphic to) a dual space, then $\mathcal{C}(K)$ is isometrically injective.*

*(ii) If $(\Omega, \Sigma, \mu)$ is any $\sigma$-finite measure space, then $L_\infty(\Omega, \Sigma, \mu)$ is isometrically injective.*

*(iii) For every compact Hausdorff space $K$, the space $\mathcal{C}(K)^{**}$ is isometrically injective.*

*Proof.* For $(i)$ we will first show that $P = \{f \in \mathcal{C}(K) : f \geq 0\}$, the positive cone of $\mathcal{C}(K)$, is closed for the weak* topology of $\mathcal{C}(K)$ (regarded now as a dual Banach space by hypothesis). By the Banach–Dieudonné theorem it suffices to show that $P \cap \lambda B_{\mathcal{C}(K)}$ is weak* closed for each $\lambda > 0$. But $P \cap \lambda B_{\mathcal{C}(K)} = \{f : \|f - \frac{1}{2}\lambda \cdot 1\| \leq \frac{1}{2}\lambda\}$ is simply a closed ball, which must be weak* closed.

Let us see that $\mathcal{C}(K)$ is order-complete, and then we will invoke Theorem 4.3.6 to deduce that $\mathcal{C}(K)$ is isometrically injective. Suppose $A, B$ are nonempty subsets of $\mathcal{C}(K)$ such that $f \in A$, $g \in B$ implies $f \leq g$. For each $f \in A$ and $g \in B$, put

$$C_{f,g} = \{h \in \mathcal{C}(K) : f \leq h \leq g\}.$$

Every $C_{f,g}$ is a (nonempty) bounded and weak* closed set. If $f_1, \ldots, f_n \in A$ and $g_1, \ldots, g_n \in B$, then $\cap_{k=1}^n C_{f_k, g_k}$ is nonempty, because it contains, for example, $\max(f_1, \ldots, f_n)$. Hence, by weak* compactness, the intersection $\cap_{\{f \in A, g \in B\}} C_{f,g}$ is nonempty. If we pick $h$ in the intersection, we are done.

$(ii)$ follows directly from $(i)$, since $L_\infty(\mu) = L_1(\mu)^*$.

$(iii)$ Here we observe that $\mathcal{M}(K)$ is actually a vast $\ell_1$-sum of $L_1(\mu)$-spaces. Precisely, using Zorn's lemma, one can produce a maximal collection $(\mu_i)_{i \in \mathcal{I}}$ of probability measures on $K$ with the property that every two members of the collection are mutually singular.

If $\nu \in \mathcal{M}(K)$, then for each $i \in \mathcal{I}$ we define $f_i \in L_1(K, \mu_i)$ to be the Radon–Nikodym derivative $d\nu/d\mu_i$. Thus, $d\nu = f_i d\mu_i + d\gamma$, where $\gamma$ is singular with respect to $\mu_i$. Then it is easy to show (we leave the details to the reader) that for every finite set $\mathbb{A} \subset \mathcal{I}$ we have

$$\sum_{i \in \mathbb{A}} \|f_i\|_{L_1(\mu_i)} \leq \|\nu\|.$$

Hence,

$$\sum_{i \in \mathcal{I}} \|f_i\|_{L_1(\mu_i)} \leq \|\nu\|.$$

Notice that the last statement implies that only countably many terms in the sum are nonzero. Put

$$\nu_0 = \sum_{i \in \mathcal{I}} f_i d\mu_i,$$

where the series converges in $\mathcal{M}(K)$. It is clear that the measure $\nu - \nu_0$ is singular with respect to every $\mu_i$, and as a consequence, it must vanish on $K$. It follows that the map $\nu \mapsto (f_i)_{i \in \mathcal{I}}$ defines an isometric isomorphism between $\mathcal{M}(K)$ and the $\ell_1$-sum of the spaces $L_1(\mu_i)$ for $i \in \mathcal{I}$.

This yields that $\mathcal{C}(K)^{**}$ can be identified with the $\ell_\infty$-sum of the spaces $L_\infty(\mu_i)$. Using (*ii*) we deduce that $\mathcal{C}(K)^{**}$ is isometrically injective. □

*Remark 4.3.9.* We should note here that there are order-complete $\mathcal{C}(K)$-spaces that are not isometric to dual spaces. The first example was given in 1951 (in a slightly different context) by Dixmier [71], and we refer to Problem 4.8 and Problem 4.9 for details.

There is an easy but surprising application of the preceding proposition to the isomorphic theory [239]:

**Theorem 4.3.10.** *$L_\infty[0,1]$ is isomorphic to $\ell_\infty$.*

*Proof.* First, observe that $\ell_\infty$ embeds isometrically into $L_\infty[0,1]$ via the map

$$(\xi(n))_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} \xi(n)\chi_{A_n}(t),$$

where $(A_n)_{n=1}^{\infty}$ is a partition of $[0,1]$ into sets of positive measure. Since $\ell_\infty$ is an injective space, it follows that $\ell_\infty$ is complemented in $L_\infty[0,1]$.

On the other hand, $L_\infty[0,1]$ also embeds isometrically into $\ell_\infty$. To see this, pick $(\varphi_n)_{n=1}^{\infty}$, a dense sequence in the unit ball of $L_1$, and map $f \in L_\infty[0,1]$ to $(\int_0^1 \varphi_n f\,dt)_{n=1}^{\infty}$. Therefore, being an injective space, $L_\infty[0,1]$ is complemented in $\ell_\infty$.

Furthermore, $\ell_\infty \approx \ell_\infty \oplus \ell_\infty$ and

$$L_\infty[0,1] \approx L_\infty[0,1/2] \oplus L_\infty[1/2,1] \approx L_\infty[0,1] \oplus L_\infty[0,1].$$

Using Theorem 2.2.3 (a) (the Pełczyński decomposition technique), we deduce that $L_\infty[0,1]$ is isomorphic to $\ell_\infty$. □

We conclude this section by showing that a separable isometrically injective space is necessarily finite-dimensional.

**Proposition 4.3.11.** *For every infinite compact Hausdorff space $K$, $\mathcal{C}(K)$ contains a subspace isometric to $c_0$. If $K$ is metrizable, this subspace is complemented.*

*Proof.* Let $(U_n)$ be a sequence of nonempty, disjoint open subsets of $K$. Such a sequence can be found by induction: simply pick $U_1$ such that $K_1 = K \setminus \overline{U_1}$ is infinite, and then take $U_2 \subset K_1$ such that $K_2 = K_1 \setminus \overline{U_2}$ is infinite and so on. Next, pick a sequence $(\varphi_n)_{n=1}^\infty$ of continuous functions on $K$ such that $0 \le \varphi_n \le 1$, $\max_{s\in K} \varphi_n(s) = 1$, and $\{s \in K : \varphi_n(s) > 0\} \subset U_n$, for all $n \in \mathbb{N}$. Then for every $(a_n) \in c_{00}$ we have

$$\Big\| \sum_{n=1}^\infty a_n \varphi_n \Big\| = \max_n |a_n|.$$

Thus $(\varphi_n)_{n=1}^\infty$ is a basic sequence isometrically equivalent to the unit vector basis of $c_0$.

If $K$ is metrizable, Theorem 4.1.3 implies that $\mathcal{C}(K)$ is separable, and we can apply Sobczyk's theorem (Theorem 2.5.8) to deduce that the space $[\varphi_n]_{n=1}^\infty$ is complemented by a projection of norm at most two. □

**Proposition 4.3.12.** *If $\mathcal{C}(K)$ is order-complete and $K$ is metrizable, then $K$ is finite.*

*Proof.* If $K$ is infinite, $\mathcal{C}(K)$ contains a complemented copy of $c_0$ by Proposition 4.3.11. But if, moreover, $\mathcal{C}(K)$ is isometrically injective, this would make $c_0$ injective, which is false, because $c_0$ is uncomplemented in $\ell_\infty$, as we saw in Theorem 2.5.5. □

**Corollary 4.3.13.** *The only isometrically injective separable Banach spaces are finite-dimensional and isometric to $\ell_\infty^n$ for some $n \in \mathbb{N}$.*

*Proof.* If $X$ is an isometrically injective Banach space, by Theorem 4.3.7, $X$ can be identified with an order-complete $\mathcal{C}(K)$-space for some compact Hausdorff $K$. Since $X$ is separable, Theorem 4.1.3 yields that $K$ is metrizable, and by Proposition 4.3.12, $K$ must be finite. Therefore $\mathcal{C}(K)$ is (isometrically isomorphic to) $\ell_\infty^{|K|}$. □

In fact, there are no infinite-dimensional injective separable Banach spaces (even dropping isometrically), but this is substantially harder, and we will see it in the next chapter.

## 4.4 Spaces of Continuous Functions on Uncountable Compact Metric Spaces

We now turn to the problem of the isomorphic classification of $\mathcal{C}(K)$-spaces. The Banach–Stone theorem (Theorem 4.1.5) asserts that if $K$ and $L$ are non-homeomorphic compact Hausdorff spaces, then the corresponding spaces of continuous functions $\mathcal{C}(K)$ and $\mathcal{C}(L)$ cannot be linearly isometric. However, it is quite a different question to ask whether they can be linearly *isomorphic*. In the 1950s and 1960s, a complete classification of the isomorphism classes of $\mathcal{C}(K)$ for $K$ metrizable (i.e., for $\mathcal{C}(K)$ separable) was found through the work of Bessaga, Pełczyński, and Miljutin. We will describe some of this work in this section and the next.

Let us note before we start that it is quite possible for $\mathcal{C}(K)$ and $\mathcal{C}(L)$ to be linearly isomorphic when $K$ and $L$ are not homeomorphic. We shall need the following:

**Proposition 4.4.1.** *If $K$ is an infinite compact metric space, then $\mathcal{C}(K) \approx \mathcal{C}(K) \oplus \mathbb{R}$. Hence $\mathcal{C}(K)$ is isomorphic to its hyperplanes.*

*Proof.* By Proposition 4.3.11, $\mathcal{C}(K) \approx E \oplus c_0 \approx E \oplus c_0 \oplus \mathbb{R}$ for some subspace $E$. Hence $\mathcal{C}(K) \approx \mathcal{C}(K) \oplus \mathbb{R}$.

The latter statement of the proposition follows from the fact that every two hyperplanes (i.e., 1-codimensional subspaces) in a Banach space are isomorphic to each other and that, obviously, $\mathcal{C}(K)$ is a hyperplane of $\mathcal{C}(K) \oplus \mathbb{R}$. □

*Remark 4.4.2.* This proposition really does need metrizability of $\mathcal{C}(K)$! Indeed, a remarkable result of Koszmider [179] (followed by the subsequent work [262] of Plebanek) is that there exists a compact Hausdorff space $K$ such that $\mathcal{C}(K)$ fails to be isomorphic to its hyperplanes, thus solving in the negative the hyperplane problem for $\mathcal{C}(K)$ spaces. We recall that *Banach's hyperplane problem* asks whether every infinite-dimensional Banach space is isomorphic to its hyperplanes. The problem clearly has an affirmative answer for classical spaces like $\ell_p$ and $L_p$ $(1 \le p \le \infty)$. In 1994, Gowers [113] found the first counterexamples to the hyperplane problem, showing that this pathology can occur in separable spaces (even more, in spaces with an unconditional basis). The surprising part in Koszmider's example is that unlike Gowers's construction, whose norms are defined by a complicated process, his is of a classical space with a nice algebraic structure and a very simple norm.

Given Proposition 4.4.1, note that if $K = [0,1] \cup \{2\}$, then $\mathcal{C}(K) \approx \mathcal{C}[0,1] \oplus \mathbb{R} \approx \mathcal{C}[0,1]$, but $K$ and $[0,1]$ are not homeomorphic. Similarly, $\mathcal{C}[0,1]$ is isomorphic to its (hyperplane) subspace $\{f : f(0) = f(1)\}$, which is trivially isometric to $\mathcal{C}(\mathbb{T})$. But it is more difficult to make general statements. In Banach's 1932 book [18] he raised the question whether $\mathcal{C}[0,1]$ and $\mathcal{C}[0,1]^2$ are linearly isomorphic. We will see that they are, but at this stage it is far from obvious.

To study $\mathcal{C}(K)$-spaces with $K$ infinite and compact metric, we must consider two cases, namely $K$ countable and $K$ uncountable. Of course, $K$ must be separable, but it could actually be already countable. Indeed, the simplest infinite $K$ is the one-point

compactification of $\mathbb{N}$, $\gamma\mathbb{N}$, which consists of the terms of a convergent sequence and its limit; e.g., we can take $K = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\}$. Then $\mathcal{C}(K)$ can be identified with the space $c$ of convergent sequences. This is linearly isomorphic to $c_0$, since $c \approx c_0 \oplus \mathbb{R}$. If $K$ is countable, then $\mathcal{M}(K)$ consists only of purely atomic measures and is immediately seen to be isometric to $\ell_1$. Thus $\mathcal{C}(K)^*$ is separable. However, $\mathcal{C}[0,1]^*$ is nonseparable (since $\mathcal{C}[0,1]$ contains a copy of $\ell_1$ by the Banach–Mazur theorem (Theorem 1.4.4)).

In this section we will restrict to the case of uncountable $K$. The main result is the remarkable theorem of Miljutin [220], which asserts that for every uncountable compact metric space $K$, the space $\mathcal{C}(K)$ is isomorphic to $\mathcal{C}[0,1]$. This result was obtained by Miljutin in his thesis in 1952, but was not published until 1966. Miljutin's mathematical interests changed after his thesis, and he apparently did not regard the result as important enough to merit publication. In fact, the result was discovered in Miljutin's thesis by Pełczyński on a visit to Moscow in the 1960s, and it was only at his urging that a paper finally appeared in 1966.

The key players in the proof will be the Cantor set $\Delta = \{0,1\}^{\mathbb{N}}$, the unit interval $[0,1]$, and the Hilbert cube $[0,1]^{\mathbb{N}}$. We will need the following basic topological facts:

**Proposition 4.4.3.** *(i) If $K$ is a compact metric space, then $K$ is homeomorphic to a closed subset of the Hilbert cube* $[0,1]^{\mathbb{N}}$.

*(ii) If $K$ is an uncountable compact metric space, then $\Delta$ is homeomorphic to a closed subset of $K$.*

*Proof.* We have already shown $(i)$ in the proof of Theorem 1.4.4. Just take $(f_n)_{n=1}^{\infty}$ a dense sequence in $\{f \in \mathcal{C}(K) : 0 \le f \le 1\}$ and define the map $\sigma : K \to [0,1]^{\mathbb{N}}$ by $\sigma(s) = (f_n(s))_{n=1}^{\infty}$. Then $\sigma$ is continuous and one-to-one, hence a homeomorphism onto $\sigma(K)$. (We repeatedly use the standard fact that a one-to-one continuous map from a compact space to a Hausdorff space is a homeomorphism onto its range, since closed sets must be mapped to compact, therefore closed, sets.)

To show part $(ii)$, we first note that since $K$ is uncountable, given any $\epsilon > 0$ we can find two disjoint uncountable closed subsets $K_0, K_1$ each with diameter at most $\epsilon$. In fact, the set $E$ of all $s \in K$ with a countable neighborhood is necessarily countable by an application of Lindelöf's theorem (every open covering of a separable metric space has a countable subcover). If we take two distinct points $s_0, s_1$ outside $E$, we can then choose $K_0$ and $K_1$ as suitable neighborhoods of $s_0, s_1$.

Now we proceed by induction: for $n \in \mathbb{N}$ and $t = (t_1, \dots, t_n) \in \{0,1\}^n$ define $K_{t_1,t_2,\dots,t_n}$ to be an uncountable compact subset of $K$ of diameter at most $2^{-n}$ such that for each $n \in \mathbb{N}$, the sets $K_{t_1,\dots,t_n,0}$ and $K_{t_1,\dots,t_n,1}$ are disjoint subsets of $K_{t_1,\dots,t_n}$. For each $t = (t_k)_{k=1}^{\infty} \in \Delta$ define $\sigma(t)$ to be the unique point in $\cap_{n=1}^{\infty} K_{t_1,\dots,t_n}$. It is simple to see that $\sigma$ is one-to-one and continuous and thus is an embedding. □

Let us use this proposition. Suppose that $K$ is a compact metric Hausdorff space and let $E$ be a closed subset of $K$. We can naturally identify $\mathcal{C}(E)$ as a *quotient* of $\mathcal{C}(K)$ by considering the restriction operator

$$R : \mathcal{C}(K) \to \mathcal{C}(E), \qquad Rf = f|_E.$$

This is a genuine quotient map by the Tietze extension theorem.[1] Let us suppose that we can find a bounded linear operator $T : \mathcal{C}(E) \to \mathcal{C}(K)$ that selects an element of each coset. Then $T$ is a *linear* extension operator that defines an extension of each $f \in \mathcal{C}(E)$ to a member of $\mathcal{C}(K)$; note that $RT$ is nothing other than the identity map $I$ on $\mathcal{C}(E)$. The map $T$ is an isomorphism of $\mathcal{C}(E)$ onto a subspace of $\mathcal{C}(K)$, and the subspace is complemented by the projection $TR$. Thus we could conclude that $\mathcal{C}(E)$ is isomorphic to a complemented subspace of $\mathcal{C}(K)$. Note that the kernel of the projection is $\{f \in \mathcal{C}(K) : f|_E = 0\}$, and this must also be a complemented subspace via $I - TR$.

We have met this problem in two special cases already. In the proof of the Banach–Mazur theorem, we considered the case $K = [0, 1]$ and $E$ a closed subset, and defined an extension operator by linear interpolation on the intervals of $K \setminus E$. Now, if we regard $\ell_\infty$ as $\mathcal{C}(\beta\mathbb{N})$, then the subspace $c_0$ is identified with $\{f : f_{\beta\mathbb{N}\setminus\mathbb{N}} = 0\}$ (here $\mathbb{N}$ is an open subset of $\beta\mathbb{N}$, since each point is isolated). This is uncomplemented (Theorem 2.5.5), so no linear extension operator can exist from $\beta\mathbb{N} \setminus \mathbb{N}$.

On the other hand, recall Sobczyk's theorem (Theorem 2.5.8). If we consider a separable closed subalgebra of $\ell_\infty$ containing $c_0$ (which corresponds to a metrizable compactification), then we have no problem with the extension. This suggests that metrizability of $K$ is important here and leads us to the following classical theorem, which actually implies Sobczyk's theorem. It was proved in 1933 by Borsuk [27].

**Theorem 4.4.4 (Borsuk).** *Let $K$ be a compact metric space and suppose that $E$ is a closed subset of $K$. Then there is a linear operator $T : \mathcal{C}(E) \to \mathcal{C}(K)$ such that $(Tf)|_E = f$, $\|T\| = 1$, and $T1 = 1$. In particular, $\mathcal{C}(E)$ is isometric to a norm-one complemented subspace of $\mathcal{C}(K)$.*

Let us remark that the projection onto the kernel of $T$ has then norm at most 2, and this explains the constant in Sobczyk's theorem.

*Proof.* The key point in the argument is that $U = K \setminus E$ is metrizable and hence paracompact, i.e., every open covering of $U$ has a locally finite refinement. Let us consider the covering of $U$ by the sets $V_u = \{s \in U : d(s, u) < \frac{1}{2}d(u, E)\}$. There is a locally finite refinement of $(V_u)_{u\in U}$, which implies that we can find a partition of unity subordinate to $(V_u)_{u\in U}$, that is, a family of continuous functions $(\phi_j)_{j\in J}$ on $U$ such that

- $0 \le \phi_j \le 1$,
- $\{\phi_j > 0\}$ is a locally finite covering of $U$,

[1] *The Tietze Extension theorem* states that given a normal topological space $X$ (i.e., a topological space satisfying the $T_4$ separation axiom), a closed subspace $E$ of $X$ and a continuous real-valued function on $E$, there exists a continuous real-valued function $\tilde{f}$ on $X$ such that $\tilde{f}(x) = f(x)$ for all $x \in E$.

- $\sum_{j \in J} \phi_j(s) = 1$ for all $s \in U$,
- For each $j \in J$ there exists $u_j \in U$ such that $\{\phi_j > 0\} \subset V_{u_j}$.

For each $j \in J$ pick $v_j \in E$ with $d(u_j, E) = d(u_j, v_j)$ (possible by compactness).

If $f \in \mathcal{C}(E)$, we define

$$Tf(s) = \begin{cases} f(s) & \text{if } s \in E \\ \sum_{j \in J} \phi_j(s) f(v_j) & \text{if } s \in U. \end{cases}$$

The theorem will be proved once we have shown that $Tf$ is a continuous function on $K$, because $T$ clearly is linear, $T1 = 1$, and $\|T\| = 1$. It is also clear that $Tf$ is continuous on $U$.

Now suppose $t \in E$. If $\epsilon > 0$, fix $\delta > 0$ such that $d(s,t) < 4\delta$ implies $|f(s) - f(t)| < \epsilon$. Assume $d(s,t) < \delta$. If $s \in E$, then $|Tf(s) - Tf(t)| < \epsilon$. If $s \in U$, then

$$|Tf(s) - Tf(t)| = \sum_{\phi_j(s)>0} \phi_j(s)|f(v_j) - f(t)| \le \max_{\phi_j(s)>0} |f(v_j) - f(t)|.$$

If $\phi_j(s) > 0$, then

$$d(s, u_j) < \frac{1}{2} d(u_j, E) \le \frac{1}{2}(d(s, u_j) + d(s,t)),$$

so $d(s, u_j) < d(s,t) < \delta$ and $d(u_j, E) = d(u_j, v_j) < 2\delta$. Thus,

$$d(t, v_j) \le d(s,t) + d(s, u_j) + d(u_j, v_j) < 4\delta.$$

Therefore, $|Tf(s) - Tf(t)| < \epsilon$, and the proof is complete. □

If we combine Borsuk's theorem with Proposition 4.4.3, we see that an arbitrary $\mathcal{C}(K)$ with $K$ an uncountable compact metric space (a) is isomorphic to a complemented subspace of $\mathcal{C}([0,1]^{\mathbb{N}})$ and (b) contains a complemented subspace isomorphic to $\mathcal{C}(\Delta)$, where $\Delta = \{0,1\}^{\mathbb{N}}$. To complete the proof of Miljutin's theorem we need to set up the conditions for the Pełczyński decomposition technique (Theorem 2.2.3). The first step is easy:

**Proposition 4.4.5.** $\mathcal{C}(\Delta) \approx c_0(\mathcal{C}(\Delta))$.

*Proof.* Since $\mathcal{C}(\Delta)$ is isomorphic to its hyperplanes (Proposition 4.4.1), it is isomorphic to the subspace $Z = \{f \in \mathcal{C}(\Delta) : f(0,0,\dots) = 0\}$.

For each $n \in \mathbb{N}$ let $\Delta_n = \{(s_k)_{k=1}^\infty \in \Delta : s_k = 0 \text{ if } k < n \text{ and } s_n = 1\}$. Each $\Delta_n$ is homeomorphic to $\Delta$ and is a clopen subset of $\Delta$.

If we define the map $S : Z \to \ell_\infty(\mathcal{C}(\Delta_n))$ by $Sf = (f|_{\Delta_n})_{n=1}^\infty$, then it is clear from continuity at $(0,0,\dots)$ that $S$ maps into $c_0(\mathcal{C}(\Delta_n))$ and, in fact, defines an isometric isomorphism between $Z$ and this space. □

At this point we need only one more ingredient, but it is the crux of the argument. We must show that $\mathcal{C}([0,1]^{\mathbb{N}})$ can be embedded complementably into $\mathcal{C}(\Delta)$. In order to understand the difficulty, we will first look at the problem of embedding $\mathcal{C}[0,1]$ complementably into $\mathcal{C}(\Delta)$.

It is easy to embed $\mathcal{C}[0,1]$ into $\mathcal{C}(\Delta)$. Indeed, we saw in the proof of the Banach–Mazur theorem that there is a continuous surjection $\varphi : \Delta \to [0,1]$ defined by

$$\varphi((s_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{s_n}{2^n}.$$

This induces an isometric embedding,

$$\mathcal{C}[0,1] \to \mathcal{C}(\Delta), \qquad f \mapsto f \circ \varphi.$$

Unfortunately, the image of this embedding is *not* complemented in $\mathcal{C}(\Delta)$. We will detour from the proof of Miljutin's theorem to explain this.

Let $\mathcal{B}[0,1]$ be the space of bounded Borel functions on $[0,1]$ with the usual supremum norm,

$$\|f\| = \sup_{0 \le t \le 1} |f(t)|.$$

Let $\mathcal{D}$ be the set of dyadic rationals in $(0,1)$, i.e., $q \in \mathcal{D}$ if $q = k/2^n$, where $1 \le k \le 2^n - 1$. We will consider the subspace $E$ of $\mathcal{B}[0,1]$ of all functions $f$ that are right-continuous everywhere, continuous at all points $t \notin \mathcal{D}$, and have left-hand limits at each $t \in \mathcal{D}$. The subspace $E$ consists of exactly those functions $f \in \mathcal{B}[0,1]$ such that

- $f(t) = \lim_{s \to t+} f(s)$ for all $0 \le t < 1$,
- $f(t-) = \lim_{s \to t-} f(s)$ exists for all $0 < t \le 1$, and
- $f(t-) = f(t)$ if $t \notin \mathcal{D}$.

Then $E$ can be identified with $\mathcal{C}(\Delta)$. We utilize the fact that $\varphi$ is quite close to a homeomorphism. In fact, $\varphi^{-1}(t)$ consists of at most two points and is unique for $t \notin \mathcal{D}$. Let $\rho : [0,1] \to \Delta$ be the map defined by taking $\rho(t) = \varphi^{-1}(t)$ for $t \notin \mathcal{D}$ then extending it to be right-continuous. Thus $\varphi \circ \rho$ is the identity map on $[0,1]$, and $\rho$ is right-continuous. We can define an isometry of $\mathcal{C}(\Delta)$ onto $E$ by $Tf(t) = f(\rho(t))$.

For $s_1, s_2, \dots, s_n \in \{0,1\}$ let

$$\Delta_{s_1,\dots,s_n} = \{t = (t_k)_{k=1}^{\infty} \in \Delta : t_k = s_k \text{ for } 1 \le k \le n\}.$$

Then $\Delta_{s_1,\dots,s_n}$ is a clopen subset of $\Delta$. Let

$$q(s_1,\dots,s_n) = \varphi(s_1,\dots,s_n,0,\dots) = \sum_{k=1}^{n} \frac{s_k}{2^k}.$$

Then for $n \in \mathbb{N}$ and $q$ of the form $k/2^n$ with $0 \le k \le 2^n - 1$ let $I_{n,q}$ be the half-open interval $[q, q + 2^{-n})$ when $q + 2^{-n} < 1$ and the closed interval $[q, 1]$ when $q + 2^{-n} = 1$. In this language we have

$$T\chi_{\Delta_{s_1,\dots,s_n}} = \chi_{I_{n,q(s_1,\dots,s_n)}}.$$

Now, the embedding of $\mathcal{C}[0,1]$ into $\mathcal{C}(\Delta)$ using $\varphi$ is isometrically equivalent to the embedding of $\mathcal{C}[0,1]$ into $E$ in the sense that there is an isometry of $\mathcal{C}(\Delta)$ onto $E$ that sends $\mathcal{C}[0,1]$ to $\mathcal{C}[0,1]$.

**Proposition 4.4.6.** *There is no bounded projection from $E$ onto $\mathcal{C}[0,1]$.*

*Proof.* We start by identifying the quotient space $E/\mathcal{C}[0,1]$. Define the map $S : E \to \ell_\infty(\mathcal{D})$ by

$$Sf(q) = \frac{1}{2}(f(q) - f(q-)).$$

If we consider a function in $E$ of the form

$$f = \sum_{k=0}^{2^n-1} a_k \chi_{I_{n,k}}, \qquad n \in \mathbb{N},\ a_0, \dots, a_{2^n-1} \in \mathbb{R},$$

it is clear that $\|Sf\| = d(f, \mathcal{C}[0,1])$ and that $S$ maps this space onto the subspace of all finitely nonzero functions on $\mathcal{D}$. Thus it follows that $S$ maps onto $c_0(\mathcal{D})$, and the quotient may be identified isometrically with $c_0(\mathcal{D})$.

If $\mathcal{C}[0,1]$ is complemented in $E$, then there is a *lifting* of $S$, i.e., a bounded linear map $R : c_0(\mathcal{D}) \to E$ such that $SR = I_{c_0(\mathcal{D})}$. Let $e_d$ denote a canonical basis element in $c_0(\mathcal{D})$ and let $f_d = Re_d$. We will inductively select $(d_n)_{n=1}^\infty$ in $\mathcal{D}$, open intervals $(J_n)_{n=1}^\infty$ in $(0,1)$, and signs $(\epsilon_n)_{n=1}^\infty$ such that

$$\sum_{k=1}^n \epsilon_k f_{d_k}(t) \ge \frac{n}{2}, \qquad n \in \mathbb{N}, t \in J_n.$$

To start the induction pick $d_1 = \frac{1}{2}$, and then either $|f_{d_1}(d_1)|$ or $|f_{d_1}(d_1-)|$ is at least one. Hence we may pick a sign $\epsilon_1$ and an open interval $J_1$ (with $d_1$ as an endpoint) such that $\epsilon_1 f_{d_1}(t) > \frac{1}{2}$ for $t \in J_1$.

If $d_1, \dots, d_{n-1}$, $\epsilon_1, \dots, \epsilon_{n-1}$ and $J_1, \dots, J_{n-1}$ have been chosen, we pick $d_n \in J_{n-1}$, and then $\epsilon_n$ so that either

$$\sum_{k=1}^n \epsilon_k f_{d_k}(d_n) \ge \frac{n-1}{2} + 1$$

or

$$\sum_{k=1}^{n} \epsilon_k f_{d_k}(d_n-) \geq \frac{n-1}{2} + 1.$$

Thus we can find an open interval $J_n$ with $d_n$ as an endpoint such that

$$\sum_{k=1}^{n} \epsilon_k f_{d_k}(t) \geq \frac{n}{2}, \qquad t \in J_n.$$

This completes the induction.

It follows that

$$\frac{n}{2} \leq \|R(\epsilon_1 e_{d_1} + \cdots + \epsilon_n e_{d_n})\| \leq \|R\|, \quad n \in \mathbb{N},$$

which is clearly absurd. □

The next result, known as Miljutin's lemma, is the key step in the argument. Miljutin was able to show that $\mathcal{C}[0,1]$ can be embedded as a complemented subspace of $\mathcal{C}(\Delta)$. Indeed, we can construct an alternative continuous surjection $\psi : \Delta \to [0,1]$ such that there is a norm-one linear operator $R : \mathcal{C}(\Delta) \to \mathcal{C}[0,1]$ with $R(f \circ \psi) = f$.

**Lemma 4.4.7 (Miljutin's Lemma).** *There exist a continuous surjection $\phi : \Delta \times \Delta \to [0,1]$ and a norm-one operator $S : \mathcal{C}(\Delta \times \Delta) \to \mathcal{C}[0,1]$ such that $S(f \circ \phi) = f$ for all $f \in \mathcal{C}[0,1]$.*

*Proof.* We begin with a very similar approach to that in the previous case. This time we consider an isometric embedding $T$ of $\mathcal{C}(\Delta \times \Delta)$ into $\mathcal{B}[0,1]^2$ induced by the formula

$$Tf(s,t) = f(\rho(s), \rho(t)), \qquad 0 \leq s, t \leq 1,$$

where $\rho$ is the right-continuous left inverse of the function $\varphi$ that we considered above. Thus,

$$T(\chi_{\Delta(r_1,\ldots,r_m)\times\Delta(s_1,\ldots,s_n)}) = \chi_{I_{m,q(r_1,\ldots,r_m)} \times I_{n,q(s_1,\ldots,s_n)}},$$

where $r_1, \ldots, r_m, s_1, \ldots s_n \in \{0,1\}$; $T$ maps $\mathcal{C}(\Delta \times \Delta)$ isometrically onto a subspace $F$ of $\mathcal{B}[0,1]^2$.

Let us define a homeomorphism $\theta$ of $[0,1]^2$ onto itself by the formula

$$\theta(t,u) = (t, u^2 t + (1-t)u), \qquad (t,u) \in [0,1]^2.$$

Notice that for each fixed choice of $t$ the map $u \mapsto u^2t + u(1-t)$ is a monotone increasing homeomorphism of $[0,1]$ onto itself and that $(t,u) \mapsto (t, u^2t + u(1-t))$ is a homeomorphism of the square onto itself. Let the (continuous) inverse map be given by $(t,v) \mapsto (t, \sigma(t,v))$, where for each fixed $t$ the map $v \mapsto \sigma(t,v)$ is an increasing homeomorphism of $[0,1]$ onto itself.

Let $\phi : \Delta \times \Delta \to [0,1]$ be given by $\phi(r,s) = \sigma(\varphi(r), \varphi(s))$.

Next define a norm-one operator $V : \mathcal{B}[0,1]^2 \to \mathcal{B}[0,1]$ via the formula

$$Vf(u) = \int_0^1 f \circ \theta(t,u)dt.$$

Notice that $VT(f \circ \phi) = f$ if $f \in \mathcal{C}[0,1]$. Indeed, if $g \in C(\Delta \times \Delta)$ and $(t,u) \in [0,1]^2$, then $Tg(t,u) = g(\rho(t), \rho(u))$ and hence $Tf \circ \phi(t,u) = f \circ \phi(\rho(t), \rho(u)) = f \circ \sigma(t,u)$ and thus $T(f \circ \phi)(\theta(t,u)) = f \circ \sigma \circ \theta(t,u) = f(u)$ for all $0 \le t \le 1$.

All that remains is to show that $VT$ actually maps $\mathcal{C}(\Delta \times \Delta)$ into $\mathcal{C}[0,1]$. To this end we need to show that $V$ maps $F$ into $\mathcal{C}[0,1]$, and it is therefore more than enough to show that $g = V(\chi_{[0,a)\times[0,b)}) = V(\chi_{[0,a]\times[0,b]}) \in \mathcal{C}[0,1]$ for every $0 < a \le 1$ and $0 < b \le 1$.

Notice that $g(u)$ can be computed as the measure of the set of $t$ such that $0 \le t \le a$ and $u^2t + u(1-t) \le b$. The later inequality reduces to $t \ge (u-b)(u-u^2)^{-1}$. The single nonnegative solution of the quadratic equation $u - b = (u-u^2)a$ will be denoted by $h(a,b)$. Note that $h(a,b) > b$ unless $a = 0$. We thus have

$$g(u) = \begin{cases} a & \text{if } u \le b, \\ a - \frac{u-b}{u-u^2} & \text{if } b < u \le h(a,b), \\ 0 & \text{if } h(a,b) < u < 1. \end{cases}$$

Since $g$ is continuous, this completes our proof. □

We are now in position to complete Miljutin's theorem:

**Theorem 4.4.8 (Miljutin's Theorem).** *Suppose $K$ is an uncountable compact metric space. Then $\mathcal{C}(K)$ is isomorphic to $\mathcal{C}[0,1]$.*

*Proof.* The first step is to show that $\mathcal{C}([0,1]^{\mathbb{N}})$ is isomorphic to a complemented subspace of $\mathcal{C}(\Delta)$. By Lemma 4.4.7 there is a continuous surjection $\psi : \Delta \to [0,1]$, so that we can find a norm-one operator $R : \mathcal{C}(\Delta) \to \mathcal{C}[0,1]$ with $Rf \circ \psi = f$ for $f \in \mathcal{C}[0,1]$. Then $R(\chi_\Delta) = \chi_{[0,1]}$. For fixed $t \in [0,1]$ the linear functional $f \to Rf(t)$ is given by a probability measure $\mu_t$, so that

$$Rf(t) = \int_\Delta f \, d\mu_t.$$

The map $\tilde{\psi} : \Delta^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ given by

$$\tilde{\psi}(s_1, \ldots, s_n, \ldots) = (\psi(s_1), \ldots, \psi(s_n), \ldots)$$

is a continuous surjection. We will define $\tilde{R} : \mathcal{C}(\Delta^{\mathbb{N}}) \to \mathcal{C}([0,1]^{\mathbb{N}})$ in such a way that $\tilde{R}f \circ \tilde{\psi} = f$ for $f \in \mathcal{C}([0,1]^{\mathbb{N}})$. Indeed, the subalgebra $\mathcal{A}$ of $\mathcal{C}(\Delta^{\mathbb{N}})$ of all $f$ that depend only on a finite number of coordinates is dense by the Stone–Weierstrass theorem. If $f \in \mathcal{A}$ depends only on $s_1, \dots, s_n$, we define

$$\tilde{R}f(t_1, \dots, t_n) = \int_{\Delta} \cdots \int_{\Delta} f(s_1, \dots, s_n)\, d\mu_{t_1}(s_1) \cdots d\mu_{t_n}(s_n).$$

This map is clearly linear into $\ell_\infty[0,1]$ and has norm one. It therefore extends to a norm-one operator $\tilde{R} : \mathcal{C}(\Delta^{\mathbb{N}}) \to \ell_\infty[0,1]$. If $f \in \mathcal{C}(\Delta^{\mathbb{N}})$ is of the form $f_1(s_1)\dots f_n(s_n)$, then

$$\tilde{R}f(t) = Rf_1(t)\dots Rf_n(t),$$

so $\tilde{R}f \in \mathcal{C}[0,1]$. The linear span of such functions is again dense by the Stone–Weierstrass theorem, so $\tilde{R}$ maps into $\mathcal{C}[0,1]$.

If $f \in \mathcal{C}([0,1]^{\mathbb{N}})$ is of the form $f_1(t_1)\dots f_n(t_n)$, then it is clear that $\tilde{R}f \circ \tilde{\psi} = f$. It follows that this equation holds for all $f \in \mathcal{C}([0,1]^{\mathbb{N}})$.

Thus $\mathcal{C}([0,1]^{\mathbb{N}})$ is isomorphic to a norm-one complemented subspace of $\mathcal{C}(\Delta^{\mathbb{N}})$ or $\mathcal{C}(\Delta)$, since $\Delta$ is homeomorphic to $\Delta^{\mathbb{N}}$.

Now, suppose $K$ is an uncountable compact metric space. Then $\mathcal{C}(K)$ is isomorphic to a complemented subspace of $\mathcal{C}([0,1]^{\mathbb{N}})$ by combining Proposition 4.4.3 and Theorem 4.4.4. Hence, by the preceding argument, $\mathcal{C}(K)$ is isomorphic to a complemented subspace of $\mathcal{C}(\Delta)$. On the other hand, $\mathcal{C}(\Delta)$ is isomorphic to a complemented subspace of $\mathcal{C}(K)$, again by Proposition 4.4.3 and Theorem 4.4.4. We also have Proposition 4.4.5, which gives $c_0(\mathcal{C}(\Delta)) \approx \mathcal{C}(\Delta)$. We can apply Theorem 2.2.3 to deduce that $\mathcal{C}(K) \approx \mathcal{C}(\Delta)$. Of course, the same reasoning gives $\mathcal{C}[0,1] \approx \mathcal{C}(\Delta)$. □

## 4.5 Spaces of Continuous Functions on Countable Compact Metric Spaces

We will now briefly discuss the case that $K$ is countable. The simplest such example, as we saw in the previous section, is $K = \gamma\mathbb{N}$, the one-point compactification of the natural numbers $\mathbb{N}$, in which case $\mathcal{C}(\gamma\mathbb{N}) = c \approx c_0$.

In 1960, Bessaga and Pełczyński [25] gave a complete classification of all $\mathcal{C}(K)$-spaces when $K$ is countable and compact. To fully describe this classification requires some knowledge of ordinals and ordinal spaces, and we prefer to simply discuss the case in which $K$ has the simplest structure.

If $K$ is any countable compact metric space, the Baire category theorem implies that the union of all its isolated points, $U$, is dense and open in $K$. The *Cantor–Bendixson derivative of* $K$ is the set $K' = K \setminus U$ of accumulation points of $K$. Analogously, we can define $K'' = (K')'$ and, in general, for any natural number $n$, $K^{(n)} = (K^{(n-1)})'$.

The set $K$ is said to have finite Cantor–Bendixson index if $K^{(n)}$ is finite for some $n$ and hence $K^{(n+1)}$ is empty. When this happens, $\sigma(K)$ will denote the first $n$ for which $K^{(n)}$ is finite.

*Example 4.5.1.* It is easy to construct examples of spaces $K$ without finite Cantor–Bendixson index. Let us note, first, that if $E$ is any closed subset of $K$ then $E' \subset K'$, and therefore $\sigma(E) \leq \sigma(K)$. If $K$ is a countable compact metric space, then $K_1 = K \times \gamma\mathbb{N}$ has the property that $(K_1)'$ contains a subset homeomorphic to $K$, so $\sigma(K_1) > \sigma(K)$. In this way we can build a sequence $(K_r)_{r=1}^{\infty}$ with $\sigma(K_r) \to \infty$. If we let $K_\infty$ be the one-point compactification of the disjoint union $\bigsqcup_{r=1}^{\infty} K_r$, then $K_\infty$ does not have finite Cantor–Bendixson index.

If $K$ does not have finite index, then its index can be defined as a countable ordinal. This was used by Bessaga and Pełczyński to give a complete classification, up to linear isomorphism, of all $\mathcal{C}(K)$ for $K$ countable. But we will not pursue this; instead we will give one result in the direction of classifying such $\mathcal{C}(K)$-spaces.

**Theorem 4.5.2.** *Let $K$ be a compact metric space. The following conditions are equivalent:*

*(i) $K$ is countable and has finite Cantor–Bendixson index;*
*(ii) $\mathcal{C}(K) \approx c_0$;*
*(iii) $\mathcal{C}(K)$ embeds in a space with unconditional basis;*
*(iv) $\mathcal{C}(K)$ has property (u).*

Let us point out that this theorem greatly extends Karlin's theorem (see Proposition 3.5.4 (*ii*)) that $\mathcal{C}[0,1]$ has no unconditional basis.

*Proof.* $(i) \Rightarrow (ii)$. Let us suppose, first, that $\sigma(K) = 1$. Then $K'$ is a finite set, say $K' = \{s_1, \dots, s_n\}$. Let $V_1, \dots, V_n$ be disjoint open neighborhoods of $s_1, \dots, s_n$, respectively. Then $V_1, V_2, \dots, V_n$ must also be closed sets, since for each $j$, no sequence in $V_j$ can converge to a point that does not belong to $V_j$. If we define $V_{n+1} = K \setminus (V_1 \cup \dots \cup V_n)$, $V_{n+1}$ must be a finite set of isolated points and is also clopen; we therefore can absorb it into, say, $V_1$ without changing the conditions. Now, $K$ splits into $n$-clopen sets $V_1, \dots, V_n$ and each $V_j$ is homeomorphic to $\gamma\mathbb{N}$. Hence $\mathcal{C}(K)$ is isometric to the $\ell_\infty$-product of $n$ copies of $c$; thus it is isomorphic to $c_0$.

The proof of this implication is completed by induction. Assume we have shown that $\mathcal{C}(K) \approx c_0$ if $\sigma(K) < n$, $n \geq 2$, and suppose that $\sigma(K) = n$. Then $\mathcal{C}(K') \approx c_0$. Consider the restriction map $f \to f|_{K'}$. By Theorem 4.4.4, $\mathcal{C}(K)$ is isomorphic to $\mathcal{C}(K') \oplus E$, where $E$ denotes the kernel of the restriction $f \mapsto f|_{K'}$. If $U = K \setminus K'$ is the set of isolated points of $K$, then $E$ can be identified with $c_0(U)$, which is isometric to $c_0$. Hence $\mathcal{C}(K)$ is isomorphic to $c_0$.

$(ii) \Rightarrow (iii)$ is trivial, and $(iii) \Rightarrow (iv)$ is a consequence of Proposition 3.5.3.

$(iv) \Rightarrow (i)$ First observe that if $\mathcal{C}(K)$ has property (u), then it immediately follows that $K$ is countable by combining Theorem 4.4.8 with the fact that the space $\mathcal{C}[0,1]$ fails to have property (u). This means that $\mathcal{M}(K)$ contains only purely atomic measures and that $\mathcal{C}(K)^* = \ell_1(K)$ is separable. Thus $\mathcal{C}(K)^{**} = \ell_\infty(K)$.

Suppose $h$ is an arbitrary element in $\ell_\infty(K)$ with $\|h\| \leq 1$. Then, since $B_{\mathcal{C}(K)}$ is weak* dense in $B_{\ell_\infty(K)}$ by Goldstine's theorem, and $B_{\ell_\infty(K)}$ is weak* metrizable by Lemma 1.4.1, it follows that we can find a sequence $(g_n)_{n=1}^\infty$ in $\mathcal{C}(K)$ with $\|g_n\| \leq 1$ that converges weak* to $h$. The sequence $(g_n)_{n=1}^\infty$ is weakly Cauchy in $\mathcal{C}(K)$, so by property (u) we can find a WUC series $\sum_{n=1}^\infty f_n$ such that $\big(g_n - \sum_{k=1}^n f_k\big)_n$ converges weakly to zero in $\mathcal{C}(K)$. This means that $\sum_{k=1}^\infty f_k = h$ for the weak* topology. In particular, we have that

$$\sum_{k=1}^\infty f_k(s) = h(s), \qquad s \in K.$$

Since $\sum f_n$ is a WUC series, there is a constant $M$ such that

$$\sup_N \sup_{\epsilon_j = \pm 1} \Big| \sum_{k=1}^N \epsilon_k f_k(s) \Big| = \sum_{k=1}^\infty |f_k(s)| \leq M$$

for every $s \in K$.

Put $\phi(s) = \sum_{k=1}^\infty |f_k(s)|$ and $\psi(s) = \sum_{k=1}^\infty \big(|f_k(s)| - f_k(s)\big) = \phi(s) - h(s)$. Both $\phi$ and $\psi$ are lower semicontinuous functions on $K$, that is, for every $a \in \mathbb{R}$ the sets $\phi^{-1}(a,\infty)$ and $\psi^{-1}(a,\infty)$ are open. We also have $\|\phi\|, \|\psi\| \leq M$ and $h = \phi - \psi$.

Suppose that $K$ fails to have finite Cantor–Bendixson index. Then each of the sets $E_n = K^{(n-1)} - K^{(n)}$ is nonempty for $n = 1, 2, \ldots$ (here, $K^{(0)} = K$). We pick a particular $h \in \ell_\infty(K)$ with $\|h\| \leq 1$ such that

$$h(s) = (-1)^n, \qquad s \in E_n.$$

Since $K$ fails to have finite index, the set $K \setminus \cup_{n=1}^\infty E_n$ is nonempty, and we can define $h$ to be zero on this set. Thus, we can write $h = \phi - \psi$ as above. If we put

$$a_n = \sup_{s \in E_{2n}} \phi(s), \qquad n = 1, 2, \ldots,$$

then $|a_n| \leq M$ for all $n$.

Suppose $\epsilon > 0$ and that $n \geq 1$. Then, there exists $s_0 \in E_{2n}$ such that $\phi(s_0) > a_n - \epsilon$. Thus by the lower semicontinuity of $\phi$ there is an open set $U_0$ containing $s_0$ such that $\phi(s) > a_n - \epsilon$ for every $s \in U_0$. In particular, $U_0 \cap K^{(2n-2)}$ is relatively open in $K^{(2n-2)}$ and $U_0 \cap E_{2n-1} \neq \emptyset$. Hence there exists $s_1 \in U_0 \cap E_{2n-1}$ such that $\phi(s_1) > a_n - \epsilon$. Thus $\psi(s_1) > a_n + 1 - \epsilon$. Next we find an open set $U_1$ containing $s_1$ such that $\psi(s) > a_n + 1 - \epsilon$ for $s \in U_1$. Reasoning as above, we can find $s_2 \in U_1 \cap E_{2n-2}$ with $\psi(s_2) > a_n + 1 - \epsilon$. But this implies $\phi(s_2) > a_n + 2 - \epsilon$, and so $a_{n-1} \geq a_n + 2 - \epsilon$. Since $\epsilon > 0$ is arbitrary, we have

$$a_n \leq a_{n-1} - 2, \qquad n = 1, 2, \ldots.$$

Clearly this contradicts the lower bound of $-M$ on the sequence $(a_n)_{n=1}^\infty$. The contradiction shows that $K$ has finite Cantor–Bendixson index. $\square$

If $K$ and $L$ are countable compact metric spaces with different but finite Cantor–Bendixson indices, then $K$ and $L$ are not homeomorphic, but the spaces $\mathcal{C}(K)$ and $\mathcal{C}(L)$ are both isomorphic to $c_0$. Later we will see that, up to equivalence, there is only one unconditional basis of $c_0$, in the sense that every normalized unconditional basis is equivalent to the canonical basis.

*Remark 4.5.3.* Notice that since $\mathcal{C}(K)^*$ is isometric to $\ell_1$ for *every* countable compact metric space $K$, the Banach space $\ell_1$ is isometric to the dual of many nonisomorphic Banach spaces.

## Problems

**4.1.** Let $K$ be a compact Hausdorff space. Show that every extreme point of $B_{\mathcal{C}(K)^*}$ is of the form $\pm\delta_s$, where $\delta_s$ is the probability measure defined on the Borel sets of $K$ by $\delta_s(B) = 1$ if $s \in B$ and 0 otherwise.

**4.2 (The Banach–Stone Theorem).** Suppose $K$ and $L$ are compact Hausdorff spaces such that $\mathcal{C}(K)$ and $\mathcal{C}(L)$ are isometric. Show that $K$ and $L$ are homeomorphic. [*Hint*: Argue that if $U : \mathcal{C}(K) \to \mathcal{C}(L)$ is any (onto) isometry, then $U^*$ maps extreme points of the dual ball to extreme points.]

**4.3 (Ransford's Proof of the Stone–Weierstrass Theorem [266]).**

(a) If $E$ is a closed subset of $K$, let $\|f\|_E = \sup\{|f(t)| : t \in E\}$. Assume $A \neq \mathcal{C}(K)$; pick $f \in \mathcal{C}(K)$ with $d(f, A) = \inf\{\|f - a\| : a \in K\} = 1$. Show by a Zorn's lemma argument that there is a minimal compact subset $E$ of $K$ with $d_E(f, A) = \inf\{\|f - a\|_E : a \in A\} = 1$.
(b) Show that $E$ cannot consist of one point and that there exists $h \in A$ with $\min_{s\in E} h(s) = 0$ and $\max_{s\in E} h(s) = 1$.
(c) Let $E_0 = \{s \in E : h(s) \leq 2/3\}$ and $E_1 = \{s \in E : h(s) \geq 1/3\}$. Show that there exist $a_0, a_1 \in E$ such that $\|f - a_0\|_{E_0} < 1$ and $\|f - a_1\|_{E_1} < 1$.
(d) Let $g_n = (1 - (1 - h)^n)^{2^n} \in A$. Show that for large enough $n$ we have $\|(1 - g_n)a_0 + g_n a_1 - f\|_E < 1$. This contradiction proves the theorem.

**4.4 (De Branges's Proof of the Stone–Weierstrass Theorem [58]).**

(a) Let $\mu$ be a regular probability measure on $K$ and let $E$ be the intersection of all compact sets $F \subset K$ with $\mu(F) = 1$. Show that $\mu(E) = 1$. (The set $E$ is called the *support* of $\mu$.)
(b) Suppose $A \neq \mathcal{C}(K)$. Let $V = B_{\mathcal{M}(K)} \cap A^\perp \subset \mathcal{C}(K)^*$. Show that $A$ is weak* compact and convex and deduce that it has an extreme point $\nu$ with $\|\nu\| = 1$.

(c) If $a \in A$ with $0 \le a \le 1$, show that $\nu_a \in A^\perp$, where

$$\int h\,d\nu_a = \int ha\,d\nu.$$

Show that $\|\nu_a\| = \int a\,d|\nu|$. Deduce from the fact that $\nu$ is an extreme point that $a$ is constant $\nu$-a.e. on the support of $|\nu|$.

(d) Deduce that the support of $|\nu|$ is a single point and hence obtain a contradiction.

**4.5.** A compact Hausdorff space $K$ is called *extremally disconnected* if the closure of every open set is again open (and hence clopen!). Prove that if $\mathcal{C}(K)$ is order-complete, then $K$ is extremally disconnected. [*Hint*: If $U$ is open, apply order-completeness to the set of $f \in \mathcal{C}(K)$ with $f \ge \chi_U$.]

**4.6.** (a) If $K$ is extremally disconnected, show that for every bounded lower semicontinuous function $f$, the upper semicontinuous regularization

$$\tilde{f}(s) = \inf\{g(s) : g \in \mathcal{C}(K), g \ge f\}$$

is continuous.

(b) Deduce that if $K$ is extremally disconnected, then $\mathcal{C}(K)$ is order-complete.

**4.7.** Let $K$ be any topological space.

(a) Show that for every Borel set there is an open set $U$ such that the symmetric difference $B \Delta U$ is of first category. (Of course, this is vacuous unless $K$ is of second category in itself!)

(b) Deduce that for every real Borel function $f$ on $K$ there is a lower semicontinuous function $g$ such that $\{f \neq g\}$ is of first category.

(c) Show that if $K$ is compact and extremally disconnected, then for every bounded Borel function there is a continuous function $g$ such that $\{f \neq g\}$ is of first Baire category.

**4.8.** Let $K$ be a compact Hausdorff space and consider the space $\mathcal{B}(K)$ of all bounded Borel functions on $K$. Consider $\mathcal{B}(K)$ modulo the equivalence relation $f \sim g$ if $\{s \in K : f(s) \neq g(s)\}$ is of first category. Define a norm on the space $\mathcal{B}^\sim(K) = \mathcal{B}(K)/\sim$ by

$$\|f\| = \inf\{\lambda : \{|f| > \lambda\} \text{ is of first category}\}.$$

Show that $\mathcal{B}^\sim(K)$ is a Banach space that can be identified with a space $\mathcal{C}(L)$ where $L$ is compact Hausdorff. Show further that $\mathcal{C}(L)$ is order-complete and hence $L$ is extremally disconnected.

Note that if $K$ is extremally disconnected, then $\mathcal{B}^\sim(K) = \mathcal{C}(K)$ (in the sense that there is a unique continuous function in each equivalence class).

**4.9 (Continuation of 4.8).**

(a) Now suppose $\mathcal{B}^{\sim}(K)$ is isometrically a dual space. Show that if $\varphi$ belongs to the predual, then there is a regular Borel measure $\mu$ on $K$ such that $\mu(B) = \varphi(\chi_B)$ for every Borel set. Show that $\mu$ must vanish on every set of first category. [*Hint*: Use the fact that the positive cone must be closed for the weak* topology.]

(b) Deduce that if $K$ is compact and metrizable and has no isolated points (e.g., $K = [0, 1]$), then $\mathcal{B}^{\sim}(K)$ cannot be a dual space.

**4.10.** Let $K$ be metrizable and let $E$ denote the smallest subspace of $\mathcal{C}(K)^{**}$ containing $\mathcal{C}(K)$ that is weak* sequentially closed (i.e., is closed under the weak* convergence of sequences). Show that $E = \mathcal{B}(K)$, where $\mathcal{B}(K)$ is considered a subspace of $\mathcal{C}(K)^{**}$ via the action

$$\langle f, \mu \rangle = \int f \, d\mu, \qquad \mu \in \mathcal{M}(K).$$

**4.11 (The Amir–Cambern Theorem [12, 39]).** Let $K$ and $L$ be compact spaces and suppose $T : \mathcal{C}(K) \to \mathcal{C}(L)$ is an isomorphism such that $\|T\| = 1$ and $\|T^{-1}\| < c < 2$. For the proof of the theorem that we outline here we shall impose the additional assumption that $K$ and $L$ are metrizable.

(a) Show that $T^{**}$ maps $\mathcal{B}(K)$ onto $\mathcal{B}(L)$.

(b) For $t \in K$ define $e_t \in \mathcal{B}(K)$ by $e_t(t) = 1$ and $e_t(s) = 0$ for $s \neq t$. Show that for fixed $t \in K$,

$$|T^{**}e_t(x)| > \frac{1}{c}$$

for exactly one choice of $x \in L$. [*Hint:* If this holds for $x \neq y$, consider $T^*(a\delta_x + b\delta_y)$, where $a, b$ are chosen suitably.]

Show also that for fixed $x \in L$, $|T^{**}e_t(x)| > \frac{1}{2}$ for at most one $t \in K$.

(c) Use (b) to define an injective map $\phi : K \to L$ such that

$$|\langle T^* \delta_{\phi(t)}, e_t \rangle| > \frac{1}{c}, \qquad t \in K.$$

Show that $\phi$ is continuous and that

$$\|Tf - f \circ \phi\| \leq 2(1 - c^{-1})\|f\|, \qquad f \in \mathcal{C}(K).$$

(d) Deduce that $\phi$ is onto and $K$ and $L$ are homeomorphic.

The Amir–Cambern theorem is an extension of the Banach–Stone theorem. Of course, Miljutin's theorem means that we must have some restriction on $\|T^{-1}\|$; in fact, 2 is sharp in the sense that one can find nonhomeomorphic $K$ and $L$ and $T$ with $\|T\| = 1$, $\|T^{-1}\| = 2$; this is due to Cohen [51].

# Chapter 5
# $L_1(\mu)$-Spaces and $\mathcal{C}(K)$-Spaces

In this chapter we will prove some very classical results concerning weak compactness and weakly compact operators on $\mathcal{C}(K)$-spaces and $L_1(\mu)$-spaces, and exploit them to give further information about complemented subspaces of such spaces. We have proved forerunners of these results in Chapter 2 for the corresponding sequence spaces. If $T : c_0 \to X$ or $T : X \to \ell_1$ is weakly compact, then $T$ is in fact compact (Theorem 2.4.10 and Theorem 2.3.7). These results are essentially consequences of the fact that $\ell_1$ is a Schur space.

We can regard $c_0$ as being a space of continuous functions (it is isomorphic to $c$, which is isometrically a space of continuous functions) and $\ell_1$ is a very special example of a space $L_1(\mu)$, where $\mu$ is counting measure on the natural numbers. It is therefore natural to consider to what extent we can find substitutes for more general $\mathcal{C}(K)$-spaces and $L_1(\mu)$-spaces.

Much of the material in this chapter dates back in some form or other to some remarkable and very early work of Dunford and Pettis [74] in 1940, later developed by Grothendieck [120]. However, we will take a modern approach based on the techniques we have built up in the preceding chapters; this approach to the study of function spaces may be said to date to the paper of Kadets and Pełczyński [147].

## 5.1 General Remarks About $L_1(\mu)$-Spaces

Let $(\Omega, \Sigma, \mu)$ be a probability measure space, that is, $\mu$ is a measure on the $\sigma$-algebra $\Sigma$ of subsets of $\Omega$, where $\mu(\Omega) = 1$. Although it might appear restrictive to consider probability spaces, this covers much more general situations. Indeed, if $\nu$ is assumed to be merely a $\sigma$-finite measure on $\Sigma$, then we can always find a $\nu$-integrable function $\varphi$ such that $\varphi > 0$ everywhere and $\int \varphi \, d\nu = 1$. If we define $d\mu = \varphi \cdot d\nu$, then $\mu$ is a probability measure and $L_1(\Omega, \mu)$ is isometric to $L_1(\Omega, \nu)$ via the isometry $U : L_1(\nu) \to L_1(\mu)$ given by $Uf(\omega) = f(\omega)(\varphi(\omega))^{-1}$.



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In most practical examples $\Omega$ is a complete separable metric space $K$ (also called a *Polish space*), $\Sigma$ coincides with the Borel sets $\mathcal{B}$, and $\mu$ is nonatomic. In this case it is important to note that there is only one such space $L_1(K, \mathcal{B}, \mu)$. More precisely, if $\mu$ is a nonatomic probability measure on $K$, then there is a bijection $\sigma : [0, 1] \to K$ such that both $\sigma$ and $\sigma^{-1}$ are Borel maps and

$$\mu(B) = \lambda(\sigma^{-1}B), \qquad B \in \mathcal{B}(K),$$

where $\lambda$ denotes Lebesgue measure on $[0, 1]$. Thus $f \mapsto f \circ \sigma$ defines an isometry between $L_1(K, \mu)$ and $L_1 = L_1([0, 1], \lambda)$. See, e.g., [237] or [276].

Let us first note that unlike $\ell_1$, the space $L_1$ is not a Schur space. To see this, take for example the sequence of functions $f_n(x) = \sqrt{2}\sin n\pi x$, $n \in \mathbb{N}$. Then $(f_n)_{n=1}^\infty$ is orthonormal in $L_2[0, 1]$, and by Bessel's inequality we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)\,dx = 0$$

for all $g \in L_2[0, 1]$. In particular, $(f_n)_{n=1}^\infty$ converges to 0 weakly in $L_1$ but not in norm.

On the other hand, since it is separable and its dual is nonseparable, $L_1$ is not reflexive. Therefore the relatively weakly compact sets of $L_1[0, 1]$ are not simply the bounded sets.

We start by trying to imitate the techniques that we developed to handle sequence spaces. First we give an analogue for Lemma 2.1.1:

**Lemma 5.1.1.** *Let $(f_n)_{n=1}^\infty$ be a sequence of norm-one, disjointly supported functions in $L_1(\mu)$. Then $(f_n)_{n=1}^\infty$ is a norm-one complemented basic sequence, isometrically equivalent to the canonical basis of $\ell_1$.*

*Proof.* For any scalars $(\alpha_i)_{i=1}^n$ and any $n \in \mathbb{N}$,

$$\begin{aligned}
\Big\| \sum_{i=1}^n \alpha_i f_i \Big\|_1 &= \int_\Omega \Big| \sum_{i=1}^n \alpha_i f_i \Big| \, d\mu \\
&= \int_\Omega \Big( \sum_{i=1}^n |\alpha_i f_i| \Big) \, d\mu \\
&= \sum_{i=1}^n |\alpha_i| \int_\Omega |f_i| \, d\mu \\
&= \sum_{i=1}^n |\alpha_i|.
\end{aligned}$$

Let us consider the operator $P : L_1(\mu) \to L_1(\mu)$ given by

$$P(f) = \sum_{n=1}^{\infty} \Big( \int_{\Omega} f h_n \, d\mu \Big) f_n,$$

where for each $n$,

$$h_n(\omega) = \begin{cases} \dfrac{\overline{f_n(\omega)}}{|f_n(\omega)|} & \text{if } |f_n(\omega)| > 0, \\ 0 & \text{if } f_n(\omega) = 0. \end{cases}$$

This covers the cases of real and complex scalars. The map $P$ is a projection onto $[f_n]$. Furthermore,

$$\begin{aligned} \|Pf\|_1 &= \sum_{n=1}^{\infty} \Big| \int_{\Omega} f h_n \, d\mu \Big| \\ &= \sum_{n=1}^{\infty} \int_{\{|f_n|>0\}} |f| \, d\mu \\ &= \int_{\cup_{n=1}^{\infty} \{|f_n|>0\}} |f| \, d\mu \\ &\le \int_{\Omega} |f| \, d\mu. \end{aligned}$$

□

## 5.2 Weakly Compact Subsets of $L_1(\mu)$

In this section we will consider the problem of identifying the weakly compact subsets of $L_1(\mu)$ when $(\Omega, \Sigma, \mu)$ is a probability measure space. Our approach is through certain subsequence principles. In Chapters 1 and 2 we made heavy use of so-called *gliding hump techniques*. For example, a sequence in $\ell_1$ that converges coordinatewise to zero but not in norm has a subsequence that is basic and equivalent to the canonical basis of $\ell_1$. The appropriate generalization to $L_1(\mu)$-spaces replaces coordinatewise convergence by almost everywhere convergence or convergence in measure.

**Lemma 5.2.1.** *Let $(h_n)_{n=1}^{\infty}$ be a bounded sequence in $L_1(\mu)$ that converges to 0 in measure. Then there exist a subsequence $(h_{n_k})_{k=1}^{\infty}$ of $(h_n)_{n=1}^{\infty}$ and a sequence of disjoint measurable sets $(A_k)_{k=1}^{\infty}$ such that $\|h_{n_k} - h_{n_k}\chi_{A_k}\|_1 \to 0$.*

*Proof.* We are going to extract such a subsequence by an inductive procedure based on a technique similar to the *gliding hump* argument for sequences. Let us first note that $(h_n)_{n=1}^\infty$ has a subsequence that converges to 0 a.e., and so we may assume without loss of generality that $\lim_{n\to\infty} h_n(\omega) = 0$ $\mu$-a.e.

Let $h_{n_1} = h_1$ and take $F_1 = \{\omega\colon |h_{n_1}(\omega)| > \frac{1}{2}\}$. The function $h_{n_1}$ is integrable, so there exists $\delta_1 > 0$ such that $\mu(E) < \delta_1$ implies $\int_E |h_{n_1}|\, d\mu < \frac{1}{2}$. Next, pick $n_2 > n_1$ such that $\mu(|h_{n_2}| > \frac{1}{2^2}) < \delta_1$ and $F_2 = \{\omega\colon |h_{n_2}(\omega)| > \frac{1}{2^2}\}$. Similarly there exists $\delta_2 > 0$ such that $\mu(E) < \delta_2$ implies $\int_E |h_{n_i}|\, d\mu < \frac{1}{2^2}$ for $i = 1, 2$. Pick $n_3 > n_2$ such that $\mu(|h_{n_3}| > \frac{1}{2^3}) < \delta_2$ and $F_3 = \{\omega\colon |h_{n_3}(\omega)| > \frac{1}{2^3}\}$. Continuing by induction, we produce a subsequence $(h_{n_k})_{k=1}^\infty$ of $(h_n)_{n=1}^\infty$ and a sequence of sets $(F_k)_{k=1}^\infty$ such that $\|h_{n_k} - h_{n_k}\chi_{F_k}\|_1 \le 2^{-k}$ for all $k$.

Now we take the sequence $(A_j)_{j=1}^\infty$ of disjoint subsets of $\Omega$ given by

$$A_1 = F_1 \setminus \bigcup_{k>1} F_k, \quad A_2 = F_2 \setminus \bigcup_{k>2} F_k, \quad \dots \quad A_j = F_j \setminus \bigcup_{k>j} F_k, \quad \dots .$$

Clearly, for each $k$ we have

$$\int_{F_k} |h_{n_k}|\, d\mu - \int_{A_k} |h_{n_k}|\, d\mu \le \sum_{j>k} \int_{F_j} |h_{n_k}|\, d\mu \le \sum_{j>k} \frac{1}{2^{j-1}} = \frac{1}{2^{k-1}},$$

i.e.,

$$\|h_{n_k}\chi_{F_k} - h_{n_k}\chi_{A_k}\|_1 \le \frac{1}{2^{k-1}}.$$

Hence

$$\|h_{n_k} - h_{n_k}\chi_{A_k}\|_1 \le \|h_{n_k} - h_{n_k}\chi_{F_k}\|_1 + \|h_{n_k}\chi_{F_k} - h_{n_k}\chi_{A_k}\|_1 \le \frac{1}{2^k} + \frac{1}{2^{k-1}},$$

and so $\|h_{n_k} - h_{n_k}\chi_{A_k}\|_1 \to 0$. □

**Definition 5.2.2.** A bounded subset $\mathcal{F} \subset L_1(\mu)$ is called *equi-integrable* (or *uniformly integrable*) if given $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that for every set $E \subset \Omega$ with $\mu(E) < \delta$ we have $\sup_{f\in\mathcal{F}} \int_E |f| d\mu < \epsilon$, i.e.,

$$\lim_{\mu(E)\to 0} \sup_{f\in\mathcal{F}} \int_E |f| d\mu = 0.$$

In this definition we can omit the word *bounded* if $\mu$ is nonatomic, since then given any $\delta > 0$ it is possible to partition $\Omega$ into a finite number of sets of measure $< \delta$.

*Example 5.2.3.* (*i*) For $h \in L_1(\mu)$ with $h \ge 0$ the set $\mathcal{F} = \{f \in L_1(\mu)\,;\, |f| \le h\}$ is equi-integrable.

*(ii)* The closed unit ball of $L_2(\mu)$ is an equi-integrable subset of $L_1(\mu)$. Indeed, for every $f \in B_{L_2(\mu)}$ and measurable set $E$, by the Cauchy–Schwarz inequality,

$$\int_E |f| d\mu \leq \left(\int_E 1 d\mu\right)^{1/2} \left(\int_E |f|^2 d\mu\right)^{1/2} \leq \big(\mu(E)\big)^{1/2}.$$

Then,

$$\lim_{\mu(E)\to 0} \sup_{f\in F} \int_E |f| d\mu = 0.$$

*(iii)* The closed unit ball of $L_1(\mu)$ is not equi-integrable, as one can easily check by taking the subset $\mathcal{F} = \{\delta^{-1}\chi_{[0,\delta]}\ ;\ 0 < \delta < 1\}$.

**Lemma 5.2.4.** *Let $\mathcal{F}$ and $\mathcal{G}$ be bounded sets of equi-integrable functions in $L_1(\mu)$. Then the sets $\mathcal{F} \cup \mathcal{G}$ and $\mathcal{F} + \mathcal{G} = \{f + g\,; f \in \mathcal{F}, g \in \mathcal{G}\} \subset L_1(\mu)$ are (bounded and) equi-integrable.*

This is a very elementary deduction from the definition, and we leave the proof to the reader. Next we give an alternative formulation of equi-integrability.

**Lemma 5.2.5.** *Suppose $\mathcal{F}$ is a bounded subset of $L_1(\mu)$. Then the following are equivalent:*

*(i)* *$\mathcal{F}$ is equi-integrable;*

*(ii)* $\displaystyle\lim_{M\to\infty} \sup_{f\in\mathcal{F}} \int_{\{|f|>M\}} |f|\, d\mu = 0.$

*Proof.* $(i) \Rightarrow (ii)$ Since $\mathcal{F}$ is bounded, there is a constant $A > 0$ such that $\sup_{f\in\mathcal{F}} \|f\|_1 \leq A$. Given $f \in \mathcal{F}$, by Chebyshev's inequality,

$$\mu(\{|f| > M\}) \leq \frac{\|f\|_1}{M} \leq \frac{A}{M}.$$

Therefore, $\lim_{M\to\infty} \mu(\{|f| > M\}) = 0$. Using the equi-integrability of $\mathcal{F}$, we conclude that

$$\lim_{M\to\infty} \sup_{f\in\mathcal{F}} \int_{\{|f|>M\}} |f|\, d\mu = 0.$$

$(ii) \Rightarrow (i)$ Given $f \in \mathcal{F}$ and $E \in \Sigma$, for every finite $M > 0$ we have

$$\begin{aligned}\int_E |f|\, d\mu &= \int_{E\cap\{|f|\leq M\}} |f|\, d\mu + \int_{E\cap\{|f|>M\}} |f|\, d\mu \\ &\leq M\mu(E) + \int_{E\cap\{|f|>M\}} |f|\, d\mu\end{aligned}$$

$$\leq M\mu(E) + \int_{\{|f|>M\}} |f|\, d\mu$$
$$\leq M\mu(E) + \sup_{f\in F} \int_{\{|f|>M\}} |f|\, d\mu.$$

Hence,

$$\sup_{f\in\mathcal{F}} \int_E |f|\, d\mu \leq M\mu(E) + \sup_{f\in F} \int_{\{|f|>M\}} |f|\, d\mu.$$

Given $\epsilon > 0$, let us pick $M = M(\epsilon)$ such that

$$\sup_{f\in\mathcal{F}} \int_{\{|f|>M\}} |f|\, d\mu < \frac{\epsilon}{2}.$$

Then if $\mu(E) < \frac{\epsilon}{2M}$, we obtain

$$\sup_{f\in\mathcal{F}} \int_E |f|\, d\mu \leq M\frac{\epsilon}{2M} + \frac{\epsilon}{2} = \epsilon.$$

□

Note that whenever $(f_n)_{n=1}^{\infty}$ is a sequence bounded above by an integrable function, then in particular, $(f_n)_{n=1}^{\infty}$ is equi-integrable. The next lemma establishes that conversely, equi-integrability is a condition that can replace the existence of a dominating function in the dominated convergence theorem:

**Lemma 5.2.6.** *Suppose $(f_n)_{n=1}^{\infty}$ is an equi-integrable sequence in $L_1(\mu)$ that converges a.e. to some $g \in L_1(\mu)$. Then*

$$\lim_{n\to\infty} \int_\Omega f_n\, d\mu = \int_\Omega g\, d\mu.$$

*Proof.* For each $M > 0$ let us consider the truncations

$$f_n^{(M)} = \begin{cases} M & \text{if } f_n > M, \\ f_n & \text{if } |f_n| \leq M, \\ -M & \text{if } f_n < -M, \end{cases} \qquad g^{(M)} = \begin{cases} M & \text{if } g > M, \\ g & \text{if } |g| \leq M, \\ -M & \text{if } g < -M, \end{cases}$$

and write

$$\left|\int_\Omega (f_n - g)\, d\mu\right| \leq \left|\int_\Omega (f_n - f_n^{(M)})\, d\mu\right| + \left|\int_\Omega (f_n^{(M)} - g^{(M)})\, d\mu\right| + \left|\int_\Omega (g - g^{(M)})\, d\mu\right|.$$

Now,

$$\Big|\int_\Omega (f_n - f_n^{(M)})\,d\mu\Big| \le \int_{\{|f_n|>M\}} \big(|f_n| - M\big)\,d\mu \le \int_{\{|f_n|>M\}} |f_n|\,d\mu \to 0$$

uniformly in $n$ as $M \to \infty$ by Lemma 5.2.5. Analogously, since $g \in L_1(\mu)$,

$$\Big|\int_\Omega (g - g^{(M)})\,d\mu\Big| \le \int_{\{|g|>M\}} \big(|g| - M\big)\,d\mu \le \int_{\{|g|>M\}} |g|\,d\mu \overset{M\to\infty}{\longrightarrow} 0.$$

And finally, for each $M$ we have

$$\lim_{n\to\infty} \int_\Omega f_n^{(M)}\,d\mu = \int_\Omega g^{(M)}\,d\mu$$

by the bounded convergence theorem. The combination of these three facts finishes the proof. □

We now come to an important technical lemma that is often referred to as the *subsequence splitting lemma.* This lemma enables us to take an arbitrary bounded sequence in $L_1(\mu)$ and extract a subsequence that can be split into two sequences, the first disjointly supported and the second equi-integrable. It is due to Kadets and Pełczyński and provides a very useful bridge between sequence space methods (gliding hump techniques) and function spaces.

**Lemma 5.2.7 (Subsequence Splitting Lemma [147]).** *Let $(f_n)_{n=1}^\infty$ be a bounded sequence in $L_1(\mu)$. Then there exist a subsequence $(g_n)_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ and a sequence of disjoint measurable sets $(A_n)_{n=1}^\infty$ such that if $B_n = \Omega \setminus A_n$, then $(g_n\chi_{B_n})_{n=1}^\infty$ is equi-integrable.*

*Proof.* Without loss of generality we can assume $\|f_n\|_1 \le 1$ for all $n$. We will first find a subsequence $(f_{n_s})_{s=1}^\infty$ and a sequence of measurable sets $(F_s)_{s=1}^\infty$ such that if $E_s = \Omega \setminus F_s$, then $(f_{n_s}\chi_{E_s})_{s=1}^\infty$ is equi-integrable and $\lim_{s\to\infty} f_{n_s}\chi_{F_s} = 0$ $\mu$-a.e.

For every choice of $k \in \mathbb{N}$, Chebyshev's inequality gives

$$0 \le \mu(|f_n| > k) \le \frac{1}{k}, \qquad \forall\ n \in \mathbb{N}.$$

Since $(\mu(|f_n| > k))_{n=1}^\infty$ is a bounded sequence, by passing to a subsequence we can assume that $(\mu(|f_n| > k))_{n=1}^\infty$ converges for each $k$. Let us call $\alpha_k$ its limit. Our first goal is to see that the series $\sum_{k=1}^\infty \alpha_k$ is convergent with sum no bigger than 1.

For each $n$,

$$1 \ge \int_\Omega |f_n|\,d\mu = \int_0^\infty \mu(|f_n| > t)\,dt$$

$$= \sum_{k=1}^{\infty} \int_{k-1}^{k} \mu(|f_n| > t)\, dt$$
$$\geq \sum_{k=1}^{\infty} \mu(|f_n| > k).$$

Therefore the partial sums of $\sum_{k=1}^{\infty} \alpha_k$ are uniformly bounded:

$$\sum_{k=1}^{N} \alpha_k = \sum_{k=1}^{N} \lim_{n\to\infty} \mu(|f_n| > k) = \lim_{n\to\infty} \sum_{k=1}^{N} \mu(|f_n| > k) \leq 1.$$

For each $k$ we want to speed up the convergence of the sequence $\big(\mu(|f_n| > k)\big)_{n=1}^{\infty}$ to $\alpha_k$. Let us extract a subsequence $(f_{n_s})_{s=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ in such a way that for $s \in \mathbb{N}$,

$$\mu(|f_{n_s}| > k) < \alpha_k + 2^{-2s} \text{ if } 1 \leq k \leq 2^s. \tag{5.1}$$

For each $s$ put

$$E_s = \{\omega \in \Omega : |f_{n_s}(\omega)| \leq 2^s\} \quad \text{and} \quad F_s = \{\omega \in \Omega : |f_{n_s}(\omega)| > 2^s\}.$$

Notice that

$$\sum_{s=1}^{\infty} \mu(F_s) \leq \sum_{s=1}^{\infty} \frac{\|f_{n_s}\|_1}{2^s} \leq \sum_{s=1}^{\infty} \frac{1}{2^s} = 1.$$

This implies that for almost every $\omega \in \Omega$, there is just a finite number of sets such that $\omega \in F_s$. Thus $(f_{n_s}\chi_{F_s})_{s=1}^{\infty}$ converges to 0 $\mu$-a.e. Next we will prove that $(f_{n_s}\chi_{E_s})_{s=1}^{\infty}$ is equi-integrable. For the sake of simplicity in the notation, we will set $h_s = f_{n_s}\chi_{E_s}$. It suffices to show that

$$\sup_s \int_{\{|h_s|>2^r\}} |h_s|\, d\mu \overset{r\to\infty}{\longrightarrow} 0.$$

Clearly $\mu(|h_s| > k) = 0$ if $k > 2^s$, which implies that for fixed $r \in \mathbb{N}$, if $s < r$, then

$$\int_{\{|h_s|>2^r\}} |h_s|\, d\mu = 0.$$

For values of $s \geq r$,

$$\int_{\{|h_s|>2^r\}} |h_s|\, d\mu \leq \int_{\{|h_s|>2^r\}} \big(|h_s| - 2^r\big)\, d\mu + 2^r \mu(|h_s| > 2^r).$$

By (5.1) we have $2^r\mu(|h_s| > 2^r) \le 2^r\alpha_{2^r} + 2^{r-2s}$. On the other hand,

$$\begin{aligned}
\int_{\{|h_s|>2^r\}} \big(|h_s| - 2^r\big)\, d\mu &= \int_0^\infty \mu\big(|h_s| - 2^r > t\big)\, dt \\
&= \sum_{k=1}^\infty \int_{k-1}^k \mu\big(|h_s| - 2^r > t\big)\, dt \\
&\le \sum_{k=1}^\infty \mu\big(|h_s| - 2^r > k-1\big) \\
&= \sum_{k=0}^\infty \mu\big(|h_s| > 2^r + k\big) \\
&= \sum_{k=2^r}^{2^s} \mu\big(|h_s| > k\big) \\
&\le \sum_{k=2^r}^{2^s} (\alpha_k + 2^{-2s}) \\
&\le 2^{-r} + \sum_{k=2^r}^\infty \alpha_k.
\end{aligned}$$

Summing up, if $s \ge r$, we get

$$\int_{\{|h_s|>2^r\}} |h_s|\, d\mu \le 2\cdot 2^{-r} + 2^r\alpha_{2^r} + \sum_{k=2^r}^\infty \alpha_k \stackrel{r\to\infty}{\longrightarrow} 0.$$

This establishes the equi-integrability of $(h_s)_{s\in\mathbb{N}}$.

Note that $\lim_{s\to\infty}(f_{n_s} - h_s) = 0$ $\mu$-a.e. Thus we can apply Lemma 5.2.1 to the sequence $h_s' = f_{n_s} - h_s$ to deduce the existence of a further subsequence $(h_{s_r}')_{r=1}^\infty$ and a sequence of disjoint sets $(A_r)_{r=1}^\infty$ in $\Sigma$ such that $\lim_{r\to\infty}\|h_{s_r}'\chi_{B_r}\| = 0$, where $B_r = \Omega\setminus A_r$. Clearly we may assume that $A_r \subset F_{s_r}$. Then the set $\{h_{s_r}'\chi_{B_r}\}_{r=1}^\infty$ is equi-integrable, and so $\{h_{s_r} + h_{s_r}'\chi_{B_r}\}_{r=1}^\infty$ is also equi-integrable. If we write $g_r = f_{n_{s_r}}$, then the subsequence $(g_r)_{r=1}^\infty$ gives us the conclusion, since $g_r\chi_{B_r} = h_{s_r} + h_{s_r}'\chi_{B_r}$.

□

Now we come to our main result on weak compactness. The main equivalence, $(i) \Leftrightarrow (ii)$, in Theorem 5.2.8 is due to Dunford and Pettis [74].

**Theorem 5.2.8.** *Let $\mathcal{F}$ be a bounded set in $L_1(\mu)$. The following conditions on $\mathcal{F}$ are equivalent:*

*(i) $\mathcal{F}$ is relatively weakly compact;*
*(ii) $\mathcal{F}$ is equi-integrable;*

*(iii)* *$\mathcal{F}$ does not contain a basic sequence equivalent to the canonical basis of $\ell_1$;*
*(iv)* *$\mathcal{F}$ does not contain a complemented basic sequence equivalent to the canonical basis of $\ell_1$;*
*(v)* *for every sequence $(A_n)_{n=1}^{\infty}$ of disjoint measurable sets,*

$$\lim_{n\to\infty} \sup_{f\in\mathcal{F}} \int_{A_n} |f|\, d\mu = 0.$$

*Proof.* It is clear that $(i) \Rightarrow (iii)$, since the unit vector basis of $\ell_1$ contains no weakly convergent subsequences. Trivially, $(iii) \Rightarrow (iv)$; $(ii) \Rightarrow (v)$ is also immediate, since if $(A_n)$ are disjoint measurable sets, then $\mu(A_n) \to 0$, and so $\lim_{n\to 0} \sup_{f\in\mathcal{F}} \int_{A_n} |f|\, d\mu = 0$ by equi-integrability. We shall complete the circle by showing that $(iv) \Rightarrow (ii)$, $(v) \Rightarrow (ii)$, and $(ii) \Rightarrow (i)$.

If $(ii)$ fails, by Lemma 5.2.5 there exist a sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{F}$ and some $\delta > 0$ such that for each $n \in \mathbb{N}$,

$$\int_{\{|f_n|>n\}} |f_n|\, d\mu \geq \delta. \tag{5.2}$$

We may suppose, using Lemma 5.2.7 and passing to a subsequence, that every $f_n$ can be written as $f_n = f_n\chi_{A_n} + f_n\chi_{B_n}$, where $(A_n)_{n=1}^{\infty}$ is a sequence of disjoint sets in $\Sigma$, $B_n = \Omega \setminus A_n$, and $(f_n\chi_{B_n})_{n=1}^{\infty}$ is equi-integrable. Then observe that since $\mu(|f_n| > n) \to 0$, we must have

$$\lim_{n\to\infty} \int_{B_n\cap\{|f_n|>n\}} |f_n| d\mu = 0.$$

By deleting finitely many terms in the sequence $(f_n)_{n=1}^{\infty}$, we can assume that

$$a_n = \int_{A_n} |f_n| d\mu \geq \frac{1}{2}\delta, \qquad \forall\, n \in \mathbb{N}. \tag{5.3}$$

By Lemma 5.1.1, the sequence $(a_n^{-1} f_n\chi_{A_n})_{n=1}^{\infty}$ is a norm-one complemented basic sequence in $L_1(\mu)$ isometrically equivalent to the canonical $\ell_1$-basis. Let $(h_n)_{n=1}^{\infty}$ in $L_\infty(\mu)$ be the norm-one biorthogonal functionals chosen in the proof of Lemma 5.1.1; each $h_n$ is supported on $A_n$. Since $\mu A_n \to 0$ and the set $\{f\chi_{B_k}\}_{k=1}^{\infty}$ is equi-integrable, we can pass to yet a further subsequence and assume that

$$\int_{A_n\cap B_m} |f_m|\, d\mu < \frac{1}{4} 2^{-n}\delta, \qquad m, n \in \mathbb{N}.$$

Define $T : L_1(\mu) \to \ell_1$ by

$$Tf = \left(\int_\Omega fh_n d\mu\right)_{n=1}^\infty$$

and $R : \ell_1 \to L_1(\mu)$ by

$$R(\xi) = \sum_{n=1}^\infty \xi_n a_n^{-1} f_n.$$

Then

$$TRe_k - e_k = \left(a_k^{-1} \int_{A_n \cap B_k} f_k h_n d\mu\right)_{n=1}^\infty,$$

and we obtain the estimate

$$\left|a_k^{-1} \int_{A_n \cap B_k} f_k h_n d\mu\right| \leq 2^{-n-1}.$$

Hence

$$\|TRe_k - e_k\| \leq a_k^{-1} \sum_{n=1}^\infty \frac{1}{4}\delta 2^{-n} \leq \frac{1}{2},$$

which yields $\|TR - I\| \leq \frac{1}{2}$, where $I$ is the identity operator on $\ell_1$. This implies that $TR$ is invertible, so there is $U : \ell_1 \to \ell_1$ such that $UTR = I$. The mapping $RUT$ is a projection onto the range of $R$; hence $R$ maps $\ell_1$ isomorphically onto a complemented subspace of $L_1(\mu)$; this shows that $(f_n)_{n=1}^\infty$ is a complemented basic sequence equivalent to the $\ell_1$-basis. Thus $(iv)$ is contradicted, and so $(iv)$ implies $(ii)$.

Let us point out that equation (5.3), which we obtained with the sole assumption that $\mathcal{F}$ failed to be equi-integrable, contradicts $(v)$; hence on our way we also obtained the implication $(v) \Rightarrow (ii)$.

Finally, let us prove $(ii) \Rightarrow (i)$. We must show that $\overline{\mathcal{F}}^{w^*}$, the weak$^*$ closure of $\mathcal{F}$ in the bidual of $L_1(\mu)$, is contained in $L_1(\mu)$.

For $M \in (0, \infty)$ put

$$\mathcal{F}_M = \{f\chi_{\{|f|\leq M\}} : f \in \mathcal{F}\} \quad \text{and} \quad \mathcal{F}^M = \{f\chi_{\{|f|>M\}} : f \in \mathcal{F}\}.$$

It is obvious that $\mathcal{F} \subset \mathcal{F}_M + \mathcal{F}^M$; therefore, $\overline{\mathcal{F}}^{w^*} \subset \overline{\mathcal{F}_M}^{w^*} + \overline{\mathcal{F}^M}^{w^*}$. Let us notice that if $f \in \mathcal{F}_M$, we have

$$\|f\|_2 \leq \|f\|_\infty \leq M.$$

Then, $\mathcal{F}_M \subset MB_{L_2(\mu)}$. Since $L_2(\mu)$ is reflexive, its closed unit ball is weakly compact. Therefore $MB_{L_2(\mu)}$ is weakly compact for each $M > 0$, and so $\mathcal{F}_M$ is a relatively weakly compact set in $L_2(\mu)$. Being norm-to-norm continuous, the inclusion $\iota : L_2(\mu) \to L_1(\mu)$ is weak-to-weak continuous, so $\iota(\mathcal{F}_M) = \mathcal{F}_M$ is a relatively weakly compact set in $L_1(\mu)$ for each $M > 0$. This is equivalent to saying that for each positive $M$, the weak* closure of $\mathcal{F}_M$ in the bidual of $L_1(\mu)$ is a subset of $L_1(\mu)$, i.e.,

$$\overline{\mathcal{F}_M}^{w^*} \subset L_1(\mu) \quad \text{for all } M > 0.$$

On the other hand, if $f \in \mathcal{F}^M$, then $\|f\|_1 \leq \epsilon(M)$, where

$$\epsilon(M) = \sup_{f \in \mathcal{F}} \int_{\{|f| > M\}} |f| \, d\mu.$$

Hence, $\mathcal{F}^M \subset \epsilon(M) B_{L_1(\mu)}$. Using Goldstine's theorem, we deduce that

$$\overline{\mathcal{F}^M}^{w^*} \subset \epsilon(M) B_{L_1(\mu)^{**}}.$$

Hence if $f \in \overline{\mathcal{F}}^{w^*}$, then we can write $f = \psi + \phi$, with $\psi \in L_1(\mu)$ and $\phi \in \epsilon(M) B_{L_1(\mu)^{**}}$. Therefore, for an arbitrary $M > 0$,

$$d(f, L_1(\mu)) \leq \epsilon(M).$$

Since $\lim_{M\to\infty} \epsilon(M) = 0$ by Lemma 5.2.5, $d(f, L_1(\mu)) = 0$ and $f \in L_1(\mu)$. □

We conclude this section with a simple deduction from this theorem.

**Theorem 5.2.9.** *The space $L_1(\mu)$ is weakly sequentially complete.*

*Proof.* Let $(f_n)_{n=1}^{\infty} \subset L_1(\mu)$ be a weakly Cauchy sequence. Then, no subsequence of $(f_n)_{n=1}^{\infty}$ can be equivalent to the canonical $\ell_1$-basis, which is not weakly Cauchy. Hence the set $\{f_n\}_{n=1}^{\infty}$ is relatively weakly compact by Theorem 5.2.8, and this implies that the sequence must actually be weakly convergent. □

**Corollary 5.2.10.** *The space $c_0$ is not isomorphic to a subspace of $L_1(\mu)$.*

*Proof.* Since $L_1(\mu)$ is weakly sequentially complete, by Corollary 2.4.15 every WUC series in $L_1(\mu)$ is unconditionally convergent, so by Theorem 2.4.11, $L_1(\mu)$ does not contain a copy of $c_0$. □

## 5.3 Weak Compactness in $\mathcal{M}(K)$

Suppose now that $K$ is a compact Hausdorff space (not necessarily metrizable). The space $\mathcal{M}(K) = \mathcal{C}(K)^*$ as a Banach space is a "very large" $\ell_1$-sum of spaces $L_1(\mu)$, where $\mu$ is a probability measure on $K$. This fact has already been observed in the proof of Proposition 4.3.8 (*iii*). Using this, it is possible to extend Theorem 5.2.8 to the spaces $\mathcal{M}(K)$; however, we need some additional characterizations of weak compactness in spaces of measures.

**Definition 5.3.1.** A subset $\mathcal{A}$ of $\mathcal{M}(K)$ is said to be *uniformly regular* if given any open set $U \subset K$ and $\epsilon > 0$, there is a compact set $H \subset U$ such that $|\mu|(U \setminus H) < \epsilon$ for all $\mu \in \mathcal{A}$.

The next equivalences are due to Grothendieck [120].

**Theorem 5.3.2.** *Let $\mathcal{A}$ be a bounded subset of $\mathcal{M}(K)$. The following are equivalent:*

*(i)* *$\mathcal{A}$ is relatively weakly compact;*
*(ii)* *$\mathcal{A}$ is uniformly regular;*
*(iii)* *for every sequence of disjoint Borel sets $(B_n)_{n=1}^\infty$ in $K$ and every sequence of measures $(\mu_n)_{n=1}^\infty$ in $\mathcal{A}$, $\lim_{n\to\infty} |\mu_n|(B_n) = 0$;*
*(iv)* *for every sequence of disjoint open sets $(U_n)_{n=1}^\infty$ in $K$ and every sequence of measures $(\mu_n)_{n=1}^\infty$ in $\mathcal{A}$, $\lim_{n\to\infty} \mu_n(U_n) = 0$;*
*(iv)′* *for every sequence of disjoint open sets $(U_n)_{n=1}^\infty$ in $K$ and every sequence of measures $(\mu_n)_{n=1}^\infty$ in $\mathcal{A}$, $\lim_{n\to\infty} |\mu_n|(U_n) = 0$.*

*Proof.* $(iii) \Rightarrow (iv)$ This is immediate, because an open set is a Borel set and

$$0 \le |\mu_n(U_n)| \le |\mu_n|(U_n) \overset{n\to\infty}{\longrightarrow} 0.$$

$(iv) \Rightarrow (iv)'$ If $(iv)'$ fails, there exist a sequence of open sets $(U_n)_{n=1}^\infty$ in $K$ and a sequence of regular signed measures $(\mu_n)_{n=1}^\infty$ on $K$ such that $(|\mu_n|(U_n))_{n=1}^\infty$ does not converge to 0. For each $n$ we write $\mu_n = \mu_n^+ - \mu_n^-$ as the difference of its positive and negative parts. The total variation of $\mu_n$ is the sum $|\mu_n| = \mu_n^+ + \mu_n^-$. Therefore, without loss of generality, we will suppose that the sequence $(\mu_n^+(U_n))_{n=1}^\infty$ does not converge to 0. By passing to a subsequence, we can assume that there exists $\delta > 0$ such that $\mu_n^+(U_n) \ge \delta > 0$ for all $n$. Let us fix $n \in \mathbb{N}$. Using the Hahn decomposition theorem, there is a Borel set $B_n \subset U_n$ such that $\mu_n(B_n) = \mu_n^+(U_n) \ge \delta$. Now, by the regularity of $\mu_n$, there is an open set $O_n$ such that $B_n \subset O_n \subset U_n$ and $\mu_n(O_n) \ge \frac{\delta}{2}$. In this way, we have a sequence of disjoint open sets $(O_n)_{n=1}^\infty \subset K$ such that $(\mu_n(O_n))_{n=1}^\infty$ does not converge to 0, contradicting $(iv)$.

$(iv)' \Rightarrow (ii)$ Let us assume that $\mathcal{A}$ fails to be uniformly regular. Then there is an open set $U \subset K$ such that for some $\delta > 0$ we have

$$\sup_{\mu\in\mathcal{A}} |\mu|(U \setminus H) > \delta,$$

for all compact sets $H \subset K$. Given $H_0 = \emptyset$, pick $\mu_1 \in \mathcal{A}$ such that $|\mu_1|(U \setminus H_0) > \delta$. By regularity of the measure $\mu_1$ there exists a compact set $F_1 \subset U \setminus H_0$ such that $|\mu_1|(F_1) > \delta$. Using the $T_4$ separation property, we find an open set $V_1$ satisfying $F_1 \subset V_1 \subset \overline{V_1} \subset U \setminus H_0$. Given the compact set $H_1 = \overline{V_1}$, there is $\mu_2 \in \mathcal{A}$ such that $|\mu_2|(U \setminus H_1) > \delta$. By regularity of $\mu_2$ there exists a compact set $F_2 \subset U \setminus H_1$ such that $|\mu_2|(F_2) > \delta$, and the $T_4$ separation property yields an open set $V_2$ such that $F_2 \subset V_2 \subset \overline{V_2} \subset U \setminus H_1$. For the next step in this recurrence argument we would pick $H_2 = \overline{V_1} \cup \overline{V_2}$ and repeat the previous procedure. In this way, by induction we obtain a sequence of disjoint open sets $(V_n)_{n=1}^{\infty} \subset K$ and a sequence $(\mu_n)_{n=1}^{\infty} \subset \mathcal{A}$ such that $|\mu_n|(V_n) > \delta$ for all $n$, contradicting $(ii)$.

$(ii) \Rightarrow (i)$ By the Eberlein–Šmulian theorem it suffices to show that every sequence $(\mu_n)_{n=1}^{\infty} \subset \mathcal{A}$ is relatively weakly compact. Let us consider the (positive) finite measure on the Borel sets of $K$ given by

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} |\mu_n|.$$

Every $\mu_n$ is absolutely continuous with respect to $\mu$. By the Radon–Nikodym theorem, for each $n$ there exists a unique $f_n \in L_1(K, \mu)$ such that $d\mu_n = f_n \, d\mu$ and $\|\mu_n\| = \int_K |f_n| \, d\mu$. This provides an isometric isomorphism from $L_1(\mu)$ onto the closed subspace of $\mathcal{M}(K)$ consisting of the regular signed measures on $K$ that are absolutely continuous with respect to $\mu$. The isometry, in particular, takes each $f_n$ in $L_1(K, \mu)$ to $\mu_n$. Therefore we need only show that $(f_n)_{n=1}^{\infty}$ is equi-integrable in $L_1(K, \mu)$. If it is not, using $(ii)$ we find a sequence $(U_n)_{n=1}^{\infty}$ of open sets and $\epsilon > 0$ such that $\mu(U_n) < 2^{-n}$ and $\sup_k \int_{U_n} |f_k| \, d\mu > \epsilon$. For each $n$ put $V_n = \bigcup_{k>n} U_k$. The sets $(V_n)_{n=1}^{\infty}$ form a decreasing sequence of open sets such that $\mu(V_n) < 2^{-n}$ and

$$\sup_k \int_{V_n} |f_k| \, d\mu > \epsilon. \tag{5.4}$$

Now, for each $n$ there exists $E_n$ compact, $E_n \subset V_n$, for which

$$\sup_k \int_{V_n \setminus E_n} |f_k| \, d\mu < \frac{\epsilon}{2^{n+2}}.$$

Obviously, $\mu(\bigcap_{n=1}^{\infty} E_n) = 0$. The uniform regularity yields an open set $W$ such that $\bigcap_{n=1}^{\infty} E_n \subset W$ and $\sup_k \int_W |f_k| \, d\mu < \frac{\epsilon}{2}$. By compactness there exists $N$ such that $\bigcap_{n=1}^{N} E_n \subset W$, and so

$$\int_{\bigcap_{n=1}^{N} E_n} |f_k| \, d\mu < \frac{\epsilon}{2} \quad \text{for each } k.$$

Thus, for each $k$ we have

$$\int_{V_{N+1}} |f_k|\,d\mu \le \int_{\bigcap_{n=1}^{N} E_n} |f_k|\,d\mu + \sum_{n=1}^{N} \int_{V_n \setminus E_n} |f_k|\,d\mu < \frac{\epsilon}{2} + \sum_{k=1}^{N} \frac{\epsilon}{2^{k+2}} < \epsilon,$$

which contradicts (5.4).

$(i) \Rightarrow (iii)$ Let $(B_n)_{n=1}^{\infty}$ be an arbitrary sequence of disjoint Borel sets in $K$ and let $(\mu_n)_{n=1}^{\infty}$ be an arbitrary sequence of measures in $\mathcal{A}$. Put

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} |\mu_n|.$$

Reasoning as we did in the previous implication, for each $n$ there exists a unique $g_n \in L_1(K,\mu)$ such that $d\mu_n = g_n\,d\mu$. If $\mathcal{A}$ is relatively weakly compact in $\mathcal{M}(K)$, the sequence $(g_n)_{n=1}^{\infty}$ is relatively weakly compact in $L_1(K,\mu)$, hence equi-integrable. Thus, since $\mu(B_n) \to 0$, we have

$$|\mu_n|(B_n) = \int_{B_n} |g_n|\,d\mu \to 0.$$

$\square$

*Remark 5.3.3.* This theorem is true for either real or complex scalars. We gave the proof in the real case, but it is easy to extend this to the complex case by the simple procedure of splitting a complex measure into real and imaginary parts.

## 5.4 The Dunford–Pettis Property

**Definition 5.4.1.** Let $X$ and $Y$ be Banach spaces. A bounded linear operator $T : X \to Y$ is *completely continuous* or a *Dunford–Pettis* operator if whenever $W$ is a weakly compact subset of $X$, then $T(W)$ is a norm-compact subset of $Y$.

Clearly, if an operator is compact, then it is Dunford–Pettis. If $X$ is reflexive, then an operator $T: X \to Y$ is compact if and only if $T$ is Dunford–Pettis.

**Proposition 5.4.2.** *Suppose that $X$ and $Y$ are Banach spaces. A linear operator $T: X \to Y$ is Dunford–Pettis if and only if $T$ is weak-to-norm sequentially continuous, i.e., whenever $(x_n)_{n=1}^{\infty} \subset X$ converges to $x$ weakly, then $(Tx_n)_{n=1}^{\infty}$ converges to $Tx$ in norm.*

*Proof.* Let $T : X \to Y$ be Dunford–Pettis and suppose that there is a weakly null sequence $(x_n)_{n=1}^{\infty} \subset X$ such that $\|Tx_n\| \ge \delta > 0$ for some positive $\delta$. Since the subset $W = \{x_n : n \in \mathbb{N}\} \cup \{0\}$ is weakly compact, its image under $T$ is norm-compact, and therefore it contains a subsequence $(T(x_{n_k}))_{k=1}^{\infty}$ that converges in norm to some

$y \in Y$. From the fact that $T$ is in particular weak-to-weak continuous, it follows that the sequence $(T(x_n))_{n=1}^{\infty}$ is weakly null, so $y$ must be 0, which contradicts our assumption.

For the converse implication, suppose $T$ is weak-to-norm sequentially continuous. Let $W$ be a weakly compact subset of $X$ and let $(y_n)_{n=1}^{\infty}$ be a sequence in $T(W)$. Pick $(x_n)$ in $X$ such that $y_n = Tx_n$ for all $n$. By the Eberlein–Šmulian theorem $(x_n)$ contains a subsequence $(x_{n_k})$ that converges weakly to some $x$ in $W$. Hence $(y_{n_k})_{k=1}^{\infty}$ converges in norm to $Tx$. We conclude that $T(W)$ is norm-compact. □

The following definition was introduced by Grothendieck [120] as an abstraction of ideas originally developed by Dunford and Pettis [74].

**Definition 5.4.3.** A Banach space $X$ is said to have the *Dunford–Pettis property* (or, in short, *$X$ has (DPP)*) if every weakly compact operator $T$ from $X$ into a Banach space $Y$ is Dunford–Pettis.

*Example 5.4.4.* (a) The space $c_0$ has (DPP), because if $Y$ is a Banach space and $T : c_0 \to Y$ is a weakly compact operator, then $T$ is compact (Theorem 2.4.10), hence Dunford–Pettis.
(b) The space $\ell_1$ has also (DPP), because $\ell_1$ has the Schur property, which implies, as we saw, that weakly compact subsets in $\ell_1$ are actually compact.
(c) No infinite-dimensional reflexive Banach space $X$ has (DPP), since the identity operator $I_X: X \to X$ is weakly compact but cannot be a Dunford–Pettis operator because the closed unit ball of $X$ is not compact.

**Theorem 5.4.5.** *Suppose that $X$ is a Banach space. Then $X$ has (DPP) if and only if for every sequence $(x_n)_{n=1}^{\infty}$ in $X$ converging weakly to 0 and every sequence $(x_n^*)_{n=1}^{\infty}$ in $X^*$ converging weakly to 0, the sequence of scalars $(x_n^*(x_n))_{n=1}^{\infty}$ converges to 0.*

*Proof.* Let $Y$ be a Banach space and $T : X \to Y$ a weakly compact operator. Let us suppose that $T$ is not Dunford–Pettis. Then there is $(x_n)_{n=1}^{\infty}$ in $X$ such that $x_n \overset{w}{\to} 0$ but $\|Tx_n\| \geq \delta > 0$ for all $n$. Pick $(y_n^*)_{n=1}^{\infty} \subset Y^*$ such that $y_n^*(Tx_n) = \|Tx_n\|$ and $\|y_n^*\| = 1$ for all $n$. By Gantmacher's theorem $T^*$ is weakly compact; hence $T^*(B_{Y^*})$ is a relatively weakly compact subset of $X^*$. By the Eberlein–Šmulian theorem the sequence $(T^*y_n^*)_{n=1}^{\infty} \subset T^*(B_{Y^*})$ can be assumed weakly convergent to some $x^*$ in $X^*$. Then $(T^*y_n^* - x^*)_{n=1}^{\infty}$ is weakly convergent to 0, which implies $(T^*y_n^* - x^*)(x_n) \to 0$. But since $x^*(x_n) \to 0$, it would follow that $(T^*y_n^*(x_n))_{n=1}^{\infty} = (\|Tx_n\|)_{n=1}^{\infty}$ must converge to 0, which is absurd.

For the converse, let $(x_n)_{n=1}^{\infty}$ in $X$ be such that $x_n \overset{w}{\to} 0$ and let $(x_n^*)_{n=1}^{\infty}$ in $X^*$ be such that $x_n^* \overset{w}{\to} 0$. Consider the operator

$$T: X \longrightarrow c_0, \qquad T(x) = (x_n^*(x))_{n=1}^{\infty}.$$

The adjoint operator $T^*$ of $T$ satisfies $T^*(e_k) = x_k^*$ for all $k \in \mathbb{N}$, where $(e_k)_{k=1}^{\infty}$ denotes the canonical basis of $\ell_1$. This implies that $T^*(B_{\ell_1})$ is contained in the

convex hull of the weakly null sequence $(x_n^*)_{n=1}^\infty$. Therefore $T^*$ is weakly compact, hence by Gantmacher's theorem so is $T$.

Since $T$ is weakly compact, $T$ is also Dunford–Pettis by the hypothesis. Then, by Proposition 5.4.2, $\|Tx_n\|_\infty \to 0$. Thus $(x_n^*(x_n))_{n=1}^\infty$ converges to 0, since

$$|x_n^*(x_n)| \le \max_k |x_k^*(x_n)| = \|Tx_n\|_\infty$$

for all $n$. □

We now reach the main result of the chapter. The fact that $L_1(\mu)$-spaces have (DPP) is due to Dunford and Pettis [74] (at least for the case that $L_1(\mu)$ is separable) and to Phillips [252]. The case of $\mathcal{C}(K)$-spaces was covered by Grothendieck in [120].

**Theorem 5.4.6 (The Dunford–Pettis Theorem).**

*(i) If $\mu$ is a $\sigma$-finite measure, then $L_1(\mu)$ has (DPP).*
*(ii) If $K$ is a compact Hausdorff space, then $\mathcal{C}(K)$ has (DPP).*

*Proof.* Let us first prove part (*ii*). Take any weakly null sequence $(f_n)_{n=1}^\infty$ in $\mathcal{C}(K)$ and any weakly null sequence $(\mu_n)_{n=1}^\infty$ in $\mathcal{M}(K)$. Without loss of generality both sequences can be assumed to lie inside the unit balls of the respective spaces. Define the (positive) measure $\nu = \sum_{n=1}^\infty \frac{1}{2^n}|\mu_n|$. The Radon–Nikodym theorem provides an isometry from $L_1(\nu)$ onto the closed subspace of $\mathcal{M}(K)$ consisting of the regular signed measures on $K$ that are absolutely continuous with respect to $\nu$. Clearly each $\mu_n$ is absolutely continuous with respect to $\nu$; hence for each $n$ there exists $g_n \in L_1(\nu)$ such that $d\mu_n = g_n\,d\nu$. Since the sequence $(\mu_n)_{n=1}^\infty$ is weakly null in $\mathcal{M}(K)$, it follows that $(g_n)_{n=1}^\infty$ is weakly null in $L_1(\nu)$. Thus the set $\{g_n : n \in \mathbb{N}\}$ is relatively weakly compact in $L_1(\nu)$, hence equi-integrable by Theorem 5.2.8.

Note that since $(f_n)_{n=1}^\infty$ is weakly null in $\mathcal{C}(K)$, it converges to 0 pointwise. Now for every $M > 0$, by the bounded convergence theorem, we have that

$$\lim_{n\to\infty} \int_{|g_n|\le M} f_n g_n\,d\nu = 0.$$

Hence

$$\left|\limsup_{n\to\infty} \int_K f_n g_n\,d\nu\right| \le \sup_n \int_{|g_n|>M} |g_n|\,d\nu.$$

Note that the right-hand-side term tends to zero as $M \to \infty$ by Lemma 5.2.5. Then

$$\lim_{n\to\infty} \int f_n\,d\mu_n = 0,$$

as required.

(*i*) follows from (*ii*), since the dual space of $L_1(\mu)$, $L_\infty(\mu)$, can be regarded as a $\mathcal{C}(K)$-space for a suitable compact Hausdorff space $K$. Hence if $(f_n)_{n=1}^\infty$ is weakly null in $L_1(\mu)$ and $(g_n)_{n=1}^\infty$ is weakly null in $L_\infty(\mu)$, then $\lim_{n\to\infty} \int f_n g_n \, d\mu = 0$ by the preceding argument. □

**Corollary 5.4.7.** *If $K$ is a compact Hausdorff space then $\mathcal{M}(K)$ has (DPP).*

The Dunford–Pettis theorem was a remarkable achievement in the early history of Banach spaces. The motivation of Dunford and Pettis came from the study of integral equations, and their hope was to develop an understanding of linear operators $T : L_p(\mu) \to L_p(\mu)$ for $p \geq 1$. In fact, the Dunford–Pettis theorem immediately gives the following application.

**Theorem 5.4.8.** *Let $T: L_1(\mu) \to L_1(\mu)$ or $T: \mathcal{C}(K) \to \mathcal{C}(K)$ be a weakly compact operator. Then $T^2$ is compact.*

*Proof.* This is immediate. For example, in the first case, $T(B_{L_1(\mu)})$ is relatively weakly compact; hence $T^2(B_{L_1(\mu)})$ is relatively norm-compact. □

It is well known that compact operators have very nice spectral properties. For instance, every nonzero $\lambda$ in the spectrum is an eigenvalue, and the only possible accumulation point of the spectrum is 0. These properties extend in a very simple way to an operator whose square is compact, so the previous result means that weakly compact operators on $L_1(\mu)$-spaces or $\mathcal{C}(K)$-spaces have similar properties. The Dunford–Pettis theorem was thus an important step in the development of the theory of linear operators in the first half of the twentieth century; this theory reached its apex in the publication of a three-volume treatise by Dunford and Schwartz between 1958 and 1971 [75–77]. The first of these volumes alone runs to more than 1000 pages!

## 5.5 The Emergence of the Radon–Nikodym Property

The original proof of Dunford and Pettis relied heavily on the theory of representations for operators on $L_1$. In order to study an operator $T: L_1(\mu) \to X$ one can associate it to a *vector measure* $\nu: \Sigma \to X$ given by $\nu(E) = T\chi_E$. Thus $\|\nu(E)\| \leq \|T\|\mu(E)$. Dunford and Pettis [74], and Phillips [252] showed that if $T$ is weakly compact, one can prove a vector-valued Radon–Nikodym theorem and thus produce a Bochner integrable function $g: \Omega \to X$ such that

$$\mu(E) = \int_E g(\omega) \, d\mu(\omega).$$

This permits a representation for the operator $T$ in the form

$$Tf = \int g(\omega) f(\omega) d\mu(\omega),$$

and they established the Dunford–Pettis theorem from this representation. This was the springboard for the definition of the *Radon–Nikodym property* for Banach spaces, which led to a remarkable theory developed largely between 1965 and 1980.

**Definition 5.5.1.** A Banach space $X$ has the *Radon–Nikodym property*, (RNP) for short, provided every operator from $L_1$ into $X$ is representable, i.e., for every bounded linear operator $T\colon L_1[0,1] \to X$ there is a bounded and strongly measurable function $g\colon [0,1] \to X$ such that

$$T(f) = \int_0^1 f(s)g(s)\,ds, \quad f \in L_1[0,1].$$

The Radon–Nikodym property appears in a variety of contexts within Banach space theory and has many equivalent characterizations involving vector-valued measures, the convergence of vector-valued martingales, or the extremal structure of closed bounded convex sets, to name just a few. Most important from our point of view, especially with an eye to Chapter 14, is the connection with the existence of derivatives of $X$-valued Lipschitz maps (see Definition 14.1.5). Recall that a map $f\colon \mathbb{R} \to X$ with values in a Banach space is differentiable at a point $t$ if the limit

$$\lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

exists in the norm topology of $X$. In this case it is called the *derivative of f at t* and is denoted by $f'(t)$.

**Theorem 5.5.2.** *A Banach space X has the Radon–Nikodym property if and only if every Lipschitz map from the unit interval* $[0,1]$ *into X is differentiable almost everywhere.*

*Proof.* Assume every Lipschitz map on $[0,1]$ is differentiable almost everywhere. Given a bounded linear operator $T\colon L_1[0,1] \to X$, consider

$$f\colon [0,1] \to X, \quad t \mapsto f(t) = T(\chi_{[0,t]}).$$

The map $f$ is Lipschitz, so its derivative $f'$ is bounded and strongly measurable; hence $f'$ is Bochner integrable. Therefore, for every functional $x^* \in X^*$ and every $t \in [0,1]$,

$$\begin{aligned} x^*(T(\chi_{[0,t]})) = x^* \circ f(t) &= \int_0^t (x^* \circ f)'(s)\,ds = x^*\left(\int_0^t f'(s)\,ds\right) \\ &= x^*\left(\int_0^t f'(s)\chi_{[0,t]}(s)\,ds\right). \end{aligned}$$

Consequently,

$$T(g) = \int_0^1 f'(s)g(s)\,ds,$$

for all $g$ in the set $\{\chi_{[0,t]} : t \in [0,1]\}$. By linearity and continuity, this identity extends to every $g$ in $L_1[0,1]$.

Conversely, assume that every operator mapping $L_1[0,1]$ into $X$ is representable. Denote by $\mathcal{S}$ the linear span of $\{\chi_{[0,t]} : t \in [0,1]\}$. Every function in $\mathcal{S}$ has (essentially) a unique representation in the form $\sum_{i=1}^N a_i\chi_{(s_{i-1},s_i]}$, where $(a_i)_{i=1}^N$ are scalars and $0 = s_0 < s_1 < \cdots < s_N = 1$. Let $f : [0,1] \to X$ be a Lipschitz map. Define

$$T : \mathcal{S} \to X, \quad \sum_{i=1}^N a_i\chi_{(s_{i-1},s_i]} \mapsto \sum_{i=1}^N a_i(f(s_i) - f(s_{i-1})).$$

The operator $T$ is linear and bounded; hence it extends univocally to a bounded linear operator from $L_1$ into $X$. Using the hypothesis, $T$ is representable via a strongly measurable bounded function $g : [0,1] \to X$. Then,

$$\frac{f(u) - f(t)}{u - t} = \frac{1}{u - t}\int_t^u g(s)\,ds, \quad 0 \le t < u \le 1.$$

By the Lebesgue differentiation theorem there exists $f'(t) = g(t)$ a.e. on $[0,1]$. □

In 1936, Clarkson introduced the class of uniformly convex Banach spaces and provided a link between geometry and differentiation by proving that if $X$ is a uniformly convex Banach space, then every Lipschitz map from $[0,1]$ into $X$ is differentiable almost everywhere [50]. He also noted that $\ell_1$ enjoys this property, while $c_0$ and $L_1[0,1]$ do not.

*Example 5.5.3.* Neither of the spaces $L_1$ nor $c_0$ has the Radon–Nikodym property. Indeed, in $L_1$ the Lipschitz map

$$f(t) = \chi_{(0,t)}, \qquad 0 \le t \le 1,$$

is nowhere differentiable. In $c_0$ we can consider the Lipschitz map

$$g(t) = \Big(\frac{1}{n}\sin(nt)\Big)_{n=1}^\infty, \qquad 0 \le t \le 1,$$

which is again nowhere differentiable (note that formally, differentiating takes us into the bidual!).

Moreover, since (RNP) is inherited by subspaces, a space with (RNP) has no subspace isomorphic to $c_0$ or $L_1$.

In the same issue of the *Transactions of the American Mathematical Society* that Clarkson's paper appeared, Dunford and Morse [73] extended Clarkson observation about $\ell_1$ by showing that Lipschitz maps on the real line taking values in a Banach space with a boundedly complete basis (see Definition 3.2.12) are differentiable almost everywhere and can be recovered through the integrals of their derivatives.

**Theorem 5.5.4 (Dunford and Morse [73]).** *Suppose $X$ is a Banach space with a boundedly complete basis. If $f\colon [a,b] \to X$ is a Lipschitz map, then $f$ is differentiable almost everywhere, the derivative $f'(t)$ is Bochner integrable, and*

$$f(t) = f(a) + \int_a^t f'(s)\, ds, \qquad t \in [a,b].$$

*Proof.* Suppose $(e_n)_{n=1}^\infty$ is a boundedly complete basis in $X$ with biorthogonal functionals $(e_n^*)_{n=1}^\infty$. Let $S_N(x) = \sum_{n=1}^N e_n^*(x)e_n$, $x \in X$, be the $N$th partial sum projection associated to the basis and put $\mathsf{K}_\mathsf{b} = \sup_N \|S_N\|$ for the basis constant of $(e_n)_{n=1}^\infty$. Let $f\colon [0,1] \to X$ be a Lipschitz map, and without loss of generality assume $f(0) = 0$. For each $n \in \mathbb{N}$ there exist $A_n \subset I$ with $|A_n| = 0$ such that the Lipschitz map $f_n = e_n^* \circ f\colon [0,1] \to \mathbb{R}$ is differentiable for all $s \in I \setminus A_n$. Let $f_n'\colon [0,1] \setminus A_n \to \mathbb{R}$ be the derivative of $f_n$. The set $A = \cup_{n=1}^\infty A_n$ satisfies $|A| = 0$, and if $s \in [0,1] \setminus A$, then

$$\left\| \sum_{n=1}^N f_n'(s)e_n \right\| = \|(S_N \circ f)'(s)\| \le \operatorname{Lip}(S_N \circ f) \le \mathsf{K}_\mathsf{b} \operatorname{Lip}(f),$$

where $\operatorname{Lip}(S_N \circ f)$ and $\operatorname{Lip}(f)$ denote the Lipschitz constants (see Definition 14.1.5) of $S_N \circ f$ and $f$, respectively. Since the basis $(e_n)_{n=1}^\infty$ is boundedly complete, for $s \in [0,1] \setminus A$ the series $\sum_{n=1}^\infty f_n'(s)e_n$ converges in $X$ to a vector with norm smaller than $\mathsf{K}_\mathsf{b} \operatorname{Lip}(f)$. Let $g$ be the function defined by the sum of this series, i.e.,

$$g(s) = \sum_{n=1}^\infty f_n'(s)e_n, \qquad s \in [0,1] \setminus A.$$

Now that we have a candidate for the derivative of $f$, we just need to show that $g(s) = f'(s)$ for all $s \in [0,1] \setminus A$.

The function $g$ is bounded and strongly measurable, hence Bochner integrable. Let $G\colon [0,1] \to X$ be the function

$$G(t) = \int_0^t g(u)\, du.$$

For every $n \in \mathbb{N}$ and all $t \in [0,1]$ we have

$$e_n^*(G(t)) = \int_0^t e_n^* \circ g(u)\, du = \int_0^t (e_n^* \circ f)'(u)\, du = e_n^*(f(t)).$$

Therefore $G = f$, and we finish the proof with the aid of the Lebesgue differentiation theorem for the Bochner integral (see Appendix K). □

*Example 5.5.5.* (a) The space $\ell_p$ for $1 \leq p < \infty$ has the Radon–Nikodym property, since its unit vector basis is boundedly complete (Example 3.2.13).

(b) The space $L_p$ for $1 < p < \infty$ has the Radon–Nikodym property, because by James' theorem (Theorem 3.2.19), the Haar system is a boundedly complete basis.

These two classical examples of spaces with (RNP) can also be deduced from the following general result, originally shown by Dunford and Pettis [74].

**Proposition 5.5.6 (Dunford and Pettis [74]).**

*(a) Separable dual spaces have the Radon–Nikodym property.*
*(b) Reflexive spaces have the Radon–Nikodym property.*

*Proof.* (a) Let $Z$ be a Banach space such that $Z$ is the dual $X^*$ of some Banach space $X$. Suppose $f\colon \mathbb{R} \to Z = X^*$ is Lipschitz with $f(0) = 0$. Pick a sequence $(h_n)_{n=1}^{\infty}$ of positive scalars converging to 0. Given $t \in \mathbb{R}$, the sequence $(h_n^{-1}(f(t+h_n)-f(t)))_{n=1}^{\infty}$ is bounded by Lip($f$), the Lipschitz constant of $f$. By the Banach–Alaoglu theorem,

$$\exists g(t) := \lim_{\mathcal{U}} \frac{f(t+h_n) - f(t)}{h_n}$$

in the weak$^*$ sense, where $\mathcal{U}$ is a free ultrafilter over $\mathbb{N}$. Moreover,

$$\|g(t)\| \leq \mathrm{Lip}(f), \qquad t \in \mathbb{R}.$$

For $x \in X$ fixed, the map

$$f_x := f(\cdot)(x)\colon \mathbb{R} \to \mathbb{R}, \qquad t \mapsto f_x(t) = f(t)(x),$$

is Lipschitz, hence differentiable almost everywhere. Consequently,

$$\begin{aligned} g(t)(x) &= \lim_{\mathcal{U}} \frac{f(t+h_n)(x) - f(t)(x)}{h_n} \\ &= \lim_{n} \frac{f(t+h_n)(x) - f(t)(x)}{h_n} \\ &= f_x'(t), \end{aligned}$$

a.e. $t \in \mathbb{R}$. We infer that the function

$$g(\cdot)(x)\colon \mathbb{R} \to \mathbb{R}, \qquad t \mapsto g(t)(x),$$

is Lebesgue measurable. Hence, if for $x \in X$ and $a \in \mathbb{R}$ we put

$$A_{x,a} = \{x^* \in X^* : x^*(x) > a\},$$

we have that the set $g^{-1}(A_{x,a}) = \{t \in \mathbb{R} : g(t)(x) > a\}$ is Lebesgue measurable. But since the unit ball of $X^*$ is separable, the $\sigma$-algebra of Borel sets of $X^*$ coincides with the $\sigma$-algebra generated by $\{A_{x,a} : x \in X, a \in \mathbb{R}\}$. Thus $g^{-1}(A)$ is Lebesgue measurable for every Borel set $A \subset X^*$. Since $X^*$ is separable, $g$ is strongly measurable and so Bochner integrable in every compact interval. Define

$$G(t) = \int_0^t g(s)\, ds, \quad t \in \mathbb{R}.$$

We have

$$G(t)(x) = \int_0^t g(s)(x)\, ds = \int_0^t f_x'(s)\, ds = f(t)(x), \quad t \in \mathbb{R}, x \in X.$$

That is, $G = f$. Observe that now the Lebesgue differentiation theorem yields $f'(t) = G'(t) = g(t)$ almost everywhere, and the proof is complete.

(b) Let $f : [0, 1] \to X$ be a Lipschitz map. Consider a separable subspace $Y$ of $X$ such that $f(t) \in Y$ for all $t \in [0, 1]$. We know that $Y$ is reflexive, since reflexivity is inherited by subspaces. In this way, $Y = (Y^*)^*$ is a separable dual space, and we just need to use part (a) to conclude that $f$ is differentiable almost everywhere. □

We will not pursue this direction further in this book; for a very nice account of this theory we refer to the monograph [62] of Diestel and Uhl from 1977. One of the surprising aspects of this theory is the connection between the Radon–Nikodym property and the *Krein–Milman property*, or *(KMP)* for short. A Banach space $X$ has (KMP) if every closed bounded (not necessarily compact!) convex set is the closed convex hull of its extreme points. Obviously, reflexive spaces have (KMP), but remarkably, every space with (RNP) has (KMP) (Lindenstrauss [192]). The converse remains the major open problem in this area; the best results in this direction are due to Phelps [251] and Schachermayer [277]. It is probably fair to say that the subject has received relatively little attention since the 1980s, and some really new ideas seem to be necessary to make further progress.

## 5.6 Weakly Compact Operators on $\mathcal{C}(K)$-Spaces

Let us refer again to Theorem 2.4.10. In that theorem it was shown that for operators $T : c_0 \to X$ the properties of being weakly compact, compact, or strictly singular are equivalent. For general $\mathcal{C}(K)$-spaces we have seen that weak compactness implies Dunford–Pettis. Next we turn to strict singularity.

**Theorem 5.6.1.** *Let $K$ be a compact Hausdorff space. If $T : \mathcal{C}(K) \to X$ is weakly compact, then $T$ is strictly singular.*

*Proof.* Let $Y$ be a subspace of $\mathcal{C}(K)$ such that $T|_Y$ is an isomorphism onto its image. Since $T$ is weakly compact, $T(B_Y)$ is relatively weakly compact, which implies that $B_Y$ is weakly compact. But $T(B_Y)$ is actually compact by the Dunford–Pettis theorem, Theorem 5.4.6. It follows that $Y$ is finite-dimensional. □

*Remark 5.6.2.* Clearly, Theorem 5.6.1 also holds if we replace $\mathcal{C}(K)$ by $L_1(\mu)$.

The following result is a much more precise statement than Theorem 5.6.1.

**Theorem 5.6.3 (Pełczyński [243]).** *Suppose $T: \mathcal{C}(K) \to X$ is a bounded linear operator, where $K$ is a Hausdorff compact. If $T$ fails to be weakly compact, there is a closed subspace $E$ of $\mathcal{C}(K)$ isomorphic to $c_0$ such that $T|_E$ is an isomorphism.*

*Proof.* Suppose that $T: \mathcal{C}(K) \to X$ fails to be weakly compact. Then, by Gantmacher's theorem, its adjoint operator $T^* : X^* \to \mathcal{M}(K)$ also fails to be weakly compact, and so the subset $T^*(B_{X^*})$ of $\mathcal{M}(K)$ is not relatively weakly compact. By Theorem 5.3.2, there exist $\delta > 0$, a disjoint sequence of open sets $(U_n)_{n=1}^{\infty}$ in $K$, and a sequence $(x_n^*)_{n=1}^{\infty}$ in $B_{X^*}$ such that if we set $\nu_n = T^* x_n^*$, then $\nu_n(U_n) > \delta$ for all $n$.

For each $n$ there exists a compact subset $F_n$ of $U_n$ such that $|\nu|(U_n \setminus F_n) < \frac{\delta}{2}$. By Urysohn's lemma there exists $f_n \in \mathcal{C}(K)$, $0 \leq f_n \leq 1$, such that $f_n = 0$ on $K \setminus U_n$ and $f_n = 1$ on $F_n$. Then $(f_n)_{n=1}^{\infty}$ is isometrically equivalent to the canonical basis of $c_0$, which implies that $[f_n]$, the closed linear span of the basic sequence $(f_n)_{n=1}^{\infty}$, is isometrically isomorphic to $c_0$. Let $S : c_0 \to \mathcal{C}(K)$ be the isometric embedding defined by $Se_n = f_n$, where $(e_n)_{n=1}^{\infty}$ is the canonical basis of $c_0$.

Consider $TS : c_0 \to X$. We claim that $TS$ cannot be compact. Indeed, since $(e_n)_{n=1}^{\infty}$ is weakly null, if $TS$ were, compact we would have $\lim_n \|TSe_n\| = 0$. However,

$$\begin{aligned}
x_n^*(TSe_n) &= x_n^*(Tf_n) \\
&= (T^* x_n^*)(f_n) \\
&= \int_K f_n \, d\nu_n \\
&= \int_{U_n} d\nu_n + \int_{U_n} (f_n - 1) \, d\nu_n \\
&\geq \delta - |\nu_n|(U_n \setminus E_n) \\
&\geq \frac{\delta}{2}.
\end{aligned}$$

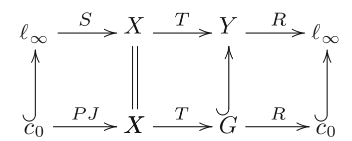
Thus $TS$ is not compact, and by Theorem 2.4.10, it is also not strictly singular. In fact, $TS$ must be an isomorphism on a subspace isomorphic to $c_0$ (Proposition 2.2.1). □

**Corollary 5.6.4.** *Let $X$ be a Banach space such that no closed subspace of $X$ is isomorphic to $c_0$. Then every operator $T : \mathcal{C}(K) \to X$ is weakly compact.*

Using the above theorem, we can now say a little bit more about injective Banach spaces.

**Theorem 5.6.5.** *Suppose $X$ is an injective Banach space and $T: X \to Y$ is a bounded linear operator. If $T$ fails to be weakly compact, then there is a closed subspace $F$ of $X$ such that $F$ is isomorphic to $\ell_\infty$ and $T|_F$ is an isomorphism.*

*Proof.* We start by embedding $X$ isometrically into an $\ell_\infty(\Gamma)$-space; this can be done by taking $\Gamma = B_{X^*}$ and using the embedding $x \mapsto \hat{x}$, where $\hat{x}(x^*) = x^*(x)$. Since $X$ is injective, there is a projection $P : \ell_\infty(\Gamma) \to X$. Now the operator $TP : \ell_\infty(\Gamma) \to Y$ is not weakly compact; since $\ell_\infty(\Gamma)$ can be represented as a $\mathcal{C}(K)$-space, we can find a subspace $E$ of $\ell_\infty(\Gamma)$ that is isomorphic to $c_0$ and such that $TP|_E$ is an isomorphism. Let $J: c_0 \to E$ be any isomorphism. Since $X$ is injective, we can find a bounded linear extension $S : \ell_\infty \to X$ of the operator $PJ : c_0 \to X$. Note also that $TPJ$ maps $c_0$ isomorphically onto a subspace $G$ of $Y$, and thus using the fact that $\ell_\infty$ is injective, we can find a bounded linear operator $R : Y \to \ell_\infty$ that extends the operator $(TPJ)^{-1} : G \to c_0$. Thus we have the following commutative diagram:



The operator in the second row, namely $RTPJ$, is the identity operator $I$ on $c_0$, and $RTS : \ell_\infty \to \ell_\infty$ is an extension. Thus the operator $RTS - I$ on $\ell_\infty$ vanishes on $c_0$. We can now refer to Theorem 2.5.4 to deduce the existence of a subset $\mathbb{A}$ of $\mathbb{N}$ such that $RTS - I$ vanishes on $\ell_\infty(\mathbb{A})$. In particular, $RTS$ is an isomorphism from $\ell_\infty(\mathbb{A})$ onto its range. This requires that $F = S(\ell_\infty)$ be isomorphic to $\ell_\infty$, and $T|_F$ is an isomorphism. □

## 5.7 Subspaces of $L_1(\mu)$-Spaces and $\mathcal{C}(K)$-Spaces

Our first result in this section is a direct application of Theorem 5.4.8.

**Proposition 5.7.1.** *$L_1(\mu)$ and $\mathcal{C}(K)$ have no infinite-dimensional complemented reflexive subspaces.*

**Proposition 5.7.2.** *If $X$ is a nonreflexive subspace of $L_1(\mu)$, then $X$ contains a subspace isomorphic to $\ell_1$ and complemented in $L_1(\mu)$.*

*Proof.* If $X$ is nonreflexive, its closed unit ball $B_X$ is not weakly compact; therefore, $B_X$ is not an equi-integrable set in $L_1(\mu)$. The proposition then follows from Theorem 5.2.8. □

Combining Proposition 5.7.1 and Proposition 5.7.2 gives us the following:

**Proposition 5.7.3.** *If $X$ is an infinite-dimensional complemented subspace of $L_1(\mu)$, then $X$ contains a complemented subspace isomorphic to $\ell_1$.*

The analogous result for $\mathcal{C}(K)$-spaces is just as easy:

**Proposition 5.7.4.** *Let $K$ be a compact metric space. If $X$ is an infinite-dimensional complemented subspace of $\mathcal{C}(K)$, then $X$ contains a complemented subspace isomorphic to $c_0$.*

*Proof.* Again by Proposition 5.7.1, $X$ is nonreflexive, and hence every projection $P$ onto it fails to be weakly compact. By Theorem 5.6.3, $X$ must contain a subspace isomorphic to $c_0$, and this subspace must be complemented, because (since $K$ is metrizable) $X$ is separable (by Sobczyk's theorem, Theorem 2.5.8). □

Note here that if $K$ is not metrizable, we can obtain a subspace isomorphic to $c_0$, but it need not be complemented. In the case of $\ell_\infty$ we can use these techniques to add this space to our list of prime spaces. This result is due to Lindenstrauss [194] and it completes our list of classical prime spaces. We remind the reader of Pełczyński's result that the sequence spaces $\ell_p$ for $1 \le p < \infty$ and $c_0$ are prime (Theorem 2.2.4).

**Theorem 5.7.5.** *The space $\ell_\infty$ is prime.*

*Proof.* Let $X$ be an infinite-dimensional complemented subspace of $\ell_\infty$. We have already seen that $X$ cannot be reflexive (Proposition 5.7.1), and hence a projection $P$ onto $X$ cannot be weakly compact. In this case we can use Theorem 5.6.5 to deduce that $X$ contains a copy of $\ell_\infty$. Since $\ell_\infty$ is injective, $X$ actually contains a complemented copy of $\ell_\infty$ (Proposition 2.5.2). We are now in position to use Theorem 2.2.3(b) in the case $p = \infty$, and we deduce that $X \approx \ell_\infty$. □

**Corollary 5.7.6.** *There are no infinite-dimensional separable injective Banach spaces.*

*Proof.* Suppose that $X$ is a separable injective space. Then $X$ embeds isometrically into $\ell_\infty$ by Theorem 2.5.7. Since $X$ is injective, it embeds complementably into $\ell_\infty$, which is a prime space. That forces $X$ to be isomorphic to $\ell_\infty$, a contradiction because $\ell_\infty$ is nonseparable. □

It is quite clear that the spaces $L_1$ and $\mathcal{C}[0,1]$ cannot be prime; the former contains a complemented subspace isomorphic to $\ell_1$, and the latter contains a complemented subspace isomorphic to $c_0$. However, the classification of the complemented subspaces of these classical function spaces remains a very intriguing and important open question.

In the case of $L_1$ the following conjecture remains open:

**Conjecture 5.7.7.** *Every infinite-dimensional complemented subspace of $L_1$ is isomorphic to $L_1$ or $\ell_1$.*

The best result known in this direction is the Lewis–Stegall theorem from 1973 that every complemented subspace of $L_1$ that is a dual space is isomorphic to $\ell_1$ [188]. (More generally, we can replace the dual space assumption by the Radon–Nikodym property.) Later we will develop techniques that show that every complemented subspace with an unconditional basis is isomorphic to $\ell_1$ (an earlier result that is due to Lindenstrauss and Pełczyński [196]).

The corresponding conjecture for $\mathcal{C}[0,1]$ is the following:

**Conjecture 5.7.8.** *Every infinite-dimensional complemented subspace of $\mathcal{C}[0,1]$ is isomorphic to a $\mathcal{C}(K)$-space for some compact metric space $K$.*

Here the best positive result known is due to Rosenthal [271], who proved that if $X$ is a complemented subspace of $\mathcal{C}[0,1]$ with nonseparable dual, then $X \approx \mathcal{C}[0,1]$. We refer to the survey article of Rosenthal [275] for a fuller discussion of this problem.

Since both these spaces fail to be prime, it is natural to weaken the notion:

**Definition 5.7.9.** A Banach space $X$ is *primary* if whenever $X \approx Y \oplus Z$, then either $X \approx Y$ or $X \approx Z$.

The spaces $L_1$ and $\mathcal{C}[0,1]$ are both primary. In the case of $L_1$ this result is due to Enflo and Starbird [89] (for an alternative approach see [152]). In the case of $\mathcal{C}[0,1]$ this was proved by Lindenstrauss and Pełczyński in 1971 [197], but of course it follows from Rosenthal's result cited above [271], which was proved slightly later, since one factor must have nonseparable dual.

## Problems

**5.1.** Show that there is a sequence $(a_n)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z})$ that is not the Fourier transform of any $f \in L_1(\mathbb{T})$.

**5.2.** Let $X$ be a Banach space that does not contain a copy of $\ell_1$. Show that every Dunford–Pettis operator $T : X \to Y$, with $Y$ any Banach space, is compact.

**5.3.** Show that the identity operator $I_{\ell_1} : \ell_1 \to \ell_1$ is Dunford–Pettis.

**5.4.** Let $X$ be a Banach space that does not contain a copy of $\ell_1$; show that every operator $T : X \to L_1$ is weakly compact.

**5.5.** Let $\mu$ be a probability measure. Show that an operator $T : L_1(\mu) \to X$ is Dunford–Pettis if and only if $T$ restricted to $L_2(\mu)$ is compact.

**5.6.** In this exercise we work in the *complex* space $L_p(\mathbb{T})$ $(1 \leq p < \infty)$, where $\mathbb{T}$ is the unit circle with the normalized Haar measure $d\theta/2\pi$. We identify functions $f$ on $\mathbb{T}$ with $2\pi$-periodic functions on $\mathbb{R}$. The Fourier coefficients of $f$ in $L_1(\mathbb{T})$ are given by

$$\hat{f}(n) = \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,\frac{d\theta}{2\pi}, \quad n \in \mathbb{Z}.$$

For measures $\mu \in \mathcal{M}(\mathbb{T})$ we write

$$\hat{\mu}(n) = \int_{-\pi}^{\pi} e^{-in\theta}\,d\mu(\theta).$$

(a) Let $\mu$ be a Borel measure on the unit circle $\mathbb{T}$ such that $\mu \in \mathcal{M}(\mathbb{T})$. Show that for $1 \leq p < \infty$ the map $T_\mu : L_p(\mathbb{T}, d\theta/2\pi) \to L_p(\mathbb{T}, d\theta/2\pi)$ defined by

$$T_\mu f(s) = \mu * f(s) = \int f(s-t)d\mu(t) \qquad \text{a.e.}$$

is a well-defined bounded operator with $\|T_\mu\| \leq \|\mu\|$. [Note that $T_\mu$ maps continuous functions and can be extended to $L_p(\mu)$ by continuity.]

(b) Show that $T_\mu e_n = \hat{\mu}(n)e_n$, where $e_n(t) = e^{\mathrm{int}}$. Deduce that $T_\mu$ is Dunford–Pettis if and only if $\lim_{n\to\infty} \hat{\mu}(n) = 0$.

(c) Show that $T_\mu : L_1(\mathbb{T}) \to L_1(\mathbb{T})$ is weakly compact if and only if $\mu$ is absolutely continuous with respect to Lebesgue measure. [*Hint*: To show that $\mu$ is absolutely continuous, consider $T_\mu f_n$, where $f_n$ is a sequence of nonnegative continuous functions with $\int f_n(t)dt/2\pi = 1$ and whose supports shrink to 0.]

**5.7.** Let $T : \ell_\infty \to X$ be a weakly compact operator that vanishes on $c_0$. Show that there exists an infinite subset $\mathbb{A}$ of $\mathbb{N}$ such that $T|_{\ell_\infty(\mathbb{A})} = 0$. [*Hint*: Mimic the argument in Theorem 2.5.4.]

**5.8.** If $T : \ell_\infty \to X$ is a weakly compact operator, show that for every $\epsilon > 0$, there exists an infinite subset $\mathbb{A}$ of $\mathbb{N}$ such that $T : \ell_\infty(\mathbb{A}) \to X$ is compact and $\|T|_{\ell_\infty(\mathbb{A})}\| < \epsilon$.

**5.9.** Show that if $X$ is a Banach space containing $\ell_\infty$ and $E$ is a closed subspace of $X$, then either $E$ contains $\ell_\infty$ or $X/E$ contains $\ell_\infty$.

**5.10.** Show that every injective Banach space $X$ contains a copy of $\ell_\infty$.

**5.11.** Suppose $X$ is a Banach space with a closed subspace $E$ such that $X/E$ is isomorphic to $L_1$. Show that $E^{\perp\perp}$ is complemented in $X^{**}$. [*Hint:* Use the injectivity of $L_\infty$.]

**5.12 (Lindenstrauss [190]).** Show that $\ell_1$ has a subspace $E$ that is not complemented in its bidual. [*Hint:* Use the kernel of a quotient map onto $L_1$.] Show that this subspace also has no unconditional basis.

# Chapter 6
# The Spaces $L_p$ for $1 \leq p < \infty$

In this chapter we will initiate the study of the Banach space structure of the spaces $L_p(\mu)$, where $1 \leq p < \infty$. We will be interested in some natural questions that ask which Banach spaces can be isomorphic to a subspace of a space $L_p(\mu)$. Questions of this type were called problems of *linear dimension* by Banach in his book [18].

If $1 < p < \infty$, the Banach space $L_p(\mu)$ is reflexive, while $L_1(\mu)$ is nonreflexive; we will see that this is just an example of a discontinuity in behavior when $p = 1$. We will also show certain critical differences between the cases $1 < p < 2$ and $2 < p < \infty$.

Before proceeding, we note that every infinite-dimensional separable $L_p(\mu)$ space, $1 \leq p < \infty$, is isomorphic either to $L_p[0,1]$ or to $\ell_p$. Indeed, if $X$ is a separable subspace of (a not necessarily separable space) $L_p(\Omega, \Sigma, \mu)$, then there is a sequence $(A_n)_{n=1}^\infty$ of sets of finite measure such that $X$ is a subspace of $L_p(\Omega_0, \Sigma_0, \mu_0)$, where $\Omega_0 = \cup_{n=1}^\infty A_n$, $\Sigma_0 = \sigma(A_n, : n \in \mathbb{N})$, $\mu_0 = \mu|_{\Sigma_0}$, and we regard $L_p(\Omega_0, \Sigma_0, \mu_0)$ as a closed subspace of $L_p(\Omega, \Sigma, \mu)$. Consequently, if $L_p(\Omega, \Sigma, \mu)$ is separable, by taking $X = L_p(\Omega, \Sigma, \mu)$, we can assume that $\mu$ is a $\sigma$-finite measure and that $\Sigma$ is a countably generated $\sigma$-algebra. Now, $L_p(\Omega, \Sigma, \mu)$ is isometric to $L_p(\Omega, \Sigma, \nu)$, where $\nu$ is a density measure given by $d\nu = \varphi \, d\mu$ for some $0 < \varphi < \infty$ in $L_1(\mu)$. By looking at the atoms of the measure space $(\Omega, \Sigma, \nu)$, we realize that $L_p(\Omega, \Sigma, \nu)$ is isometrically isomorphic to one of the following spaces:

$$\ell_p^n, \quad \ell_p, \quad L_p(\nu_1), \quad L_p(\nu_1) \oplus_p \ell_p^n, \quad L_p(\nu_1) \oplus_p \ell_p,$$

where $n = 1, 2, \ldots$, and $\nu_1$ is a nonatomic probability measure on a countably generated measurable space $(\Omega_1, \Sigma_1)$. We recursively construct an increasing sequence $(\mathcal{A}_k)_{k=1}^\infty$ of finite sub-$\sigma$-algebras of $\Sigma_1$, each of which is generated by a partition of $\Omega_1$ into sets of measure smaller than $2^{-k}$. We do it in such a way that the algebra $\mathcal{A} = \cup_{k=1}^\infty \mathcal{A}_k$ generates $\Sigma_1$. There is a (unique) algebra $\mathcal{B}$ on $[0,1)$ consisting of left-closed and right-open intervals, together with an isomorphism $\Phi: \mathcal{A} \to \mathcal{F}$ such that $|\Phi(A)| = \nu_1(A)$. We have that $\mathcal{F}$ generates the Borel $\sigma$-algebra



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in $[0, 1)$. Then, $\Phi$ induces a linear isometry from $L_p(\Omega_1, \Sigma_1, \nu_1)$ onto $L_p[0, 1]$. In particular, if $K$ is a metrizable compact space and $\mu$ is a nonatomic finite measure on $(K, \mathcal{B}(K))$, then $L_p(\mu)$ is isometrically isomorphic to $L_p[0, 1]$.

For $p \neq 2$, the spaces $\ell_p$ (and all the others), where $n = 1, 2$, are mutually nonisometric (see [137, pp. 14–15]), while for $p = 2$ the list of mutually nonisometric spaces reduces to $\ell_2$ and $\ell_2^n$, where $n = 1, 2, \ldots$. Since $\ell_p$ is a complemented subspace of $L_p[0, 1]$, we infer that the list contains at most two nonisomorphic infinite-dimensional spaces, namely $L_p[0, 1]$ and $\ell_p$.

We have already seen that $\ell_1$ and $L_1$ are not isomorphic (the former is a Schur space, while the latter is not). That $L_p[0, 1]$ and $\ell_p$ are not isomorphic for $1 < p \neq 2 < \infty$ will be a consequence of Theorem 2.2.4, which states that $\ell_p$ is prime, and Proposition 6.4.2, which tells us that $L_p[0, 1]$ contains a complemented copy of $\ell_2$.

From now on we will use the abbreviation $L_p$ for the space $L_p[0, 1]$.

## 6.1 The Haar Basis in $L_p[0, 1]$ $(1 \le p < \infty)$

The *Haar system* is the sequence of functions $(h_n)_{n=1}^\infty$ defined on $[0, 1]$ by $h_1 = 1$ and for $n = 2^k + s$, where $k = 0, 1, 2, \ldots$, and $s = 1, 2, \ldots, 2^k$,

$$h_n(t) = \chi_{[\frac{2s-2}{2^{k+1}}, \frac{2s-1}{2^{k+1}})}(t) - \chi_{[\frac{2s-1}{2^{k+1}}, \frac{2s}{2^{k+1}})}(t) = \begin{cases} 1 & \text{if } \quad \dfrac{2s-2}{2^{k+1}} \le t < \dfrac{2s-1}{2^{k+1}}, \\[2ex] -1 & \text{if } \quad \dfrac{2s-1}{2^{k+1}} \le t < \dfrac{2s}{2^{k+1}}, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Given $k = 0, 1, 2, \ldots$ and $1 \le s \le 2^k$, each interval of the form $[\frac{s-1}{2^k}, \frac{s}{2^k})$ is called *dyadic*. It is often useful to label the elements of the Haar system by their supports; thus we write $h_I$ to denote $h_n$ when $I$ is the dyadic interval support of $h_n$.

In this first section we will see that the Haar system is a (nonnormalized) Schauder basis in $L_p$ for each $1 \le p < \infty$ and that it is unconditional when $1 < p < \infty$. We shall need the concept of conditional expectation, which we introduce next.

Let $(\Omega, \Sigma, \mu)$ be a probability measure space, and $\Sigma'$ a sub-$\sigma$-algebra of $\Sigma$. Given $f \in L_1(\Omega, \Sigma, \mu)$, we define a (signed) measure, $\nu$, on $\Sigma'$ by

$$\nu(E) = \int_E f \, d\mu, \quad E \in \Sigma'.$$

The measure $\nu$ is absolutely continuous with respect to $\mu|_{\Sigma'}$; hence by the Radon–Nikodym theorem, there is a (unique, up to sets of measure zero) $\Sigma'$-measurable function $\psi \in L_1(\Omega, \Sigma', \mu)$ such that

$$\nu(E) = \int_E \psi \, d\mu, \quad E \in \Sigma'.$$

**Definition 6.1.1.** Given $f \in L_1(\Omega, \Sigma, \mu)$, the *conditional expectation of $f$ on the $\sigma$-algebra* $\Sigma'$ is the (unique) function $\psi$ that satisfies

$$\int_E f \, d\mu = \int_E \psi \, d\mu, \quad \forall \, E \in \Sigma'.$$

The function $\psi$ will be denoted by $\mathbb{E}(f \mid \Sigma')$.

Let us notice that if $\Sigma'$ consists of countably many disjoint atoms $(A_n)_{n=1}^{\infty}$, the definition of $\mathbb{E}(f \mid \Sigma')$ is especially simple:

$$\mathbb{E}(f \mid \Sigma')(t) = \sum_{j=1}^{\infty} \frac{1}{\mu(A_j)} \Big( \int_{A_j} f \, d\mu \Big) \chi_{A_j}(t).$$

We also observe that if $f \in L_1(\mu)$, for all $\Sigma'$-measurable simple functions $g$, we have

$$\int_\Omega gf \, d\mu = \int_\Omega g \mathbb{E}(f \mid \Sigma') \, d\mu$$

and

$$\mathbb{E}(fg \mid \Sigma') = g\mathbb{E}(f \mid \Sigma').$$

**Lemma 6.1.2.** *Let $(\Omega, \Sigma, \mu)$ be a probability measure space and suppose $\Sigma'$ is a sub-$\sigma$-algebra of $\Sigma$. Then $\mathbb{E}(\cdot \mid \Sigma')$ is a norm-one linear projection from $L_p(\Omega, \Sigma, \mu)$ onto $L_p(\Omega, \Sigma', \mu)$ for every $1 \le p \le \infty$.*

*Proof.* Fix $1 \le p \le \infty$. It is immediate to check that $\mathbb{E}(\cdot \mid \Sigma')^2 = \mathbb{E}(\cdot \mid \Sigma')$. If $f \in L_p(\mu)$, using Hölder's inequality in $L_p(\Omega, \Sigma', \mu)$ (see C.2 in the appendix), we have

$$\begin{aligned}
\big\| \mathbb{E}(f \mid \Sigma') \big\|_p &= \sup \Big\{ \int_\Omega \mathbb{E}(f \mid \Sigma') g \, d\mu : g \text{ simple } \Sigma'\text{-measurable with } \|g\|_q \le 1 \Big\} \\
&= \sup \Big\{ \int_\Omega fg \, d\mu : g \text{ simple } \Sigma'\text{-measurable with } \|g\|_q \le 1 \Big\} \\
&\le \sup \Big\{ \int_\Omega fg \, d\mu : g \text{ simple with } \|g\|_q \le 1 \Big\} = \|f\|_p \, .
\end{aligned}$$

We leave the case $p = \infty$ to the reader. □

**Proposition 6.1.3.** *The Haar system is a monotone basis in $L_p$ for $1 \le p < \infty$.*

*Proof.* Let us consider an increasing sequence of $\sigma$-algebras, $(\mathcal{B}_n)_{n=1}^{\infty}$, contained in the Borel $\sigma$-algebra of $[0,1]$ defined as follows: we let $\mathcal{B}_1$ be the trivial $\sigma$-algebra, $\{\emptyset, [0,1]\}$, and for $n = 2^k + s$ ($k = 0, 1, 2, \dots, 1 \le s \le 2^k$) we let $\mathcal{B}_n$ be the finite subalgebra of the Borel sets of $[0,1]$ whose atoms are the dyadic intervals of the family

$$\mathcal{F}_n = \begin{cases} \left[\frac{j-1}{2^{k+1}}, \frac{j}{2^{k+1}}\right) & \text{for } j = 1, \dots, 2s, \\ \left[\frac{j-1}{2^{k}}, \frac{j}{2^{k}}\right) & \text{for } j = s+1, \dots, 2^k. \end{cases}$$

Fix $1 \le p < \infty$. For each $n$, $\mathbb{E}_n$ will denote the conditional expectation operator on the $\sigma$-algebra $\mathcal{B}_n$. By Lemma 6.1.2, $\mathbb{E}_n$ is a norm-one projection from $L_p$ onto $L_p([0,1], \mathcal{B}_n, \lambda)$, the space of functions that are constant on intervals of the family $\mathcal{F}_n$. We will denote this space by $L_p(\mathcal{B}_n)$. Clearly, rank $\mathbb{E}_n = n$. Furthermore, $\mathbb{E}_n\mathbb{E}_m = \mathbb{E}_m\mathbb{E}_n = \mathbb{E}_{\min\{m,n\}}$ for any two positive integers $m, n$.

On the other hand, the set

$$\{f \in L_p \colon \|\mathbb{E}_n(f) - f\|_p \to 0\}$$

is closed by the partial converse of the Banach–Steinhaus theorem (see E.14 in the appendix) and contains the set $\cup_{k=1}^{\infty} L_p(\mathcal{B}_k)$, which is dense in $L_p$. Therefore $\|\mathbb{E}_n(f) - f\|_p \to 0$ for all $f \in L_p$. By Proposition 1.1.7, $L_p$ has a basis whose natural projections are $(\mathbb{E}_n)_{n=1}^{\infty}$. This basis is actually the Haar system, because for each $n \in \mathbb{N}$ we have $\mathbb{E}_m(h_n) = h_n$ for $m \ge n$ and $\mathbb{E}_m h_n = 0$ for $m < n$. The basis constant is $\sup_n \|\mathbb{E}_n\| = 1$. □

*Remark 6.1.4.* (a) Integrating the Haar system, we obtain Schauder's original basis $(\varphi_n(t))_{n=1}^{\infty}$ for $\mathcal{C}[0,1]$ (see Section 1.2). More precisely, if $n = 2^k + s$, where $k = 0, 1, 2, \dots$, and $s = 1, 2, \dots, 2^k$, then

$$\varphi_n(t) = 2^{k+1} \int_0^t h_n(u)\, du = \begin{cases} 2^{k+1}t - (2s-2) & \text{if } \frac{2s-2}{2^{k+1}} \le t \le \frac{2s-1}{2^{k+1}}, \\ -2^{k+1}t + 2s & \text{if } \frac{2s-1}{2^{k+1}} \le t \le \frac{2s}{2^{k+1}}, \\ 0 & \text{otherwise.} \end{cases}$$

(b) The Haar system as we have defined it is not normalized in $L_p$ for $1 \le p < \infty$. It is normalized in $L_\infty$, since $\|h_{2^k+s}\|_p = (1/2^k)^{1/p}$. To normalize in $L_p$ one should take $h_n/\|h_n\|_p = |I_n|^{-1/p} h_n$, where $I_n$ denotes the support of the Haar function $h_n$.

(c) Let us observe that if $f \in L_p$ $(1 \le p < \infty)$, then

$$\mathbb{E}_n(f) - \mathbb{E}_{n-1}(f) = \Big(\frac{1}{|I_n|}\int f(t)h_n(t)\,dt\Big)\,h_n.$$

We deduce that the dual functionals associated to the Haar system are given by

$$h_n^* = \frac{1}{|I_n|}h_n, \quad n \in \mathbb{N},$$

and the series expansion of $f \in L_p$ in terms of the Haar basis is

$$f = \sum_{n=1}^{\infty}\Big(\frac{1}{|I_n|}\int f(t)h_n(t)\,dt\Big)h_n.$$

Notice that if $p = 2$, then $(h_n/\|h_n\|_2)_{n=1}^{\infty}$ is an orthonormal basis for the Hilbert space $L_2$ and is thus unconditional. It is an important fact that, actually, the Haar basis is an unconditional basis in $L_p$ for $1 < p < \infty$. This was first proved by Paley [235] in 1932. Much more recently, Burkholder [37] established the *best* constant. We are going to present another proof by Burkholder from 1988 [38]. We will treat only the real case here, although, remarkably, the same proof works for complex scalars with the same constant; however, the calculations needed for the complex case are a little harder to follow. For our purposes the constant is not so important, and we simply note that if the Haar basis is unconditional for real scalars, one readily checks that it is also unconditional for complex scalars. There is one drawback to Burkholder's argument: it is simply too clever in the sense that the proof looks very much like magic.

We start with some elementary calculus.

**Lemma 6.1.5.** *Suppose $p > 2$. Then*

$$\frac{p^{p-2}}{(p-1)^{p-1}} < 1. \tag{6.1}$$

*Proof.* If we let $t = p - 1$, inequality (6.1) is equivalent to

$$H(t) = -(t-1)\log(1+t) + t\log(t) > 0, \quad \forall\, t > 1.$$

Indeed, differenting $H$ gives

$$H'(t) = \frac{2}{t+1} - \log\left(\frac{1+t}{t}\right) \ge \frac{2}{t+1} - \frac{1}{t} = \frac{t-1}{t(t+1)} > 0$$

for all $t > 1$. Therefore $H(t) > H(1) = 0$ for all $t > 1$. □

In the next lemma we introduce a mysterious function that will enable us to prove Burkholder's theorem. This function appears to be plucked out of the air, although there are sound reasons behind its selection. The use of such functions to prove sharp inequalities has been developed extensively by Nazarov, Treil, and Volberg, who termed them *Bellman functions.* We refer to [226] for a discussion of this technique.

**Lemma 6.1.6.** *Suppose $p > 2$ and define a function $\varphi$ on the first quadrant of $\mathbb{R}^2$ by*

$$\varphi(x, y) = (x + y)^{p-1}((p - 1)x - y), \quad x, y \ge 0.$$

*(a) The following inequality holds for all $(x, y)$ with $x \ge 0$ and $y \ge 0$:*

$$\frac{(p-1)^{p-1}}{p^{p-2}} \varphi(x, y) \le (p - 1)^p x^p - y^p.$$

*(b) For all real numbers $x, y, a$ and for $\varepsilon = \pm 1$,*

$$\varphi(|x + a|, |y + \varepsilon a|) + \varphi(|x - a|, |y - \varepsilon a|) \ge 2\varphi(|x|, |y|). \tag{6.2}$$

*Proof.* $(a)$ By homogeneity we can suppose that $x + y = 1$. Then it suffices to show that the function

$$G(x) = \frac{p^{p-2}}{(p-1)^{p-1}}((p - 1)^p x^p - (1 - x)^p) - px + 1, \quad 0 \le x \le 1,$$

is nonnegative. The first two derivatives of $G$ are

$$G'(x) = \frac{p^{p-1}}{(p-1)^{p-1}}((p - 1)^p x^{p-1} + (1 - x)^{p-1}) - p,$$

$$G''(x) = \frac{p^{p-1}}{(p-1)^{p-2}}((p - 1)^p x^{p-2} - (1 - x)^{p-2}).$$

Since $p > 2$, $G''$ is increasing. Furthermore,

$$G''(0) = -\frac{p^{p-1}}{(p-1)^{p-2}} < 0 < p((p - 1)^2 - 1) = G''(1/p).$$

Therefore, there is $0 < a < 1/p$ such that $G$ is concave on $[0, a]$ and convex on $[a, 1]$. Notice that

$$G(1/p) = G'(1/p) = 0.$$

Consequently, by the convexity of $G$ on $[a, 1]$, $G(x) > 0$ for all $x \in [a, 1] \setminus \{1/p\}$. In addition to $G(a) > 0$ we have, applying (6.1),

$$G(0) = 1 - \frac{p^{p-2}}{(p-1)^{p-1}} > 0.$$

Hence, by the concavity of $G$ on $[0, a]$, $G(x) > 0$ for all $x \in [0, a]$.

(*b*) Note that the case $\varepsilon = -1$ can be deduced from the case $\varepsilon = 1$ by replacing $y$ with $-y$. To prove the inequality for $\varepsilon = 1$, we consider the family of functions

$$F_s \colon \mathbb{R} \to \mathbb{R}, \quad t \mapsto \varphi(|t+s|, |t-s|),$$

defined for all $s \in \mathbb{R}$. Then, in order to obtain (6.2) it suffices to see that $F_s$ is a convex function for all $s$ real. Indeed, if this were the case, we would have

$$F_s(t+a) + F_s(t-a) \ge 2F_s(t), \quad \forall\, t, a \in \mathbb{R},$$

and choosing $s$ and $t$ such that $t + s = x$ and $t - s = y$, we would get the desired inequality.

To show the convexity of the functions $F_s$, notice that by a continuity argument, it suffices to deal with $s \neq 0$. Now, if $s \neq 0$, we have $F_s(t) = |s|^p F_1(t/s)$. Hence we need only prove that $F_1$ is a convex function. Clearly, $F_1$ is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{\pm 1\}$. A straightforward computation yields

$$F_1'(t) = \begin{cases} p2^{p-1}(-t)^{p-2}((p-2)t - (p-1)), & \text{if } t < -1, \\ -p2^{p-1}, & \text{if } -1 < t < 1, \\ p2^{p-1}t^{p-2}((p-2)t - (p-1)), & \text{if } 1 < t. \end{cases}$$

From here it is clear that $F_1'$ is increasing on $(-\infty, -1)$ and constant on $(-1, 1)$. Differentiating again at points $t > 1$, we obtain

$$F_1''(t) = p(p-1)(p-2)2^{p-1}(t-1)t^{p-2} > 0.$$

Hence $F_1'$ is also increasing on $(1, \infty)$. At the junction points $\pm 1$, where $F_1$ may fail to be smooth,

$$F_1'((-1)^-) < F_1'((-1)^+) = F_1'(1^-) = F_1'(1^+).$$

Consequently, $F_1$ is convex on the entire real line. □

**Theorem 6.1.7.** *Suppose* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $p^* = \max(p, q)$. *The Haar basis* $(h_n)_{n=1}^{\infty}$ *in* $L_p$ *is unconditional with unconditional constant at most* $p^* - 1$. *That is,*

$$\Big\| \sum_{n=1}^{N} \epsilon_n a_n h_n \Big\|_p \le (p^* - 1) \Big\| \sum_{n=1}^{N} a_n h_n \Big\|_p,$$

*whenever* $N \in \mathbb{N}$, *for any real scalars* $a_1, \dots, a_N$ *and any choice of signs* $\epsilon_1, \dots, \epsilon_N$.

*Proof.* Suppose first $p > 2$, in which case $p^* = p$. Taking into account Lemma 6.1.6(a) we need only show that

$$\int_0^1 \varphi\Big( \Big| \sum_{n=1}^{N} a_n h_n(u) \Big|, \Big| \sum_{n=1}^{N} \varepsilon_n a_n h_n(u) \Big| \Big) \, du \ge 0 \tag{6.3}$$

for all $N \in \mathbb{N}$, all $N$-tuples $(a_n)_{n=1}^N$ in $\mathbb{R}$, and all $(\varepsilon_n)_{n=1}^N$ in $\{\pm 1\}$. To that end, we proceed by induction on $N$. For $N = 1$ this is trivial, since taking $x = y = 0$ and $\varepsilon = 1$ in Lemma 6.1.6(b) yields

$$\varphi(|a|, |a|) \ge \varphi(0, 0) = 0, \quad \forall\, a \in \mathbb{R}.$$

In order to establish the inductive step, suppose $N \ge 2$ and assume that (6.3) holds for $(N-1)$-tuples. Given $(a_n)_{n=1}^N$ in $\mathbb{R}$ and $(\varepsilon_n)_{n=1}^N$ in $\{\pm 1\}$, put

$$f_{N-1} = \sum_{n=1}^{N-1} a_n h_n, \quad g_{N-1} = \sum_{n=1}^{N-1} \varepsilon_n a_n h_n,$$

$$f_N = \sum_{n=1}^{N} a_n h_n = f_{N-1} + a_N h_N, \quad g_N = \sum_{n=1}^{N} \varepsilon_n a_n h_n = g_{N-1} + \varepsilon_N a_N h_N$$

$$\tilde{f}_N = \sum_{n=1}^{N} a_n h_n = f_{N-1} - a_N h_N, \quad \tilde{g}_N = \sum_{n=1}^{N} \varepsilon_n a_n h_n = g_{N-1} - \varepsilon_N a_N h_N.$$

Since $h_N$ is supported on an interval on which $f_{N-1}$ and $g_{N-1}$ are constant and it takes opposite values on intervals of the same length (or, using probabilistic terminology, $h_N$ and $-h_N$ are equidistributed random variables, both independent with the random variable pair $(f_{N-1}, g_{N-1})$),

$$J := \int_0^1 \varphi(|f_N(u)|, |g_N(u)|) \, du = \int_0^1 \varphi(|\tilde{f}_N(u)|, |\tilde{g}_N(u)|) \, du.$$

Therefore, by Lemma 6.1.6(b),

$$\begin{aligned} J &= \int_0^1 \frac{1}{2} \Big( \varphi(|f_N(u)|, |g_N(u)|) + \varphi(|\tilde{f}_N(u)|, |\tilde{g}_N(u)|) \Big) \, du \\ &\ge \int_0^1 \varphi(|f_{N-1}(u)|, |g_{N-1}(u)|) \, du \ge 0. \end{aligned}$$

The case $p = 2$ is trivial, since the Haar system is an orthonormal basis of $L_2$; hence its unconditional basis constant is 1.

The case $1 < p < 2$ now follows by duality. With $f_N = \sum_{n=1}^{N-1} a_n h_n$ and $g_N = \sum_{n=1}^{N-1} \varepsilon_n a_n h_n$ as before, choose $g'_N \in L_q(\mathcal{B}_N)$ such that $\|g'_N\|_q = 1$ and

$$\int_0^1 g_N(u) g'_N(u)\, du = \|g_N\|_p.$$

Then $g'_N = \sum_{n=1}^{N} b_n h_n$ for some $(b_n)_{n=1}^N$. Let $f'_N = \sum_{n=1}^{N} \varepsilon_n b_n h_n$. It is clear that

$$\begin{aligned}\|g_N\|_p &= \int_0^1 f_N(u) f'_N(u)\, du \le \|f_N\|_p \|f'_N\|_q \le \|f_N\|_p (q-1) \|g'_N\|_q \\ &\le (q-1)\|f_N\|_p.\end{aligned}$$

□

*Remark 6.1.8.* The constant $p^* - 1$ in Burkholder's theorem, Theorem 6.1.7, is sharp, although we will not prove this here.

## 6.2 Averaging in Banach Spaces

In discussing unconditional bases and unconditional convergence of series in a Banach space $X$ we have frequently met the problem of estimating expressions of the type

$$\max \left\{ \left\| \sum_{i=1}^{n} \epsilon_i x_i \right\| \;:\; (\epsilon_i) \in \{-1, 1\}^n \right\},$$

where $\{x_i\}_{i=1}^n$ are vectors in $X$. In many situations it is much easier to replace the maximum by the average over all choices of signs $\epsilon_i = \pm 1$.

It turns out to be helpful to consider such averages using the Rademacher functions $(r_i)_{i=1}^\infty$, since the sequence $(r_i(t))_{i=1}^n$ gives us all possible choices of signs $(\epsilon_i)_{i=1}^n$ as $t$ ranges over $[0, 1]$. Thus,

$$\operatorname*{Average}_{\epsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \epsilon_i x_i \right\| = \frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \epsilon_i x_i \right\| = \int_0^1 \left\| \sum_{i=1}^{n} r_i(t) x_i \right\| dt.$$

Let us recall the definition of the Rademacher functions and their basic properties.

**Definition 6.2.1.** The *Rademacher functions* $(r_k)_{k=1}^\infty$ are defined on $[0, 1]$ by

$$r_k(t) = \operatorname{sgn}(\sin 2^k \pi t).$$

Alternatively, the sequence $(r_k)_{k=1}^{\infty}$ can be described as

$$r_1(t) = \begin{cases} 1 & \text{if } t \in [0, \frac{1}{2}), \\ -1 & \text{if } t \in [\frac{1}{2}, 1), \end{cases}$$

$$r_2(t) = \begin{cases} 1 & \text{if } t \in [0, \frac{1}{4}) \cup [\frac{1}{2}, \frac{3}{4}), \\ -1 & \text{if } t \in [\frac{1}{4}, \frac{1}{2}) \cup [\frac{3}{4}, 1), \end{cases}$$

$$\vdots$$

$$r_{k+1}(t) = \begin{cases} 1 & \text{if } t \in \bigcup_{s=1}^{2^k} [\frac{2s-2}{2^{k+1}}, \frac{2s-1}{2^{k+1}}), \\ -1 & \text{if } t \in \bigcup_{s=1}^{2^k} [\frac{2s-1}{2^{k+1}}, \frac{2s}{2^{k+1}}). \end{cases}$$

That is,

$$r_{k+1} = \sum_{s=1}^{2^k} h_{2^k+s}, \qquad k = 0, 1, 2, \dots.$$

Thus $(r_k)_{k=1}^{\infty}$ is a block basic sequence with respect to the Haar basis in every $L_p$ for $1 \le p < \infty$. The key properties we need are the following:

- $r_k(t) = \pm 1$ a.e. for all $k$,
- $\displaystyle\int_0^1 r_{k_1} r_{k_2}(t) \dots r_{k_m}(t) dt = 0,$ whenever $k_1 < k_2 < \cdots < k_m$.

The Rademacher functions were first introduced by Rademacher in 1922 [264] with the idea of studying the problem of finding conditions under which a series of real numbers $\sum \pm a_n$, where the signs were assigned randomly, would converge almost surely. Rademacher showed that if $\sum |a_n|^2 < \infty$, then $\sum \pm a_n$ converges almost surely. The converse was proved in 1925 by Khintchine and Kolmogorov [171]. Historically, the subject of finding estimates for averages over all choices of signs was initiated in 1923 by the classical Khintchine's inequalities [170], but the usefulness of a probabilistic viewpoint in studying the $L_p$-spaces seems to have been fully appreciated quite late (around 1970).

**Theorem 6.2.2 (Khintchine's Inequalities).** *For every $1 \le p < \infty$ there exist positive constants $A_p$ and $B_p$ such that for every finite sequence of scalars $(a_i)_{i=1}^n$ and $n \in \mathbb{N}$ we have*

$$A_p \Big( \sum_{i=1}^n |a_i|^2 \Big)^{1/2} \le \Big\| \sum_{i=1}^n a_i r_i \Big\|_p \le \Big( \sum_{i=1}^n |a_i|^2 \Big)^{1/2} \quad \text{if } 1 \le p < 2,$$

*and*

$$\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2} \le \Big\|\sum_{i=1}^n a_i r_i\Big\|_p \le B_p\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2} \quad \textit{if } p > 2.$$

We will not prove this here, but it will be derived as a consequence of a more general result below. Theorem 6.2.2 was first given in the stated form by Littlewood in 1930 [208], but Khintchine's earlier work (of which Littlewood was unaware) implied these inequalities as a consequence.

*Remark 6.2.3.* (*a*) Khintchine's inequalities tell us that $(r_i)_{i=1}^\infty$ is a basic sequence equivalent to the $\ell_2$-basis in every $L_p$ for $1 \le p < \infty$. In $L_\infty$, though, one readily checks that $(r_i)_{i=1}^\infty$ is isometrically equivalent to the canonical $\ell_1$-basis.

(*b*) $(r_i)_{i=1}^\infty$ is an orthonormal sequence in $L_2$, which yields the identity

$$\Big\|\sum_{i=1}^n a_i r_i\Big\|_2 = \Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2},$$

for any choice of scalars $(a_i)$. But $(r_i)_{i=1}^\infty$ is not a complete system in $L_2$, that is, $[r_i] \neq L_2$ (for instance, notice that the function $r_1 r_2$ is orthogonal to the subspace $[r_i]$). However, one can obtain a complete orthonormal system for $L_2$ using the Rademacher functions by adding to $(r_i)_{i=1}^\infty$ the constant function $r_0 = 1$ and the functions of the form $r_{k_1} r_{k_2} \dots r_{k_n}$ for any $k_1 < k_2 < \cdots < k_n$. This collection of functions are the *Walsh functions*.

Khintchine's inequalities can also be interpreted by saying that all the norms $\{\|\cdot\|_p \colon 1 \le p < \infty\}$ are equivalent on the linear span of the Rademacher functions in $L_p$. It turns out that in this form, the statement can be generalized to an arbitrary Banach space. This generalization was first obtained by Kahane in 1964 [150].

For our purposes it will be convenient to replace the concrete Rademacher functions by an abstract model. To that end we will use the language and methods of probability theory (see Appendix I for a quick fix on the basics that will be required in this chapter).

**Definition 6.2.4.** A *Rademacher sequence* is a sequence of mutually independent random variables $(\varepsilon_n)_{n=1}^\infty$ defined on some probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = \frac{1}{2}$ for every $n$.

The terminology is justified by the fact that the Rademacher functions $(r_n)_{n=1}^\infty$ are a Rademacher sequence on $[0, 1]$. Thus,

$$\int_0^1 \Big\|\sum_{i=1}^n r_i(t) x_i\Big\|\, dt = \mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\| = \int_\Omega \Big\|\sum_{i=1}^n \varepsilon_i(\omega) x_i\Big\|\, d\mathbb{P}.$$

**Theorem 6.2.5 (Kahane–Khintchine Inequalities).** *For each $1 \le p < \infty$ there exists a constant $C_p$ such that for every Banach space $X$ and finite sequence $(x_i)_{i=1}^n$ in $X$, the following inequality holds:*

$$\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\| \le \Big(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^p\Big)^{1/p} \le C_p\,\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|.$$

We will prove the Kahane–Khintchine inequalities (and this will imply the Khintchine inequalities by taking $X = \mathbb{R}$ or $X = \mathbb{C}$), but first we shall establish three lemmas on our way to the proof. To avoid repetitions, in all three lemmas, $(\Omega, \Sigma, \mathbb{P})$ will be a probability space and $X$ will be a Banach space. Let us recall that an *$X$-valued random variable* on $\Omega$ is a function $f\colon \Omega \to X$ such that $f^{-1}(B) \in \Sigma$ for every Borel set $B \subset X$. The random variable $f$ is *symmetric* if $\mathbb{P}(f \in B) = \mathbb{P}(-f \in B)$ for all Borel subsets $B$ of $X$.

**Lemma 6.2.6.** *Let $f : \Omega \to X$ be a symmetric random variable. Then for all $x \in X$ we have*

$$\mathbb{P}\big(\,\|f + x\| \ge \|x\|\,\big) \ge \frac{1}{2}.$$

*Proof.* Let us take any $x \in X$. For every $\omega \in \Omega$, using the convexity of the norm of $X$, clearly $\|f(\omega) + x\| + \|x - f(\omega)\| \ge 2\,\|x\|$. Then, either $\|f(\omega) + x\| \ge \|x\|$ or $\|x - f(\omega)\| \ge \|x\|$. Hence

$$1 \le \mathbb{P}\big(\,\|f + x\| \ge \|x\|\,\big) + \mathbb{P}\big(\,\|x - f\| \ge \|x\|\,\big).$$

Since $f$ is symmetric, $x + f$ and $x - f$ have the same distribution, and the lemma follows. □

Let $(\varepsilon_i)_{i=1}^\infty$ be a Rademacher sequence on $\Omega$. Given $n \in \mathbb{N}$ and vectors $x_1, \dots, x_n$ in $X$, we shall consider $\Lambda_m : \Omega \longrightarrow X$ $(1 \le m \le n)$ defined by

$$\Lambda_m(\omega) = \sum_{i=1}^m \varepsilon_i(\omega) x_i.$$

**Lemma 6.2.7.** *For all $\lambda > 0$,*

$$\mathbb{P}\big(\max_{m\le n} \|\Lambda_m\| > \lambda\big) \le 2\mathbb{P}\big(\,\|\Lambda_n\| > \lambda\big).$$

*Proof.* Given $\lambda > 0$, for $m = 1, \dots, n$ put

$$\Omega_m^{(\lambda)} = \big\{\omega \in \Omega : \|\Lambda_m(\omega)\| > \lambda \;\text{ and }\; \big\|\Lambda_j(\omega)\big\| \le \lambda \;\text{ for all } j = 1, \dots, m-1\big\}.$$

Since $\{\omega \in \Omega : \max_{m\le n} \|\Lambda_m(\omega)\| > \lambda\} = \cup_{m=1}^n \Omega_m^{(\lambda)}$, by the disjointedness of the sets $\Omega_m^{(\lambda)}$ it follows that

$$\mathbb{P}\big(\max_{m\le n} \|\Lambda_m\| > \lambda\big) = \sum_{m=1}^n \mathbb{P}(\Omega_m^{(\lambda)}). \tag{6.4}$$

Therefore,

$$\mathbb{P}\big(\|\Lambda_n\| > \lambda\big) = \sum_{m=1}^n \mathbb{P}\big(\Omega_m^{(\lambda)} \cap (\|\Lambda_n\| > \lambda)\big). \tag{6.5}$$

Notice that every $\Omega_m^{(\lambda)}$ can be written as the union of sets of the type

$$\{\omega \in \Omega : \varepsilon_j(\omega) = \delta_j \text{ for } 1 \le j \le m\}$$

for some choices of signs $\delta_j = \pm 1$. For each of these choices of signs $\delta_1, \dots, \delta_m$ we observe that by Lemma 6.2.6,

$$\mathbb{P}\left(\left\|\sum_{j=1}^m \delta_j x_j + \sum_{j=m+1}^n \varepsilon_j x_j\right\| \ge \left\|\sum_{j=1}^m \delta_j x_j\right\|\right) \ge \frac{1}{2}.$$

Summing over the appropriate signs $(\delta_1, \dots, \delta_m)$ it follows that

$$\mathbb{P}\left(\Omega_m^{(\lambda)} \cap (\|\Lambda_n\| \ge \|\Lambda_m\|)\right) \ge \frac{1}{2}\mathbb{P}(\Omega_m^{(\lambda)}).$$

Thus,

$$\mathbb{P}\big(\Omega_m^{(\lambda)} \cap (\|\Lambda_n\| > \lambda)\big) \ge \frac{1}{2}\mathbb{P}(\Omega_m^{(\lambda)}).$$

Summing over $m$ and combining (6.4) and (6.5), we finish the proof. □

**Lemma 6.2.8.** *For all* $\lambda > 0$,

$$\mathbb{P}\big(\|\Lambda_n\| > 2\lambda\big) \le 4\big(\mathbb{P}\big(\|\Lambda_n\| > \lambda\big)\big)^2.$$

*Proof.* We will keep the notation that we introduced in the previous lemma. Notice that for each $1 \le m \le n$, the random variable $\|\sum_{i=m}^n \varepsilon_i x_i\|$ is independent of each of $\varepsilon_1, \dots, \varepsilon_m$, and hence for all $\lambda > 0$ the events $\{\omega : \|\sum_{i=m}^n \varepsilon_i(\omega) x_i\| > \lambda\}$ and $\Omega_m^{(\lambda)}$ are independent. Observe as well that if some $\omega \in \Omega_m^{(\lambda)}$ further satisfies $\|\Lambda_n(\omega)\| > 2\lambda$, then $\|\Lambda_n(\omega) - \Lambda_{m-1}(\omega)\| > \lambda$ (for $m = 1$, take $\Lambda_0 = 0$). Therefore, since $\mathbb{P}(\|\sum_{i=m}^n \varepsilon_i x_i\| > \lambda) \le 2\mathbb{P}(\|\Lambda_n\| > \lambda)$ for each $m = 1, \dots, n$, by Lemma 6.2.7,

$$\mathbb{P}\big(\Omega_m^{(\lambda)} \cap (\|\Lambda_n\| > 2\lambda)\big) \le \mathbb{P}(\Omega_m^{(\lambda)})\mathbb{P}\Big(\Big\| \sum_{i=m}^{n} \varepsilon_i x_i \Big\| > \lambda\Big) \le 2\mathbb{P}(\Omega_m^{(\lambda)})\mathbb{P}\big( \|\Lambda_n\| > \lambda\big).$$

Summing over $m$ and using again Lemma 6.2.7, we obtain

$$\mathbb{P}\big( \|\Lambda_n\| > 2\lambda\big) \le 2\mathbb{P}\big( \max_{m\le n} \|\Lambda_m\| > \lambda\big)\mathbb{P}\big( \|\Lambda_n\| > \lambda\big) \le 4\big(\mathbb{P}\big( \|\Lambda_n\| > \lambda\big)\big)^2.$$

□

*Proof of Theorem 6.2.5.* Fix $1 \le p < \infty$ and let $\{x_i\}_{i=1}^n$ be any finite set of vectors in $X$. Without loss of generality we will suppose that $\mathbb{E}\| \sum_{i=1}^n \varepsilon_i x_i\| = 1$. Then, by Chebyshev's inequality,

$$\mathbb{P}\big( \|\Lambda_n\| > 8\big) \le \frac{1}{8}. \tag{6.6}$$

Using Lemma 6.2.8 repeatedly, we obtain

$$\begin{aligned} \mathbb{P}\big( \|\Lambda_n\| > 2 \cdot 8\big) &\le 4(1/8)^2, \\ \mathbb{P}\big( \|\Lambda_n\| > 2^2 \cdot 8\big) &\le 4^3(1/8)^4, \\ \mathbb{P}\big( \|\Lambda_n\| > 2^3 \cdot 8\big) &\le 4^7(1/8)^8, \end{aligned}$$

and so on. Hence, by induction, we deduce that

$$\mathbb{P}\big( \|\Lambda_n\| > 2^n \cdot 8\big) \le 4^{2^n-1}(1/8)^{2^n} \le 4^{2^n}(1/8)^{2^n} = (1/2)^{2^n}.$$

Therefore,

$$\begin{aligned} \mathbb{E}\Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\|^p &= \int_0^\infty \mathbb{P}\big( \|\Lambda_n\|^p > t\big)\, dt \\ &= \int_0^\infty p\, t^{p-1}\mathbb{P}\big( \|\Lambda_n\| > t\big)\, dt \\ &= \int_0^8 p\, t^{p-1}\mathbb{P}\big( \|\Lambda_n\| > t\big)\, dt + \sum_{n=1}^\infty \int_{2^{n-1}\cdot 8}^{2^n\cdot 8} p\, t^{p-1}\mathbb{P}\big( \|\Lambda_n\| > t\big) dt \\ &\le \int_0^8 p\, t^{p-1}\, dt + \sum_{n=1}^\infty (1/2)^{2^n-1} \int_{2^{n-1}\cdot 8}^{2^n\cdot 8} p\, t^{p-1}\, dt \\ &\le 8^p\Big(1 + \sum_{n=1}^\infty (1/2)^{2^n-1}\, 2^{np}\Big) = C_p^p. \end{aligned}$$

□

Suppose that $H$ is a Hilbert space. The well known parallelogram law states that for any two vectors $x, y$ in $H$ we have

$$\frac{\|x+y\|^2 + \|x-y\|^2}{2} = \|x\|^2 + \|y\|^2 .$$

This identity is a simple example of the power of averaging over signs and has an elementary generalization:

**Proposition 6.2.9 (Generalized Parallelogram Law).** *Suppose that $H$ is a Hilbert space. Then for every finite sequence $(x_i)_i^n$ in $H$,*

$$\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^2 = \sum_{i=1}^n \|x_i\|^2 .$$

*Proof.* For any vectors $\{x_i\}_{i=1}^n$ in $H$ we have

$$\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^2 = \mathbb{E}\Big\langle \sum_{i=1}^n \varepsilon_i x_i, \sum_{i=1}^n \varepsilon_i x_i\Big\rangle = \sum_{i,j=1}^n \langle x_i, x_j\rangle \mathbb{E}(\varepsilon_i\varepsilon_j) = \sum_{i=1}^n \|x_i\|^2 .$$

□

Next we are going to study how the averages $(\mathbb{E}\|\sum_{i=1}^n \varepsilon_i x_i\|^p)^{1/p}$ are situated with respect to the sums $(\sum_{i=1}^n \|x_i\|^p)^{1/p}$ using the concepts of *type* and *cotype* of a Banach space. These were introduced into Banach space theory by Hoffmann-Jørgensen [125], and their basic theory was developed in the early 1970s by Maurey and Pisier [215]; see [214] for historical comments. However, it should be said that the origin of these ideas was in two very early papers of Orlicz in 1933, [233, 234]. Orlicz essentially introduced the notion of cotype for the spaces $L_p$, although he did not use the more modern terminology.

**Definition 6.2.10.** (a) A Banach space $X$ is said to have *Rademacher type $p$* (in short, *type $p$*) for some $1 \le p \le 2$ if there is a constant $C$ such that for every finite set of vectors $\{x_i\}_{i=1}^n$ in $X$,

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^p\Big)^{1/p} \le C\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p}. \tag{6.7}$$

The smallest constant for which (6.7) holds is called the *type-$p$ constant of $X$* and is denoted by $T_p(X)$.

(b) A Banach space $X$ is said to have *Rademacher cotype $q$* (in short, *cotype $q$*) for some $2 \le q < \infty$ if there is a constant $C$ such that for every finite set of vectors $\{x_i\}_{i=1}^n$ in $X$,

$$\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q} \le C\Big(\mathbb{E}\Big\|\sum_{i=1}^{n} \varepsilon_i x_i\Big\|^q\Big)^{1/q}. \tag{6.8}$$

The smallest constant for which (6.8) holds is called the *cotype-q constant of X* and is denoted by $C_q(X)$.

*Remark 6.2.11.* (*a*) The restrictions on $p$ and $q$ in the definitions of type and cotype respectively are natural, since it is impossible to have type $p > 2$ or cotype $q < 2$ even in a one-dimensional space. To see this, for each $n$ take vectors $\{x_i\}_{i=1}^n$ all equal to some $x \in X$ with $\|x\| = 1$. The combination of Khintchine's inequality with (6.7) and (6.8) gives us the range of eligible values for $p$ and $q$.

(*b*) For every finite set of vectors $\{x_i\}_{i=1}^n$ in a Banach space $X$ the triangle inequality gives

$$\max_{1\le i\le n} \|x_i\| \le \mathbb{E}\Big\|\sum_{i=1}^{n} \varepsilon_i x_i\Big\| \le \sum_{i=1}^{N} \|x_i\|. \tag{6.9}$$

The right-hand-side inequality of (6.9) tells us that every Banach space $X$ has type 1 with $T_1(X) = 1$. Thus $X$ is said to have *nontrivial type* if it has type $p$ for some $1 < p \le 2$. The left-hand-side inequality of (6.9) in turn can be interpreted as saying that every Banach space $X$ has trivial cotype, or that it has cotype $\infty$ with $C_\infty(X) = 1$.

(*c*) The generalized parallelogram law (Proposition 6.2.9) says that a Hilbert space $H$ has type 2 and cotype 2 with $T_2(H) = C_2(H) = 1$. In particular, a one-dimensional space has type 2 and cotype 2. But the parallelogram law is also a characterization of Banach spaces that are linearly isometric to Hilbert spaces; hence we deduce that a Banach space $X$ is isometric to a Hilbert space if and only if $T_2(X) = C_2(X) = 1$ (see Problem 7.6).

(*d*) By Theorem 6.2.5, the $L_p$-average $(\mathbb{E}\|\sum_{i=1}^n \varepsilon_i x_i\|^p)^{1/p}$ in the definition of type can be replaced by any other $L_r$-average $(\mathbb{E}\|\sum_{i=1}^n \varepsilon_i x_i\|^r)^{1/r}$ $(1 \le r < \infty)$, and this has the effect only of changing the constant. The same comment applies to the $L_q$-average in the definition of cotype.

(*e*) If $X$ has type $p$ then $X$ has type $r$ for $r < p$ and if $X$ has cotype $q$ then $X$ has cotype $s$ for $s > q$.

(*f*) The type and cotype of a Banach space are isomorphic invariants and are inherited by subspaces.

(*g*) Consider the unit vector basis $(e_n)_{n=1}^\infty$ in $\ell_p$ $(1 \le p < \infty)$ or $c_0$. Then for any signs $(\epsilon_k)$, we have

$$\|\epsilon_1 e_1 + \cdots + \epsilon_n e_n\|_p = n^{1/p}$$

and

$$\|\epsilon_1 e_1 + \cdots + \epsilon_n e_n\|_\infty = 1.$$

Thus $\ell_p$ cannot have type greater than $p$ if $1 \leq p \leq 2$ or cotype less than $p$ if $2 \leq p \leq \infty$.

**Proposition 6.2.12.** *If a Banach space $X$ has type $p$, then $X^*$ has cotype $q$, where $\frac{1}{p} + \frac{1}{q} = 1$, and $C_q(X^*) \leq T_p(X)$.*

*Proof.* Let us pick an arbitrary finite set $\{x_i^*\}_{i=1}^n$ in $X^*$. Given $\epsilon > 0$, we can find $x_1, \dots, x_n$ in $X$ such that $\|x_i\| = 1$ and $|x_i^*(x_i)| \geq (1-\epsilon)\left\|x_i^*\right\|$ for all $1 \leq i \leq n$. Thus

$$\left(\sum_{i=1}^n |x_i^*(x_i)|^q\right)^{1/q} \geq (1-\epsilon)\left(\sum_{i=1}^n \left\|x_i^*\right\|^q\right)^{1/q}.$$

On the other hand,

$$\left(\sum_{i=1}^n |x_i^*(x_i)|^q\right)^{1/q} = \sup\Big\{\Big|\sum_{i=1}^n a_i x_i^*(x_i)\Big| : \sum_{i=1}^n |a_i|^p \leq 1\Big\}.$$

For any scalars $(a_i)_{i=1}^n$ with $\sum_{i=1}^n |a_i|^p \leq 1$ we have

$$\begin{aligned}
\sum_{i=1}^n a_i x_i^*(x_i) &= \int_\Omega \big(\sum_{i=1}^n \varepsilon_i x_i^*\big)\big(\sum_{i=1}^n \varepsilon_i a_i x_i\big)\, d\mathbb{P} \\
&\leq \int_\Omega \Big\|\sum_{i=1}^n \varepsilon_i x_i^*\Big\| \Big\|\sum_{i=1}^n \varepsilon_i a_i x_i\Big\|\, d\mathbb{P} \\
&\leq \Big(\int_\Omega \Big\|\sum_{i=1}^n \varepsilon_i x_i^*\Big\|^q d\mathbb{P}\Big)^{1/q} \Big(\int_\Omega \Big\|\sum_{i=1}^n \varepsilon_i a_i x_i\Big\|^p d\mathbb{P}\Big)^{1/p} \\
&\leq \Big(\int_\Omega \Big\|\sum_{i=1}^n \varepsilon_i x_i^*\Big\|^q d\mathbb{P}\Big)^{1/q} T_p(X)\Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p}.
\end{aligned}$$

Therefore,

$$\Big(\sum_{i=1}^n \left\|x_i^*\right\|^q\Big)^{1/q} \leq (1-\epsilon)^{-1} T_p(X)\Big(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i^*\Big\|^q\Big)^{1/q}.$$

Since $\epsilon$ was arbitrary, this shows that $C_q(X^*) \leq T_p(X)$. □

Curiously, Proposition 6.2.12 does not have a converse statement. At the end of the section we shall give an example showing that if $X$ has cotype $q$ for $q < \infty$, then $X^*$ may not have type $p$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Next we want to investigate the type and cotype of $L_p$ for $1 \le p < \infty$. To do so, we will estimate $\|(\sum_{i=1}^n |f_i|^2)^{1/2}\|_p$ in relation to the Rademacher averages $(\mathbb{E}\| \sum_{j=1}^n \varepsilon_j f_j\|_p^p)^{1/p}$ on a generic $L_p(\mu)$-space.

**Theorem 6.2.13.** *Let $1 \le p < \infty$. For every finite set of functions $\{f_i\}_{i=1}^n$ in $L_p(\mu)$,*

$$A_p \Big\| \Big( \sum_{i=1}^n |f_i|^2 \Big)^{\frac{1}{2}} \Big\|_p \le \Big( \mathbb{E} \Big\| \sum_{i=1}^n \varepsilon_i f_i \Big\|_p^p \Big)^{1/p} \le B_p \Big\| \Big( \sum_{i=1}^n |f_i|^2 \Big)^{\frac{1}{2}} \Big\|_p,$$

*where $A_p$, $B_p$ are the constants in Khintchine's inequalities (in particular, $A_p = 1$ for $2 \le p < \infty$ and $B_p = 1$ for $1 \le p \le 2$).*

*Proof.* For each $\omega \in \Omega$, from Khintchine's inequalities,

$$A_p \Big( \sum_{i=1}^n |f_i(\omega)|^2 \Big)^{1/2} \le \Big( \mathbb{E} \Big| \sum_{i=1}^n \varepsilon_i f_i(\omega) \Big|^p \Big)^{1/p},$$

where $A_p = 1$ for $2 \le p < \infty$. Now, using Fubini's theorem, we obtain

$$\begin{aligned} A_p^p \Big\| \Big( \sum_{i=1}^n |f_i|^2 \Big)^{1/2} \Big\|_p^p &\le \int_\Omega \mathbb{E} \Big| \sum_{i=1}^n \varepsilon_i f_i(\omega) \Big|^p \, d\mu \\ &= \mathbb{E} \Big( \int_\Omega \Big| \sum_{i=1}^n \varepsilon_i f_i(\omega) \Big|^p \, d\mu \Big) \\ &= \mathbb{E} \Big\| \sum_{i=1}^n \varepsilon_i f_i \Big\|_p^p. \end{aligned}$$

The converse estimate is obtained similarly. □

The next theorem is due to Orlicz for cotype [233, 234] and Nordlander for type [229]. Obviously, the language of type and cotype did not exist before the 1970s, and their results were stated differently. Note the difference in behavior of the $L_p$-spaces when $p > 2$ or $p < 2$. This is the first example in which we meet some fundamental change around the index $p = 2$, and as the reader will see, it is really because when $p/2 < 1$, the triangle inequality for positive functions in $L_{p/2}$ reverses.

**Theorem 6.2.14.**

(*a*) *If $1 \le p \le 2$, $L_p(\mu)$ has type $p$ and cotype 2.*
(*b*) *If $2 < p < \infty$, $L_p(\mu)$ has type 2 and cotype $p$.*

*Moreover,* (*a*) *and* (*b*) *are optimal.*

*Proof.* (*a*) Let us prove first that if $1 \le p \le 2$, then $L_p(\mu)$ has type $p$. We recall this elementary inequality:

**Lemma 6.2.15.** *Let* $0 < r \le 1$. *Then for any nonnegative scalars* $(\alpha_i)_{i=1}^n$ *we have*

$$(\alpha_1 + \cdots + \alpha_n)^r \le \alpha_1^r + \cdots + \alpha_n^r. \tag{6.10}$$

In this way, combining Theorem 6.2.13 with (6.10), we obtain

$$\begin{aligned}\Big(\mathbb{E}\,\Big\|\sum_{i=1}^n \varepsilon_i f_i\Big\|_p^p\Big)^{\frac{1}{p}} &\le \Big\|\big(\sum_{i=1}^n |f_i|^2\big)^{1/2}\Big\|_p \\ &= \Big\|\sum_{i=1}^n |f_i|^2\Big\|_{p/2}^{1/2} \\ &\le \Big\|\big(\sum_{i=1}^n |f_i|^p\big)^{2/p}\Big\|_{p/2}^{1/2} \\ &= \Big(\int_\Omega \sum_{i=1}^n |f_i|^p \, d\mu\Big)^{1/p} \\ &= \big(\sum_{i=1}^n \|f\|_p^p\big)^{1/p}.\end{aligned}$$

To show that $L_p(\mu)$ has cotype 2 when $1 \le p \le 2$, we need the reverse of Minkowski's inequality:

**Lemma 6.2.16.** *Let* $0 < r < 1$. *Then*

$$\|f + g\|_r \ge \|f\|_r + \|g\|_r\,,$$

*whenever f and g are nonnegative functions in* $L_r(\mu)$.

*Proof.* Without loss of generality we can assume that $\|f + g\|_r = 1$, and so $d\nu = (f + g)^r \, d\mu$ is a probability measure. This implies

$$\|f\|_r = \Big(\int_\Omega f^r \, d\mu\Big)^{1/r} = \Big(\int_{\{f+g>0\}} \frac{f^r}{(f+g)^r} \, d\nu\Big)^{1/r} \le \int_{\{f+g>0\}} \frac{f}{f+g} \, d\nu.$$

Analogously,

$$\|g\|_r \le \int_{\{f+g>0\}} \frac{g}{f+g} \, d\nu = \int_{\{f+g>0\}} \frac{g}{f+g} \, (f+g)^r d\mu.$$

Therefore $\|f\|_r + \|g\|_r \le 1 = \|f + g\|_r$. □

Now, combining Theorem 6.2.13 with Lemma 6.2.16 yields

$$\begin{aligned} A_p^{-1}\Big(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i f_i\Big\|_p^p\Big)^{1/p} &\geq \Big\|\Big(\sum_{i=1}^n |f_i|^2\Big)^{1/2}\Big\|_p \\ &= \Big\|\sum_{i=1}^n |f_i|^2\Big\|_{p/2}^{1/2} \\ &\geq \Big(\sum_{i=1}^n \|f_i^2\|_{p/2}\Big)^{1/2} \\ &= \Big(\sum_{i=1}^n \|f_i\|_p^2\Big)^{1/2}. \end{aligned}$$

In order to obtain the cotype-2 estimate we just have to replace the $L_p$-average $(\mathbb{E}\|\sum_{j=1}^n \varepsilon_j f_j\|_p^p)^{1/p}$ by $(\mathbb{E}\|\sum_{j=1}^n \varepsilon_j f_j\|_p^2)^{1/2}$ using Kahane's inequality (at the small cost of a constant).

$(b)$ For each $2 < p < \infty$, from Theorem 6.2.13 in combination with Kahane's inequality there exists a constant $C = C(p)$ such that

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i f_i\Big\|_p^2\Big)^{1/2} \leq C\Big\|\Big(\sum_{i=1}^n |f_i|^2\Big)^{1/2}\Big\|_p.$$

Since $p/2 > 1$, the triangle inequality now holds in $L_{p/2}(\mu)$, and hence

$$\Big\|\Big(\sum_{i=1}^n |f_i|^2\Big)^{1/2}\Big\|_p = \Big\|\sum_{i=1}^n |f_i|^2\Big\|_{p/2}^{1/2} \leq \Big(\sum_{i=1}^n \|f_i^2\|_{p/2}\Big)^{1/2} = \Big(\sum_{i=1}^n \|f_i\|_p^2\Big)^{1/2}.$$

This shows that $L_p(\mu)$ has type 2. Therefore, from part $(a)$ and Proposition 6.2.12 it follows that $L_p(\mu)$ has cotype $p$.

The last statement of the theorem follows from Remark 6.2.11 and the fact that $L_p(\mu)$ contains $\ell_p$ as a subspace. □

*Example 6.2.17.* To finish the section let us give an example showing that the concepts of type and cotype are not in duality, in the sense that the converse of Proposition 6.2.12 need not hold. The space $\mathcal{C}[0,1]$ fails to have nontrivial type, because it contains a copy of $L_1$, whereas its dual, $\mathcal{M}(K)$, has cotype 2 (we leave the verification of this fact to the reader).

## 6.3 Properties of $L_1$

In Section 6.1 we saw that the Haar basis is unconditional in $L_p$ when $1 < p < \infty$. It is, however, not unconditional in $L_1$, and this highlights an important difference between the cases $p = 1$ and $p > 1$.

**Proposition 6.3.1.** *The Haar basis is not unconditional in* $L_1$.

*Proof.* For each $N \in \mathbb{N}$ let

$$f_N(t) = 2^{1-2N}\chi_{[0,2^{1-2N}]}(t), \quad t \in [0,1].$$

Let us use the device of labeling the elements of the Haar system by their supports. With this convention, expanding with respect to the Haar basis gives

$$f_N = \chi_{[0,1]} + \sum_{j=0}^{2N} 2^j h_{[0,2^{-j})}.$$

If for each $N$ we suppress the odd terms in the expansion of $f_N$, we obtain the function

$$g_N = \sum_{j=0}^{N} 2^{2j} h_{[0,2^{-2j})}.$$

Then, for $2^{-2j-1} \leq t < 2^{-2j}$ and $0 \leq j \leq N$ we have

$$g_N(t) = 1 + 4 + 4^2 + \cdots + 4^{j-1} - 4^j = \frac{4^j - 1}{3} - 4^j = -\frac{2^{2j+1}+1}{3}.$$

Thus

$$\|g_N\|_1 \geq \sum_{j=0}^{N} \frac{2^{2j+1}+1}{3} 2^{-2j-1} \geq \frac{(N+1)}{3} = \frac{(N+1)}{3}\|f_N\|_1.$$

This shows immediately that the Haar system cannot be unconditional, because it violates condition (iv) in Proposition 3.1.5. □

In fact, we will show that $L_1$ cannot be embedded in a space with an unconditional basis; this result is due to Pełczyński [242]. In Theorem 4.5.2 we showed, by the technique of testing property $(u)$, that $\mathcal{C}(K)$ embeds in a space with unconditional basis if and only if $\mathcal{C}(K) \approx c_0$. For $L_1$ this approach does not work, because $L_1$ is weakly sequentially complete and therefore has property $(u)$. A more sophisticated argument is therefore required. The argument we use was originally discovered by Milman [221]; first we need a lemma:

**Lemma 6.3.2.** *For every $f \in L_1$ we have*

$$\lim_{n\to\infty} \int_0^1 f(t) r_n(t) dt = 0.$$

*Thus $(fr_n)_{n=1}^\infty$ is weakly null for every $f \in L_1$.*

*Proof.* The sequence $(r_n)_{n=1}^\infty$ is orthonormal in $L_2$, which implies (by Bessel's inequality) that for $f \in L_2$,

$$\lim_{n\to\infty} \int_0^1 f(t) r_n(t) dt = 0.$$

Since $(r_n)_{n=1}^\infty$ is uniformly bounded in $L_\infty$, and $L_2$ is dense in $L_1$, we deduce

$$\lim_{n\to\infty} \int_0^1 f(t) r_n(t) dt = 0, \qquad \forall f \in L_1.$$

Thus if $f \in L_1$ and $g \in L_\infty$, since $fg \in L_1$, we obtain

$$\lim_{n\to\infty} \int_0^1 g(t) f(t) r_n(t) dt = 0,$$

which gives the latter statement in the lemma. □

**Theorem 6.3.3.** *The space $L_1$ cannot be embedded in a Banach space with unconditional basis.*

*Proof.* Let $X$ be a Banach space with $K$-unconditional basis $(e_n)_{n=1}^\infty$ and suppose that $T : L_1 \to X$ is an embedding. We can assume that for some constant $M \ge 1$,

$$\|f\|_1 \le \|Tf\| \le M\|f\|_1, \qquad f \in L_1.$$

By exploiting the unconditionality of $(e_n)_{n=1}^\infty$ we are going to build an unconditional basic sequence in $L_1$ using a gliding-hump-type argument.

Take $(\delta_k)_{k=1}^\infty$ a sequence of positive real numbers with $\sum_{k=1}^\infty \delta_k < 1$. Let $f_0 = 1 = \chi_{[0,1]}$, $n_1 = 1$, $s_0 = 0$ and pick $s_1 \in \mathbb{N}$ such that

$$\Big\| \sum_{j=s_1+1}^{\infty} e_j^*(T(f_0 r_{n_1})) e_j \Big\| < \frac{1}{2}\delta_1.$$

Put

$$x_1 = \sum_{j=s_0+1}^{s_1} e_j^*(T(f_0 r_{n_1})) e_j.$$

Next take $f_1 = (1 + r_{n_1})f_0$. Since the sequence $(f_1 r_k)_{k=1}^{\infty}$ is weakly null by Lemma 6.3.2, $(T(f_1 r_k))_{k=1}^{\infty}$ is also weakly null; hence we can find $n_2 \in \mathbb{N}$, $n_2 > n_1$, such that

$$\Big\| \sum_{j=1}^{s_1} e_j^*(T(f_1 r_{n_2}))e_j \Big\| < \frac{1}{2}\delta_2.$$

Now pick $s_2 \in \mathbb{N}$, $s_2 > s_1$, for which

$$\Big\| \sum_{j=s_2+1}^{\infty} e_j^*(T(f_1 r_{n_2}))e_j \Big\| < \frac{1}{2}\delta_2.$$

Continuing in this way, we will inductively select two strictly increasing sequences of natural numbers $(n_k)_{k=1}^{\infty}$ and $(s_k)_{k=0}^{\infty}$, a sequence of functions $(f_k)_{k=0}^{\infty}$ in $L_1$ given by

$$f_k = (1 + r_{n_k})f_{k-1} \quad \text{for } k \geq 1,$$

and a block basic sequence $(x_k)_{k=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$ defined by

$$x_k = \sum_{j=s_{k-1}+1}^{s_k} e_j^*(T(f_{k-1} r_{n_k}))e_j, \qquad k = 1, 2, \dots.$$

This is how the inductive step goes: suppose $n_1, n_2, \dots, n_{l-1}$, $s_0, s_1, \dots, s_{l-1}$, and therefore $f_1, \dots, f_{l-1}$ have been determined. Since $(T(f_{l-1} r_k))_{k=1}^{\infty}$ is weakly null, we can find $n_l > n_{l-1}$ such that

$$\Big\| \sum_{j=1}^{s_{l-1}} e_j^*(T(f_{l-1} r_{n_l}))e_j \Big\| < \frac{1}{2}\delta_l,$$

and then we choose $s_l > s_{l-1}$ such that

$$\Big\| \sum_{j=s_l+1}^{\infty} e_j^*(T(f_{l-1} r_{n_l}))e_j \Big\| < \frac{1}{2}\delta_l.$$

Note that for $k \geq 1$ we have

$$f_k = \prod_{j=1}^{k} (1 + r_{n_j}), \tag{6.11}$$

which yields $f_k \geq 0$ for all $k$. Expanding out (6.11), it is also clear that for each $k$,

$$\|f_k\|_1 = \int_0^1 f_k(t)\, dt = 1.$$

On the other hand, for $k \geq 1$ we have

$$\left\| Tf_k - Tf_{k-1} - x_k \right\| < \delta_k,$$

and hence the estimate

$$\left\| \sum_{j=1}^{n} x_j \right\| < M + \sum_{j=1}^{n} \delta_j < M + 1$$

holds for all $n$.

Since it is a block basic sequence with respect to $(e_n)_{n=1}^{\infty}$, $(x_k)_{k=1}^{\infty}$ is an unconditional basic sequence in $X$ with unconditional constant $\leq K$ (see Problem 3.1). Therefore for all choices of signs $\epsilon_j = \pm 1$ and all $n = 1, 2, \ldots$ we have a bound:

$$\left\| \sum_{j=1}^{n} \epsilon_j x_j \right\| \leq K(M + 1),$$

which implies

$$\left\| \sum_{j=1}^{n} \epsilon_j (Tf_j - Tf_{j-1}) \right\| \leq K(M + 1) + 1,$$

and thus

$$\left\| \sum_{j=1}^{n} \epsilon_j (f_j - f_{j-1}) \right\|_1 \leq K(M + 1) + 1.$$

This shows that $\sum_{j=1}^{\infty} (f_j - f_{j-1})$ is a WUC series in $L_1$ (see Lemma 2.4.6). Since $L_1$ is weakly sequentially complete (Theorem 5.2.9), by Corollary 2.4.15 the series $\sum_{j=1}^{\infty} (f_j - f_{j-1})$ must converge (unconditionally) in norm in $L_1$, and in particular, $\lim_{j\to\infty} \|f_j - f_{j-1}\|_1 = 0$. But for $j \geq 1$ we have $\|f_j - f_{j-1}\|_1 = \|r_{n_j} f_{j-1}\|_1 = 1$, a contradiction. □

In Corollary 2.5.6 we saw that $c_0$ is not a dual space. We will show that $L_1$ is also not a dual space and, even more generally, that it cannot be embedded in a separable dual space. We know that $c_0$ is not isomorphic to a dual space, because $c_0$ is uncomplemented in its bidual. This is not the case for $L_1$, as we shall see below. Thus to show that $L_1$ is not a dual space requires another type of argument, and we will use some rather more delicate geometric properties of separable dual spaces.

**Lemma 6.3.4.** *Let $X$ be a Banach space such that $X^*$ is separable. Assume that $K$ is a weak$^*$ compact set in $X^*$. Then $K$ has a point of weak$^*$-to-norm continuity. That is, there is $x^* \in K$ such that whenever a sequence $(x_n^*)_{n=1}^\infty \subset K$ converges to $x^*$ with respect to the weak$^*$ topology of $X^*$, then $(x_n^*)_{n=1}^\infty$ converges to $x^*$ in the norm topology of $X^*$.*

*Proof.* Let $(\epsilon_n)_{n=1}^\infty$ be a sequence of scalars converging to zero. Using that $X^*$ is separable for the norm topology, for each $n$ there is a sequence of points $(x_{k,n}^*)_{k=1}^\infty \subset X^*$ such that

$$K \subset \bigcup_{k=1}^\infty B(x_{k,n}^*, \epsilon_n).$$

Let us call $V_{k,n}$ the weak$^*$ interior of $B(x_{k,n}^*, \epsilon_n) \cap K$ in the space $(K, w^*)$ (i.e., the space $K$ equipped with the weak$^*$ topology that it inherits from $X^*$). Notice that

$$V_n = \bigcup_{k=1}^\infty V_{k,n}$$

is dense and open.

Since $X^*$ is separable, $(K, w^*)$ is metrizable. Then $K$ is compact metric, therefore complete. By the Baire category theorem, the set $V = \bigcap_{n=1}^\infty V_n$ is a dense $G_\delta$-set. We are going to see that all of the elements in $V$ are points of weak$^*$-to-norm continuity. Indeed, take $v^* \in V$. We will see that the identity operator

$$I : (K, w^*) \longrightarrow (K, \|\cdot\|)$$

is continuous at $v^*$. Pick $\delta > 0$. Since $(\epsilon_n)_{n=1}^\infty$ converges to zero, there is $n$ such that $2\epsilon_n < \delta$. Now, there exists $k$ such that $v^* \in V_{k,n}$. Since the diameter of $V_{k,n}$ is at most $2\epsilon_n$, we have $V_{k,n} \subset B(v^*, \delta)$. ◻

**Lemma 6.3.5.** *Suppose $X$ is a Banach space that embeds in a separable dual space. Then every closed bounded subset $F$ of $X$ has a point of weak-to-norm continuity.*

*Proof.* Let $F$ be a closed bounded subset of $X$. Suppose $T : X \to Y^*$ is an embedding in $Y^*$, where $Y$ is a Banach space with separable dual. We can assume that $\|x\| \le \|Tx\| \le M\|x\|$ for $x \in X$, where $M$ is a constant independent of $x$. Let $W$ be the weak$^*$ closure of $T(F)$. Then by Lemma 6.3.4 there is $y^* \in W$ that is a point of weak$^*$-to-norm continuity. In particular, there is a sequence $(y_n^*)$ in $T(F)$ with $\|y_n^* - y^*\| \to 0$. If we let $y_n^* = Tx_n$ with $x_n \in F$ for each $n$, then $(x_n)_{n=1}^\infty$ is Cauchy in $X$ and so converges to some $x \in F$; hence $Tx = y^*$. Now for every $\epsilon > 0$ we can find a weak$^*$ neighborhood $U_\epsilon$ of $y^*$ such that $w^* \in U_\epsilon \cap W$ implies $\|w^* - y^*\| < \epsilon$. In particular, if $v \in T^{-1}(U_\epsilon) \cap C$, then $\|v - x\| < \epsilon$. Clearly $T^{-1}(U_\epsilon)$ is weakly open, since the map $T : X \to Y^*$ is weak-to-weak$^*$ continuous. This shows that $x$ is a point of weak-to-norm continuity. ◻

**Lemma 6.3.6.** *Suppose $X$ is a Banach space that embeds in a separable dual space and let $x \in B_X$ be a point of weak-to-norm continuity. If $(x_n)$ is a weakly null sequence in $X$ such that* $\limsup \|x + x_n\| \le 1$, *then* $\lim_{n\to\infty} \|x_n\| = 0$.

*Proof.* Put

$$u_n = \begin{cases} x + x_n & \text{if } \|x + x_n\| \le 1, \\ \dfrac{x + x_n}{\|x + x_n\|} & \text{if } \|x + x_n\| > 1, \end{cases}$$

and observe that

$$u_n - x = x_n + (1 - \alpha_n)(x + x_n),$$

where $\alpha_n = (\|x+x_n\|-1)_+ \to 0$. Thus $\lim_n u_n = x$ weakly and so $\lim_n \|u_n - x\| = 0$. This implies that $\lim_n \|x_n\| = 0$. □

**Theorem 6.3.7.** *Neither of the Banach spaces $L_1$ and $c_0$ can be embedded in a separable dual space.*

*Proof.* If $L_1$ embeds in a separable dual space, Lemma 6.3.5 yields a function $f \in B_{L_1}$ that is a point of weak-to-norm continuity. By Lemma 6.3.2 the sequence $(r_n f)_{n=1}^{\infty}$ is weakly null in $L_1$ and satisfies

$$\|f + fr_n\|_1 = \int (1 + r_n(t))|f(t)|\, dt \longrightarrow 1.$$

Therefore by Lemma 6.3.6 it must be $\lim_{n\to\infty} \|r_n f\|_1 = 0$, which implies $f = 0$. This is absurd, since $(r_n)_{n=1}^{\infty}$ is a weakly null sequence and $\|r_n\|_1 = 1$.

For $c_0$ the argument is similar. Let $\xi$ be a point of weak-to-norm continuity in $B_{c_0}$. Then if $(e_n)_{n=1}^{\infty}$ is the canonical basis, we have $\lim_{n\to\infty} \|\xi + e_n\| = 1$ and so $\lim_{n\to\infty} \|e_n\| = 0$, which is again absurd. □

*Remark 6.3.8.* The fact that $c_0$ cannot be embedded in a separable dual space can be proved in many ways, and we have already seen this in Problems 2.6 and 2.9.

**Corollary 6.3.9.** *The space $L_1$ does not have a boundedly complete basis.*

*Proof.* We need only recall that by Theorem 3.2.15, a space with a boundedly complete basis is (isomorphic to) a separable dual space. □

Theorem 6.3.7 is rather classical, and it is due to Gelfand [103]. In fact, the argument we have given is somewhat ad hoc; to be more precise, one should use the Radon–Nikodym property, which we discussed earlier in Section 5.5. The main point here is that neither $L_1$ nor $c_0$ has the Radon–Nikodym property, while separable dual spaces do.

Let us conclude this section with the promised result that $L_1$ is complemented in its bidual.

**Proposition 6.3.10.** *There is a norm-one linear projection $P : L_1^{**} \to L_1$.*

*Proof.* Let $J$ be the cannonical embedding of $L_1$ into $\mathcal{M}[0,1]$. Define $Q\colon \mathcal{M}[0,1] \to L_1$ by $Q(\mu) = f$, where $d\mu = f\,dm + d\nu$ and $\nu$ is singular with respect to the Lebesgue measure $m$ on $[0,1]$. We have $Q \circ J = I_{L_1}$, i.e., $Q$ is a projection. By the Riesz representation theorem, $\mathcal{M}[0,1]$ can be isometrically identified with $\mathcal{C}[0,1]^*$. Since $\mathcal{M}[0,1]$ is a dual space, there is a canonical norm-one projection $R\colon \mathcal{M}[0,1]^{**} \to \mathcal{M}[0,1]$. Now it is routine to check that $Q \circ R \circ J^{**}$ is a projection from $L_1^{**}$ onto $L_1$. □

## 6.4 Subspaces of $L_p$

In Chapter 2 we studied the subspace structure and the complemented subspace structure of the spaces $\ell_p$ for $1 \le p < \infty$ (see particularly Corollary 2.1.6 and Theorem 2.2.4). Now we would like to analyze the function space analogues, the $L_p$-spaces for $1 \le p < \infty$, in the same way. This is a more delicate problem, and the subspace structure is much richer, with the exception of the case $p = 2$, which is trivial, since $L_2$ is isometric to $\ell_2$. We will also see some fundamental differences between the cases $1 < p < 2$ and $2 < p < \infty$.

**Proposition 6.4.1.** *Let $(f_n)_{n=1}^{\infty}$ be a sequence of norm-one, disjointly supported functions in $L_p$. Then $(f_n)_{n=1}^{\infty}$ is a complemented basic sequence isometrically equivalent to the canonical basis of $\ell_p$.*

*Proof.* The case $p = 1$ was seen in Lemma 5.1.1. Let us fix $1 < p < \infty$. For every sequence of scalars $(a_i)_{i=1}^{\infty} \in c_{00}$, by the disjointness of the $f_i$'s we have

$$
\begin{aligned}
\Big\| \sum_{i=1}^{\infty} a_i f_i \Big\|_p^p &= \int_0^1 \Big| \sum_{i=1}^{\infty} a_i f_i(t) \Big|^p \, dt \\
&= \int_0^1 \sum_{i=1}^{\infty} |a_i f_i(t)|^p \, dt \\
&= \sum_{i=1}^{\infty} |a_i|^p \int_0^1 |f_i(t)|^p \, dt \\
&= \sum_{i=1}^{\infty} |a_i|^p .
\end{aligned}
$$

By the Hahn–Banach theorem, for each $i \in \mathbb{N}$ there exists $g_i \in L_q$ ($q$ the conjugate exponent of $p$) with $\|g_i\|_q = 1$ such that $1 = \|f_i\|_p = \int_0^1 f_i(t) g_i(t)\, dt$. Furthermore, without loss of generality, we can assume $g_i$ to have the same support as $f_i$ for all $i$. Let us define the linear operator from $L_p$ onto $[f_i]$ given by

$$P(f) = \sum_{i=1}^{\infty} \Big( \int_0^1 f(t) g_i(t)\, dt \Big) f_i, \quad f \in L_p.$$

Then,

$$\begin{aligned}
\|P(f)\|_p &= \Big( \sum_{i=1}^{\infty} \Big| \int_0^1 f(t) g_i(t)\, dt \Big|^p \Big)^{1/p} \\
&= \Big( \sum_{i=1}^{\infty} \Big| \int_{\{t : |f_i(t)| > 0\}} f(t) g_i(t)\, dt \Big|^p \Big)^{1/p} \\
&\leq \Big( \sum_{i=1}^{\infty} \int_{\{t : |f_i(t)| > 0\}} |f(t)|^p\, dt \Big)^{1/p} \\
&\leq \Big( \int_0^1 |f(t)|^p\, dt \Big)^{1/p}.
\end{aligned}$$

□

The following proposition allows us to deduce that $L_p$ is not isomorphic to $\ell_p$ for $p \neq 2$, and already hints at the fact that the $L_p$-spaces have a more complicated structure than the spaces $\ell_p$.

**Proposition 6.4.2.** *The space $\ell_2$ embeds in $L_p$ for all $1 \leq p < \infty$. Furthermore, $\ell_2$ embeds complementably in $L_p$ if and only if $1 < p < \infty$.*

*Proof.* For each $1 \leq p < \infty$ let $R_p$ be the closed subspace spanned in $L_p$ by the Rademacher functions $(r_n)_{n=1}^{\infty}$. By Khintchine's inequality, $(r_n)$ is equivalent to the canonical basis of $\ell_2$ and so $R_p$ is isomorphic to $\ell_2$.

By Proposition 5.7.1, $L_1$ has no infinite-dimensional complemented reflexive subspaces, so $R_1$ is not complemented in $L_1$. Let us prove that if $1 < p < \infty$, then $R_p$ is complemented in $L_p$.

Assume first that $2 \leq p < \infty$. Consider the map from $L_p$ onto $R_p$ given by

$$P(f) = \sum_{n=1}^{\infty} \Big( \int_0^1 f(t) r_n(t)\, dt \Big) r_n, \quad f \in L_p.$$

The map $P$ is linear and well defined. Indeed, the series is convergent in $L_p$, because $f \in L_p \subset L_2$ implies $\sum_{n=1}^{\infty} (\int f(t) r_n(t)\, dt)^2 < \infty$. Clearly, $P$ is the identity map when restricted to $R_p$. Now Khintchine's inequalities and Bessel's inequality yield

$$\|P(f)\|_p^2 = \Big\| \sum_{n=1}^{\infty} \Big( \int_0^1 f(t) r_n(t)\, dt \Big) r_n \Big\|_p^2$$

$$
\begin{aligned}
&\le B_p^2 \sum_{n=1}^{\infty} \Big| \int_0^1 f(t) r_n(t)\, dt \Big|^2 \\
&\le B_p^2 \, \|f\|_2^2 \\
&\le B_p^2 \, \|f\|_p^2 .
\end{aligned}
$$

If $1 < p < 2$, we define $P$ as before for each $f \in L_p \cap L_2$ (which is a dense subspace in $L_p$). Then, using Khintchine's inequalities, we obtain

$$
\begin{aligned}
\|P(f)\|_p &\le \Big( \sum_{n=1}^{\infty} \Big| \int_0^1 f(t) r_n(t)\, dt \Big|^2 \Big)^{1/2} \\
&= \sup \Big\{ \sum_{n=1}^{\infty} \Big( \alpha_n \int_0^1 f(t) r_n(t)\, dt \Big) \,:\, \sum_{n=1}^{\infty} \alpha_n^2 = 1 \Big\} \\
&= \sup \Big\{ \int_0^1 f(t) \Big( \sum_{n=1}^{\infty} \alpha_n r_n(t) \Big)\, dt \,:\, \sum_{n=1}^{\infty} \alpha_n^2 = 1 \Big\} \\
&\le \sup \Big\{ \|f\|_p \, \Big\| \sum_{n=1}^{\infty} \alpha_n r_n(t) \Big\|_q \,:\, \sum_{n=1}^{\infty} \alpha_n^2 = 1 \Big\} \\
&\le \sup \Big\{ \|f\|_p \, B_q \Big\| \sum_{n=1}^{\infty} \alpha_n r_n(t) \Big\|_2 \,:\, \sum_{n=1}^{\infty} \alpha_n^2 = 1 \Big\} \\
&= B_q \, \|f\|_p .
\end{aligned}
$$

By density, $P$ extends continuously to $L_p$ with preservation of norm. □

**Proposition 6.4.3.** *If $\ell_q$ embeds in $L_p$, then either $p \le q \le 2$ or $2 \le q \le p$.*

*Proof.* Let us start by noticing that if $(e_i)_{i=1}^{\infty}$ is the canonical basis of $\ell_q$, then for each $n$ we have

$$
\mathbb{E} \Big\| \sum_{i=1}^{n} \varepsilon_i e_i \Big\|_q = n^{1/q}.
$$

If $\ell_q$ embeds in $L_p$ for some $p < 2$, by Theorem 6.2.14 there exist constants $c_1$ and $c_2$ (given by the embedding and the type and cotype constants) such that

$$
c_1 \, n^{1/2} \le n^{1/q} \le c_2 \, n^{1/p}.
$$

For these inequalities to hold for all $n \in \mathbb{N}$ it is necessary that $q \in [p, 2]$. If $\ell_q$ embeds in $L_p$ for some $2 < p < \infty$, with the same kind of argument we deduce that $q$ must belong to the interval $[2, p]$. □

**Definition 6.4.4.** Suppose $(\Omega, \Sigma, \mu)$ is a probability measure space and let $X$ be a closed subspace of $L_p(\mu)$ for some $1 \le p < \infty$. The subspace $X$ is said to be *strongly embedded* in $L_p(\mu)$ if in $X$, convergence in measure is equivalent to convergence in the $L_p(\mu)$-norm; that is, a sequence of functions $(f_n)_{n=1}^\infty$ in $X$ converges to 0 in measure if and only if $\|f_n\|_p \to 0$.

**Proposition 6.4.5.** *Suppose* $(\Omega, \Sigma, \mu)$ *is a probability measure space and let* $1 \le p < \infty$. *Suppose X is an infinite-dimensional closed subspace of* $L_p(\mu)$. *Then the following are equivalent:*

*(i) X is strongly embedded in* $L_p(\mu)$.
*(ii) For each* $0 < q < p$ *there exists a constant* $C_q > 0$ *such that*

$$\|f\|_q \le \|f\|_p \le C_q \|f\|_q \quad \textit{for all } f \in X.$$

*(iii) For some* $0 < q < p$ *there exists a constant* $C_q > 0$ *such that*

$$\|f\|_q \le \|f\|_p \le C_q \|f\|_q \quad \textit{for all } f \in X.$$

*Proof.* Let us suppose that $X$ is strongly embedded in $L_p(\mu)$ but (*ii*) fails. Then there would exist a sequence $(f_n)_{n=1}^\infty$ in $X$ such that $\|f_n\|_p = 1$ and $\|f_n\|_q \to 0$ for some $0 < q < p$. Obviously, this implies that $(f_n)_{n=1}^\infty$ converges to 0 in measure, which would force $(\|f_n\|_p)_{n=1}^\infty$ to converge to 0. This contradiction shows that (*i*) $\Rightarrow$ (*ii*).

Suppose now that (*iii*) holds and there is a sequence of functions $(f_n)_{n=1}^\infty$ in $X$ such that $(f_n)_{n=1}^\infty$ converges to 0 in measure but $(\|f_n\|_p)_{n=1}^\infty$ does not tend to 0. By passing to a subsequence, we can assume that $(f_n)_{n=1}^\infty$ converges to 0 almost everywhere and $\|f_n\|_p = 1$ for all $n$.

For each $M > 0$, since $q < p$, we have

$$\begin{aligned} \int_\Omega |f_n|^q \, d\mu &= \int_{\{|f_n| \ge M\}} |f_n|^q \, d\mu + \int_{\{|f_n| < M\}} |f_n|^q \, d\mu \\ &\le \int_{\{|f_n| \ge M\}} M^{q-p} |f_n|^p \, d\mu + \int_{\{|f_n| < M\}} |f_n|^q \, d\mu \\ &\le \frac{1}{M^{p-q}} + \int_{\{|f_n| < M\}} |f_n|^q \, d\mu. \end{aligned}$$

Let $\epsilon > 0$. By the bounded convergence theorem, there is $N_0 \in \mathbb{N}$ such that $\int_{\{|f_n| < M\}} |f_n|^q \, d\mu < \epsilon/2$ for all $n > N_0$. So, if we pick $M > (2\epsilon^{-1})^{\frac{1}{p-q}}$, we get

$$\int_\Omega |f_n|^q \, d\mu < \epsilon,$$

contradicting (*iii*). Hence (*iii*) $\Rightarrow$ (*i*), and so the proof is complete because trivially, (*ii*) $\Rightarrow$ (*iii*). $\square$

*Example 6.4.6.* For each $1 \le p < \infty$ the closed subspace spanned in $L_p$ by the Rademacher functions, $R_p$, is strongly embedded in $L_p$, since using Khintchine's inequality, the $L_q$-norm and the $L_p$-norm are equivalent in $R_p$ for all $1 \le q < \infty$.

**Theorem 6.4.7.** *Suppose that $X$ is an infinite-dimensional closed subspace of $L_p$ for some $1 \le p < \infty$. If $X$ is not strongly embedded in $L_p$, then $X$ contains a subspace isomorphic to $\ell_p$ and complemented in $L_p$.*

*Proof.* If $X$ is not strongly embedded in $L_p$, there is a sequence $(f_n)_{n=1}^\infty$ in $X$ with $\|f_n\|_p = 1$ for all $n$ such that $f_n \to 0$ almost everywhere. By Lemma 5.2.1 there exist a subsequence $(f_{n_k})_{k=1}^\infty$ of $(f_n)_{n=1}^\infty$ and a sequence of disjoint subsets $(A_k)_{k=1}^\infty$ of $[0,1]$ such that if $B_k = [0,1] \setminus A_k$, then $(|f_{n_k}|^p \chi_{B_k})_{k=1}^\infty$ is equi-integrable. Lemma 5.2.6 implies $\int |f_{n_k}|^p \chi_{B_k}\, d\mu \to 0$. That is, $\|f_{n_k} - f_{n_k}\chi_{A_k}\|_p \to 0$. Now by standard perturbation arguments we obtain a subsequence $(f_{n_{k_j}})_{j=1}^\infty$ of $(f_{n_k})_{k=1}^\infty$ such that $(f_{n_{k_j}})_{j=1}^\infty$ is equivalent to the canonical basis of $\ell_p$ and $[f_{n_{k_j}}]$ is complemented in $L_p$. $\square$

The following theorem was proved by Kadets and Pełczyński [147] in a paper that really initiated the study of $L_p$ spaces by basic sequence techniques. We will see that the case $p > 2$ is quite different from the case $p < 2$, and this theorem emphasizes this point.

**Theorem 6.4.8 (Kadets and Pełczyński [147]).** *Suppose that $X$ is an infinite-dimensional closed subspace of $L_p$ for some $2 < p < \infty$. Then the following are equivalent:*

*(i) The space $\ell_p$ does not embed in $X$.*
*(ii) The space $\ell_p$ does not embed complementably in $X$.*
*(iii) $X$ is strongly embedded in $L_p$.*
*(iv) $X$ is isomorphic to a Hilbert space and is complemented in $L_p$.*
*(v) $X$ is isomorphic to a Hilbert space.*

*Proof.* (*i*) $\Rightarrow$ (*ii*) and (*iv*) $\Rightarrow$ (*v*) are obvious, and (*ii*) $\Rightarrow$ (*iii*) was proved in Theorem 6.4.7. Let us complete the circle by showing that (*iii*) $\Rightarrow$ (*iv*) and that (*v*) $\Rightarrow$ (*i*).

(*iii*) $\Rightarrow$ (*iv*) If $X$ is strongly embedded in $L_p$, Proposition 6.4.5 yields a constant $C_2$ such that $\|f\|_2 \le \|f\|_p \le C_2 \|f\|_2$ for all $f \in X$. This shows that $X$ embeds in $L_2$, and hence it is isomorphic to a Hilbert space. Let us see that $X$ is complemented in $L_2$.

Since $p > 2$, $L_p$ is contained in $L_2$, and the inclusion $\iota : L_p \to L_2$ is norm decreasing. The restriction of $\iota$ to $X$ is an isomorphism onto the subspace $\iota(X)$ of $L_2$, and $\iota(X)$ is complemented in $L_2$ by an orthogonal projection $P$. In diagram form,

$$\begin{array}{ccc} L_p & \xrightarrow{\iota} & L_2 \\ \uparrow & & \downarrow P \\ X & \xrightarrow{\iota|_X} & \iota(X) \end{array}$$

Then $\iota^{-1}P\iota$ is a projection of $L_p$ onto $X$ (this projection is simply the restriction of $P$ to $L_p$).

$(v) \Rightarrow (i)$ If $X \approx \ell_2$, then $X$ cannot contain an isomorphic copy of $\ell_p$ for any $p \neq 2$, because the classical sequence spaces are totally incomparable (Corollary 2.1.6). □

The Kadets–Pełczyński theorem establishes a dichotomy for subspaces of $L_p$ when $2 < p < \infty$:

**Corollary 6.4.9.** *Suppose $X$ is a closed subspace of $L_p$ for some $2 < p < \infty$. Then either*

*(i) $X$ is isomorphic to $\ell_2$, in which case $X$ is complemented in $L_p$, or*
*(ii) $X$ contains a subspace that is isomorphic to $\ell_p$ and complemented in $L_p$.*

Notice that in particular, this settles the question of which $L_q$ spaces for $1 \le q < \infty$ embed in $L_p$ for $p > 2$:

**Corollary 6.4.10.** *For $2 < p < \infty$ and $1 \le q < \infty$ with $q \neq p, 2$, $L_p$ does not have any subspace isomorphic to $L_q$ or $\ell_q$.*

We are now ready to find a more efficient embedding of $\ell_2$ into the $L_p$-spaces, replacing the Rademacher sequences by sequences of independent Gaussians. We consider only the real case, although modifications can be made to handle complex functions.

**Proposition 6.4.11.** *If $g$ is a Gaussian on some probability measure space $(\Omega, \Sigma, \mu)$, then $g \in L_p(\mu)$ for every $1 \le p < \infty$.*

*Proof.* This is because

$$\int_\Omega |g(\omega)|^p \, d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^p e^{-\frac{1}{2}x^2} \, dx,$$

and the last integral is finite and indeed computable in terms of the $\Gamma$ function as

$$\frac{2^{p/2}}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right).$$

□

**Proposition 6.4.12.** *The space $\ell_2$ embeds isometrically in $L_p$ for all $1 \le p < \infty$.*

*Proof.* Take $(g_j)_{j=1}^{\infty}$, a sequence of independent Gaussians on $[0, 1]$. By Proposition 6.4.11, $(g_j)_{j=1}^{\infty} \subset L_p$. We will show that $[g_j]$ is isometrically isomorphic to $\ell_2$.

For every $n \in \mathbb{N}$ and scalars $(a_j)_{j=1}^{n}$ such that $\sum_{j=1}^{n} a_j^2 = 1$, put

$$h_n = \sum_{j=1}^{n} a_j g_j.$$

By (I.2) we have

$$\phi_{h_n}(t) = e^{-(a_1^2+\cdots+a_n^2)t^2/2} = e^{-t^2/2}.$$

This means that $\mu_{h_n} = \mu_{g_1}$, and so by (I.1),

$$\|h_n\|_p = \|g_1\|_p.$$

It follows that for all $a_1, \ldots, a_n$ in $\mathbb{R}$,

$$\Big\| \sum_{j=1}^{n} a_j g_j \Big\|_p = \|g_1\|_p \Big( \sum_{j=1}^{n} |a_j|^2 \Big)^{1/2}.$$

Thus the mapping $e_n \mapsto \|g_1\|_p^{-1} g_n$ linearly extends to an isometry from $\ell_2$ onto the subspace $[g_n]$ of $L_p$. □

The connection between the Gaussians and $\ell_2$ is encoded in the characteristic function. We are now going to dig a little deeper to try to make copies of $\ell_q$ for other values of $q$ in the $L_p$-spaces. A moment's thought shows that we need a random variable $f$ with characteristic function

$$\phi_f(t) = e^{-c|t|^q}$$

for some constant $c = c(q)$. It turns out that if (and only if) $0 < q < 2$, we can construct such a random variable. This has long been known to probabilists; here we give a treatment based on some unpublished notes of Ben Garling.

We will need the following classical lemma due to Paul Lévy (see, for instance, [91]).

**Lemma 6.4.13.** *Suppose $(\mu_n)_{n=1}^{\infty}$ is a sequence of probability measures on $\mathbb{R}$ such that*

$$\lim_{n\to\infty} \hat{\mu}_n(-t) = F(t)$$

*exists for all $t \in \mathbb{R}$. If $F$ is continuous, then there is a probability measure $\mu$ on $\mathbb{R}$ such that $\hat{\mu}(-t) = F(t)$.*

*Proof.* It is convenient to compactify the real line by adding one point at $\infty$ to make the one-point compactification $K = \mathbb{R} \cup \{\infty\}$. We can regard each signed finite measure $\nu$ on $\mathbb{R}$ as a measure in the space $\mathcal{M}(K)$ that assigns zero mass to $\{\infty\}$. We also identify $\mathcal{M}(K)$ with $(\mathcal{C}(K))^*$. Notice that the functions $x \mapsto e^{itx}$ cannot be extended continuously to $K$. However, for $t \neq 0$ the functions

$$h_t(x) = \begin{cases} t & \text{if } \; x = 0, \\ \dfrac{e^{itx} - 1}{ix} & \text{if } \; x \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{if } \; x = \infty, \end{cases}$$

are continuous on $K$.

If $t > 0$, then for every finite signed measure $\nu$ on $\mathbb{R}$,

$$\begin{aligned} \int_K h_t(x) d\nu(x) &= \int_{\mathbb{R}} \left( \int_0^t e^{isx} ds \right) d\nu(x) \\ &= \int_0^t \left( \int_{\mathbb{R}} e^{isx} \, d\nu(x) \right) ds \\ &= \int_0^t \hat{\nu}(-s) \, ds. \end{aligned}$$

If $t < 0$, the same calculation works to obtain at once (with the usual conventions about definite integrals)

$$\int_K h_t(x) d\nu(x) = \int_0^t \hat{\nu}(-s) \, ds, \qquad \forall \; t \in \mathbb{R}. \tag{6.12}$$

Suppose $(\mu_n)_{n=1}^{\infty}$ is a sequence of probability measures on $\mathbb{R}$. Let $\mu \in \mathcal{M}(K)$ be any weak* cluster point of this sequence (viewed as elements of $\mathcal{C}(K)^*$; such a measure then exists by the Banach–Alaoglu theorem). Such $\mu$ is a positive measure on $K$, since the $\mu_n$'s are positive measures on $\mathbb{R}$. Besides, $\mu(K) = 1$, since $\mu_n(\mathbb{R}) = 1$ for all $n$. Applying formula (6.12) to each $\mu_n$ and appealing to the dominated convergence theorem gives

$$\int_K h_t(x) d\mu(x) = \int_0^t F(s) \, ds, \qquad \forall \; t \in \mathbb{R}. \tag{6.13}$$

We next show that $\mu$ is a probability measure on $\mathbb{R}$, i.e., $\mu(\{\infty\}) = 0$ or equivalently $\mu(\mathbb{R}) = 1$. Note that for $t > 0$, $|h_t(x)| \le t$ for all $x$ and $h_t(\infty) = 0$. Thus

$$\left| \int_K h_t(x) \, d\mu(x) \right| \le t\mu(\mathbb{R}).$$

Hence, for $t > 0$,

$$\frac{1}{t}\left|\int_0^t F(s)\,ds\right| \leq \mu(\mathbb{R}).$$

Now, $F$ is *continuous*, and obviously, $F(0) = 1$. Thus the left-hand side converges to 1 as $t \to 0$. We conclude that $\mu(\mathbb{R}) \geq 1$, as desired. Now we are in a position to define $\hat{\mu}$, and by (6.12) and (6.13) we have

$$\int_0^t \hat{\mu}(-s)ds = \int_0^t F(s)\,ds, \qquad \forall t \in \mathbb{R}.$$

By the fundamental theorem of calculus, since both $\hat{\mu}(-t)$ and $F(t)$ are continuous, $\hat{\mu}(-t) = F(t)$ for $t \in \mathbb{R}$. □

**Theorem 6.4.14.** *For every* $0 < p \leq 2$ *there is a probability measure* $\mu_p$ *on* $(\mathbb{R}, dx)$ *such that*

$$\int_{-\infty}^{\infty} e^{itx}\,d\mu_p(x) = e^{-|t|^p}, \quad t \in \mathbb{R}.$$

*Proof.* It obviously suffices to show the existence of $\mu_p$ with

$$\int_{-\infty}^{\infty} e^{itx}\,d\mu_p(x) = e^{-c_p|t|^p}, \quad t \in \mathbb{R},$$

where $c_p$ is some positive constant. For the case $p = 2$ this is achieved using a Gaussian.

Now suppose $0 < p < 2$. Let $f$ be a random variable on some probability space with probability distribution

$$d\mu_f = \frac{p}{2|x|^{p+1}}\left[\chi_{(-\infty,-1)}(x) + \chi_{(1,+\infty)}(x)\right]dx.$$

The characteristic function of $f$ is the following:

$$\begin{aligned}
\mathbb{E}\big(e^{itf}\big) &= \int_{-\infty}^{\infty} e^{itx}\,d\mu_f(x) \\
&= \frac{p}{2}\int_{-\infty}^{-1} \frac{e^{itx}}{(-x)^{p+1}}\,dx + \frac{p}{2}\int_1^{\infty} \frac{e^{itx}}{x^{p+1}}\,dx \\
&= p\int_1^{\infty} \frac{e^{itx} + e^{-itx}}{2}\,\frac{dx}{x^{p+1}} \\
&= p\int_1^{\infty} \frac{\cos(tx)}{x^{p+1}}\,dx.
\end{aligned}$$

Then, if $t > 0$, the substitution $u = tx$ in the last integral yields

$$1 - \mathbb{E}\big(e^{itf}\big) = p \int_1^\infty \frac{dx}{x^{p+1}} - p \int_1^\infty \frac{\cos(tx)}{x^{p+1}}\,dx$$
$$= p \int_1^\infty \frac{1 - \cos(tx)}{x^{p+1}}\,dx$$
$$= pt^p \int_t^\infty \frac{1 - \cos u}{u^{p+1}}\,du.$$

Let

$$\omega_p(t) = p \int_t^\infty \frac{1 - \cos u}{u^{p+1}}\,du$$

and

$$c_p = \lim_{t \to 0+} \omega_p(t) = p \int_0^\infty \frac{1 - \cos u}{u^{p+1}}\,du.$$

Note that $\int_0^\infty \frac{1-\cos u}{u^{p+1}}\,du$ is finite and positive for every $0 < p < 2$.

Since $f$ is symmetric, its characteristic function is even, and therefore the equality

$$\mathbb{E}\big(e^{itf}\big) = 1 - |t|^p \omega_p(t)$$

holds for all $t \in \mathbb{R}$.

Let $(f_j)_{j=1}^\infty$ be a sequence of independent random variables with the same distribution as $f$. Then, for every $n$, the characteristic function of the random variable $\frac{f_1 + \cdots + f_n}{n^{1/p}}$ is

$$\mathbb{E}\Big(e^{it\frac{f_1+\cdots+f_n}{n^{1/p}}}\Big) = \prod_{i=1}^n \mathbb{E}\big(e^{it\frac{f_i}{n^{1/p}}}\big) = \Big(\mathbb{E}\big(e^{it\frac{f}{n^{1/p}}}\big)\Big)^n = \Big(1 - \frac{|t|^p}{n}\,\omega_p\big(\frac{|t|}{n^{1/p}}\big)\Big)^n.$$

Since

$$\lim_{n \to \infty} \Big(1 - \frac{|t|^p}{n}\,\omega_p\big(\frac{|t|}{n^{1/p}}\big)\Big)^n = e^{-c_p|t|^p},$$

we can apply the preceding lemma to obtain the required measure $\mu_p$. □

**Definition 6.4.15.** A random variable $f$ on a probability space is called *p-stable* $(0 < p < 2)$ if

$$\hat{\mu}_f(-t) = e^{-c|t|^p}, \quad t \in \mathbb{R},$$

for some positive constant $c = c(p)$. We say that $f$ is *normalized p-stable* if $c = 1$.

Note that the normalization for Gaussians is somewhat different, i.e., the characteristic function of a normalized Gaussian would correspond to the case $c = 1/2$ in the previous definition.

**Theorem 6.4.16.** *Let $f$ be a $p$-stable random variable on a probability measure space $(\Omega, \Sigma, \mu)$ for some $0 < p < 2$. Then*

*(i) $f \in L_q(\mu)$ for all $0 < q < p$;*
*(ii) $f \notin L_p(\mu)$.*

*Proof.* Suppose that $f$ is normalized $p$-stable for some $0 < p < 2$ with distribution of probability $\mu_p$. Then

$$\int_\Omega |f(\omega)|^q \, d\omega = \int_{-\infty}^{\infty} |x|^q \, d\mu_p(x).$$

For every $x \in \mathbb{R}$ the substitution $u = |x|t$ in the integral $\int_0^\infty \frac{1-\cos tx}{t^{1+q}} \, dt$ yields

$$\int_0^\infty \frac{1 - \cos tx}{t^{1+q}} \, dt = |x|^q \alpha_q,$$

where $\alpha_q = \int_0^\infty \frac{1-\cos u}{u^{1+q}} \, du$ is a positive constant for $0 < q < 2$. Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty} |x|^q \, d\mu_p(x) &= \alpha_q^{-1} \int_{-\infty}^{\infty} \left( \int_0^\infty \frac{1 - \cos tx}{t^{1+q}} \, dt \right) d\mu_p(x) \\
&= \alpha_q^{-1} \int_0^\infty \frac{1}{t^{q+1}} \left( \int_{-\infty}^{\infty} (1 - \cos tx) \, d\mu_p(x) \right) dt \\
&= \alpha_q^{-1} \int_0^\infty \frac{1}{t^{q+1}} \left( \int_{-\infty}^{\infty} (1 - \Re e^{ixt}) \, d\mu_p(x) \right) dt \\
&= \alpha_q^{-1} \int_0^\infty \frac{1}{t^{q+1}} (1 - e^{-t^p}) \, dt.
\end{aligned}$$

The last integral is finite for $0 < q < p$ and fails to converge when $q = p$. □

**Theorem 6.4.17.** *If $1 \le p < 2$ and $p \le q \le 2$, then $\ell_q$ embeds isometrically in $L_p$.*

*Proof.* We have already seen the cases $q = p$ and $q = 2$. For $1 \le p < q < 2$, let $(f_j)_{j=1}^\infty$ be a sequence of independent normalized $q$-stable random variables on $[0, 1]$. Then we can repeat the argument we used in Proposition 6.4.12 to prove that $[f_j]$ is isometric to $\ell_q$ in $L_p$. The only constraint is that the sequence $(f_j)$ must belong to $L_p$, which requires that $p < q$. □

We can summarize our discussion in the following theorem:

**Theorem 6.4.18 ($\ell_q$-Subspaces of $L_p$).**

*(i) For $1 \le p \le 2$, $\ell_q$ embeds in $L_p$ if and only if $p \le q \le 2$.*
*(ii) For $2 < p < \infty$, $\ell_q$ embeds in $L_p$ if and only if $q = 2$ or $q = p$.*

*Moreover, if $\ell_q$ embeds in $L_p$, then it embeds isometrically.*

*Remark 6.4.19.* The alert reader will wonder for which values of $q$ the function space $L_q$ can be embedded in $L_p$. In fact, the answer is exactly the same as for the sequence space $\ell_q$, but we will postpone the proof of this until Chapter 11. A direct proof of this fact can be based on a discussion of stochastic integrals (see [164]).

**Theorem 6.4.20.** *Let $1 < p, q < \infty$. Then $\ell_q$ embeds complementably in $L_p$ if and only if $q = p$ or $q = 2$.*

*Proof.* We know (Propositions 6.4.2 and 6.4.1) that both $q = p$ and $q = 2$ allow complemented embeddings. Suppose $\ell_q$ embeds in $L_p$ complementably and $q \notin \{2, p\}$. By Theorem 6.4.18 we must have $p < q < 2$. Taking duals, it follows that $\ell_{q'}$ embeds complementably in $L_{p'}$, where $q', p'$ are the conjugate indices of $q$ and $p$. This is impossible. □

The $L_p$-spaces ($1 \le p < \infty$) are primary. Alspach, Enflo, and Odell [11] proved the result for $1 < p < \infty$ in 1977. The case $p = 1$ was established in 1979 by Enflo and Starbird [89], as we already mentioned in Chapter 5.

The problem of classifying the complemented subspaces of $L_p$ when $1 < p < \infty$ received a great deal of attention during the 1970s. At this stage we know of three isomorphism classes that we can find as complemented subspaces inside any $L_p$: $\ell_2$, $\ell_p$, and $L_p$, and it is easily seen that we can add $\ell_p \oplus \ell_2$ and $\ell_p(\ell_2)$ to that list. In fact, it turns out that $L_p$ has a very rich class of complemented subspaces, and the classification of them seems beyond reach. In 1981, Bourgain, Rosenthal, and Schechtman [31] showed the existence of uncountably many mutually nonisomorphic complemented subspaces of $L_p$; curiously, it seems unknown (unless we assume the continuum hypothesis) whether there is a continuum of such spaces!

## Problems

**6.1.** This exercise can be considered a continuation of Problem 5.6.

(a) A closed subspace $X$ of $L_p(\mathbb{T})$ is called *translation-invariant* if $f \in X$ implies $\tau_\phi(f) \in X$, where $\tau_\phi(f) = f(\theta - \phi)$. Show that if $X$ is translation-invariant and $E = \{n \in \mathbb{Z} : e^{in\theta} \in X\}$, then $X$ is the closed linear span of $\{e^{in\theta} : n \in E\}$. In this case we put $X = L_{p,E}(\mathbb{T})$.
(b) $E$ is called a $\Lambda(p)$-set if $L_{p,E}(\mathbb{T})$ is strongly embedded in $L_p(\mathbb{T})$. Show that if $E$ is a $\Lambda(p)$-set, then it is a $\Lambda(q)$-set for $q < p$.
(c) Show that if $p > 2$, $E$ is a $\Lambda(p)$-set if and only if $\{e^{in\theta} : n \in E\}$ is an unconditional basis of $L_{p,E}(\mathbb{T})$.
(d) Prove that $E = \{4^n : n \in \mathbb{N}\}$ is a $\Lambda(4)$-set. [*Hint*: Expand $|\sum_{n\in E} a_n e^{in\theta}|^4$.]
(e) $E$ is called a *Sidon set* if for every $(a_n)_{n\in E} \in \ell_\infty(E)$ there exists $\mu \in \mathcal{M}(\mathbb{T})$ with $\hat{\mu}(n) = a_n$. Show that the following are equivalent:

(i) $E$ is a Sidon set;
(ii) $(e^{in\theta})_{n\in E}$ is an unconditional basic sequence in $\mathcal{C}(\mathbb{T})$;
(iii) $(e^{in\theta})_{n\in E}$ is a basic sequence equivalent to the canonical $\ell_1$-basis in $\mathcal{C}(\mathbb{T})$.

(f) Show that a Sidon set is a $\Lambda(p)$-set for every $1 \leq p < \infty$.
(g) Show that $E = \{4^n : n \in \mathbb{N}\}$ is a Sidon set. [*Hint*: For $-1 \leq a_n \leq 1$, consider the functions $f_n(\theta) = \prod_{k=1}^{n}(1 + a_k \cos 4^k\theta)$, and let $\mu$ be a weak* cluster point of the measures $f_n \frac{d\theta}{2\pi}$.]

**6.2.** In this problem we aim to obtain **Khintchine's inequalities** directly, not as a consequence of Kahane's inequalities.

(a) Prove that $\cosh t \leq e^{t^2/2}$ for all $t \in \mathbb{R}$.
(b) Show that if $p \geq 1$, then $t^p \leq p^p e^{-p} e^t$.
(c) Let $(\epsilon_n)_{n=1}^{\infty}$ be a Rademacher sequence and suppose $f = \sum_{k=1}^{n} a_k \varepsilon_k$, where $\sum_{k=1}^{n} a_k^2 = 1$. Show that

$$\mathbb{E}(e^f) \leq e$$

and deduce that

$$\mathbb{E}(e^{|f|}) \leq 2e.$$

Hence show that

$$(\mathbb{E}(|f|^p))^{1/p} \leq 2^{1/p} e^{1/p} \frac{p}{e}.$$

Finally, obtain Khintchine's inequality for $p > 2$.
(d) Show using Hölder's inequality that (c) implies Khintchine's inequalities for $p < 2$.

**6.3 (The Classical Proof of Khintchine's Inequalities).**

(a) Let $(\Omega, \mathbb{P})$ a probability space and $(\varepsilon_k)$ be a Rademacher sequence on it. Suppose $\sum_{k=1}^{n} a_k^2 = 1$. If $p = 2m$ is an even integer, expand $\mathbb{E}(\sum_{k=1}^{n} a_k \varepsilon_k)^{2m}$ using the multinomial theorem and compare with $(\sum_{k=1}^{n} a_k^2)^m$.
(b) Deduce that

$$\mathbb{E}\Big(\sum_{k=1}^{n} a_k \varepsilon_k\Big)^{2m} \leq \frac{(2m)!}{2^m m!}.$$

(c) Obtain Khintchine's inequalities.

**6.4.** Let $(\Omega, \mathbb{P})$ be a probability space and $(\varepsilon_k)$ a Rademacher sequence on it. Consider a finite series $f = \sum_{k=1}^{N} a_k \varepsilon_k$ and let

$$M(t) = \max_{1\leq n\leq N} \Big|\sum_{k=1}^{n} a_k \varepsilon_k(t)\Big|.$$

(a) Show that $\mathbb{P}(M > \lambda) \le 2\mathbb{P}(|f| > \lambda)$.
(b) Deduce that $\mathbb{E}(M^2) \le 2\sum_{k=1}^{N} a_k^2$.

**6.5.** Suppose $\sum_{k=1}^{\infty} a_k^2 < \infty$. Let

$$M_m(t) = \sup_{n>m} \Big| \sum_{j=m+1}^{n} a_k \varepsilon_k(t) \Big|.$$

Show that $M_m(t) < \infty$ almost everywhere and $\lim_{m\to\infty} \mathbb{E}(M_m^2) = 0$. Deduce that $\sum_{k=1}^{\infty} a_k \varepsilon_k$ converges a.e.

**6.6.** Suppose the series $\sum_{k=1}^{\infty} a_k \varepsilon_k$ converges on a set of positive measure.

(a) Argue that there exist a measurable set $E$ with $\mathbb{P}(E) > 0$ and a constant $C$ such that

$$\Big| \sum_{j=m+1}^{n} a_j \varepsilon_k(\omega) \Big| \le C, \qquad \omega \in E,\ 1 \le m < n < \infty.$$

(b) Let $b_{jk} = \mathbb{E}(\chi_E \varepsilon_j \varepsilon_k)$ for $j < k$. Show that

$$\sum_{j<k} b_{jk}^2 \le \mathbb{P}(E).$$

(c) Deduce the existence of $m$ such that

$$\sum_{m\le j<k} b_{jk}^2 \le \frac{1}{100}(\mathbb{P}(E))^2.$$

(d) Deduce that $\sum_{k=1}^{\infty} a_k^2 < \infty$. [*Hint*: Estimate $\mathbb{E}|\chi_E \sum_{j=m+1}^{n} a_j \varepsilon_j|^2$.]

**6.7.** A Banach space $X$ has the *Orlicz property* if whenever a series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent in $X$, then $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$.

(a) **Orlicz's Theorem.** Prove that the spaces $L_p$ for $1 \le p \le 2$ have the Orlicz property.
(b) Show that for $2 \le p < \infty$, if $\sum_{n=1}^{\infty} f_n$ is unconditionally convergent in $L_p$, then $\sum_{n=1}^{\infty} \|f_n\|^p < \infty$.
(c) Prove that if a series $\sum_{n=1}^{\infty} f_n$ is unconditionally convergent in $L_p$ for $1 \le p < \infty$, then $f_n \to 0$ almost everywhere.

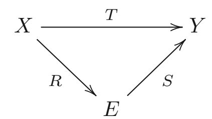
**6.8.** Prove that for $1 < p < \infty$, $\ell_2$ embeds isometrically and complementably in $L_p$.

**6.9.** Show that a quotient of a space of type $p$ is also of type $p$. Is the same statement valid for cotype?

**6.10.** Show that every operator from $\mathcal{C}(K)$ into $\ell_p$, $1 \le p < 2$, is compact.

# Chapter 7
# Factorization Theory

This chapter is devoted to some important results on factorization of operators. Suppose $X$, $Y$ are Banach spaces and that $T : X \to Y$ is a continuous operator. We say that $T$ *factorizes* through a Banach space $E$ if there are continuous operators $R : X \to E$ and $S : E \to Y$ such that $T = SR$. Pictorially, we have the following diagram:




To illustrate the importance of such theorems, consider the case in which we can factor the identity operator $I_X : X \to X$ through $E$. Then $X$ is isomorphic to a complemented subspace of $E$. Another classical example: if an operator $T : X \to Y$ between Banach spaces factors through a reflexive Banach space, then $T$ is weakly compact (actually, this property characterizes weakly compact operators, as Davis, Figiel, Johnson, and Pełczyński proved in [56]).

Most of the results of this chapter were obtained during the period 1970–1974 by Maurey, Rosenthal, and Nikishin. Factorization builds on the theory of type and cotype, as we will see. In fact, some of the work that preceded the results of this chapter and provided much of the impetus for factorization theory will be developed only in the following chapter. This particularly includes the fundamental work of Grothendieck [120] and Lindenstrauss and Pełczyński [196].



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## 7.1 Maurey–Nikishin Factorization Theorems

In this section we shall discuss factorization theory of operators with values in the $L_p$-spaces. Here factorization is related to the notion of change of density.

The first factorization result of this type, essentially discovered by Nikishin [227], establishes a criterion for an operator with values in an $L_p(\mu)$-space to factor through $L_q(\nu)$ ($q > p$), where $\nu$ is obtained from $\mu$ after a suitable change of density. Nikishin's motivation came from harmonic analysis rather than Banach space theory, where versions of this result for translation-invariant operators had been known for some time (e.g., in the work of Stein [288]). However, it was the work of Maurey [212] that combined the ideas of Nikishin with the newly evolving theory of Rademacher type to create a very powerful tool.

The proof given below is based on one presented in [303] but is similar to the proof given by Maurey.

**Definition 7.1.1.** If $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space, then a *density function h* on $\Omega$ is a measurable function such that $h \geq 0$ a.e. and $\int h\, d\mu = 1$.

**Theorem 7.1.2.** *Let* $\mu$ *be a* $\sigma$*-finite measure on some measurable space* $(\Omega, \Sigma)$*. Suppose that* $T$ *is an operator from a Banach space* $X$ *into* $L_p(\mu)$ *and that* $1 \leq p < q < \infty$*. Suppose* $0 < C < \infty$*. Then the following conditions are equivalent:*

*(a) There exists a density function* $h$ *on* $\Omega$ *such that*

$$\Big(\int_{\{h>0\}} |Tx|^q h^{1-q/p}\, d\mu\Big)^{1/q} \leq C\, \|x\|, \qquad x \in X, \tag{7.1}$$

*and*

$$\mu\{\omega :\ |Tx(\omega)| > 0,\ h(\omega) = 0\} = 0, \qquad x \in X. \tag{7.2}$$

*(b) For every finite sequence* $(x_k)_{k=1}^n$ *in* $X$*,*

$$\Big\|\Big(\sum_{k=1}^n |Tx_k|^q\Big)^{1/q}\Big\|_p \leq C\Big(\sum_{k=1}^n \|x_k\|^q\Big)^{1/q}. \tag{7.3}$$

**Interpretation** Condition $(a)$ is to be interpreted in the sense that each function $Tx$ is essentially supported on $A = \{\omega \in \Omega : h(\omega) > 0\}$. Thus the operator $Sx := h^{-1/p}Tx$ maps into $L_p(\Omega, h\, d\mu)$. However, $(a)$ asserts that $S$ actually maps boundedly into the *smaller* space $L_q(\Omega, h\, d\mu)$. This is the diagram depicting the situation:

$$\begin{array}{ccc} X & \xrightarrow{T} & L_p(\mu) \\ \downarrow & & \uparrow j \\ L_q(hd\mu) & \hookrightarrow & L_p(hd\mu) \end{array}$$

Here $j$ is an isometric embedding of $L_p(h\,d\mu)$ onto the subspace $L_p(A,\mu)$ of $L_p(\mu)$, defined by $j(f) = fh^{1/p}$.

Of course, at a very small cost we could insist that $h$ be a strictly positive density (i.e., $h > 0$ a.e.) and drop equation (7.2): simply replace $h$ by $(1+\epsilon v)^{-1}(h+\epsilon v)$, where $\epsilon > 0$ and $v$ is any strictly positive density. Then $j$ becomes a genuine isometric isomorphism. In this case, however, the norm of $S = h^{-1/p}T$ is a little greater than $C$. Since the precise value of $\|S\|$ is rarely of interest, we will often use the theorem is this form. In fact, in a formal sense we could replace (7.1) and (7.2) by

$$\Big(\int_\Omega |Tx|^q h^{1-q/p}\,d\mu\Big)^{1/q} \le C\,\|x\|\,, \qquad x \in X,$$

with the implicit understanding that $Tx = 0$ a.e. on the set $\{\omega \in \Omega : h(\omega) = 0\}$ (i.e., where $h^{-q/p} = 0$). We will use this convention later.

Before continuing, let us notice that although we have stated this for general $\sigma$-finite measures, it is enough to prove the theorem under our usual convention that $\mu$ is a probability measure. If $\mu$ is not a probability measure, we choose some strictly positive density $v$ and set $d\mu' = v\,d\mu$; then we define $T' : X \to L_p(\mu')$ by $T'x = v^{-1}Tx$. A quick inspection will show the reader that the statement of the theorem for $T'$ implies exactly the same statements for $T$. Thus we can and do resume our convention that $\mu$ is a probability measure.

*Proof.* (*a*) $\Rightarrow$ (*b*) Since $(\Omega, h\,d\mu)$ is a probability measure space and $p < q$, the $L_p(hd\mu)$-norm is smaller than the $L_q(hd\mu)$-norm, and thus we have

$$\begin{aligned}\Big(\int_\Omega \big(\sum_{k=1}^n |Tx_k|^q\big)^{p/q}\,d\mu\Big)^{1/p} &= \Big(\int_{\{h>0\}} \big(\sum_{k=1}^n |Tx_k|^q\, h^{-q/p}\big)^{p/q}\, h\,d\mu\Big)^{1/p} \\ &\le \Big(\int_{\{h>0\}} \sum_{k=1}^n |Tx_k|^q h^{-\frac{q}{p}}\, h\,d\mu\Big)^{1/q} \\ &= \Big(\sum_{k=1}^n \int_{\{h>0\}} |Tx_k|^q h^{-\frac{q}{p}}\, hd\mu\Big)^{1/q} \\ &\le C\Big(\sum_{k=1}^n \|x_k\|^q\Big)^{1/q}.\end{aligned}$$

$(b) \Rightarrow (a)$ Let us assume that $C$ is the best constant such that (7.3) holds. Then

$$\sup\left\{\Big\|\big(\sum_{k=1}^{n}|Tx_k|^q\big)^{1/q}\Big\|_p \,:\, (x_k)_{k=1}^n \subset X,\ \sum_{k=1}^{n}\|x_k\|^q \leq C^{-q},\ n\in\mathbb{N}\right\} = 1.$$

Let $W_0$ be the set of all nonnegative functions in $L_1$ that are bounded above by functions of the form $(\sum_{k=1}^{n}|Tx_k|^q)^{p/q}$, where $n \in \mathbb{N}$ and $(x_k)_{k=1}^n \subset X$ with $\sum_{k=1}^{n}\|x_k\|^q \leq C^{-q}$, i.e.,

$$0 \leq f \leq \Big(\sum_{k=1}^{n}|Tx_k|^q\Big)^{p/q}.$$

Let $W$ be the norm closure of $W_0$.

The sets $W_0$ and $W$ have the following property.

**Lemma 7.1.3.** *Let $r = q/p > 1$. Given $f_1,\dots,f_n \in W_0$ [respectively, $W$] and $c_1,\dots,c_n \geq 0$ with $c_1+\dots+c_n \leq 1$, then $(c_1f_1^r+\dots+c_nf_n^r)^{1/r} \in W_0$ [respectively, $W$].*

*Proof.* It suffices to consider the case of $W_0$. Suppose

$$0 \leq f_k \leq \left(\sum_{j=1}^{m_k}|Tx_{jk}|^q\right)^{p/q}, \quad 1 \leq k \leq n,$$

where $\sum_{j=1}^{m_k}\|x_{jk}\|^q \leq C^{-q}$ for $1 \leq k \leq n$. Then we also have

$$0 \leq \left(\sum_{k=1}^{n}c_kf_k^r\right)^{1/r} \leq \left(\sum_{k=1}^{n}\sum_{j=1}^{m_k}|T(c_k^{\frac{1}{q}}x_{jk})|^q\right)^{p/q},$$

with

$$\sum_{k=1}^{n}c_k\sum_{j=1}^{m_k}\|x_{jk}\|^q \leq C^{-q},$$

and this establishes the claim. □

Lemma 7.1.3 immediately yields that $W_0$ (and hence its norm-closure $W$) is convex. Indeed, if $f_1,\dots,f_n \in W_0$ and $c_1,\dots,c_n \geq 0$ with $c_1+\dots+c_n = 1$, then

$$\sum_{j=1}^{n}c_jf_j \leq \left(\sum_{j=1}^{n}c_jf_j^r\right)^{1/r} \in W_0.$$

By Mazur's theorem, $W$ is therefore weakly closed. Note that from the choice of $C$, we have

$$\sup_{f\in W_0}\int f\,d\mu=\sup_{f\in W}\int f\,d\mu=1,$$

so in particular, $W$ is bounded. We next show that $W$ is weakly compact. This requires showing that it is equi-integrable.

Suppose $W$ is not equi-integrable. Then there exist some $\delta>0$, a sequence $(f_n)_{n=1}^{\infty}$ in $W$, and a sequence of disjoint measurable sets $(E_n)_{n=1}^{\infty}$ such that

$$\int_{E_n} f_n\,d\mu>\delta>0,\qquad n\in\mathbb{N}.$$

Thus for every $N$ we have

$$\delta N^{1-\frac{1}{r}}\le N^{-\frac{1}{r}}\int\max(f_1,f_2,\dots,f_N)d\mu\le\int\left(\frac{1}{N}\sum_{j=1}^{N}f_j^r\right)^{1/r}d\mu\le 1,$$

using Lemma 7.1.3. This is a contradiction for large enough $N$.

Hence $W$ is weakly compact, and since integration is a weakly continuous functional on $L_1(\mu)$, it follows that there exists $h\in W$ with

$$\int h\,d\mu=1. \tag{7.4}$$

Now suppose $f\in W$. On the one hand, for every $\tau>0$ we have

$$(1+\tau)^{-\frac{1}{r}}(h^r+\tau f^r)^{\frac{1}{r}}\in W,$$

and therefore, by Lemma 7.1.3,

$$\int(h^r+\tau f^r)^{\frac{1}{r}}d\mu\le(1+\tau)^{\frac{1}{r}}. \tag{7.5}$$

On the other hand,

$$\int(h^r+\tau f^r)^{\frac{1}{r}}d\mu\ge 1+\tau^{\frac{1}{r}}\int_{h=0}f\,d\mu. \tag{7.6}$$

Since $1/r<1$, combining (7.5) and (7.6) yields

$$\int_{\{h=0\}}f\,d\mu=0. \tag{7.7}$$

From (7.4) and (7.5) we have

$$\int_{\{h>0\}} h\frac{(1+\tau f^r h^{-r})^{\frac{1}{r}}-1}{\tau}d\mu \le \frac{(1+\tau)^{\frac{1}{r}}-1}{\tau}, \qquad \tau > 0.$$

Letting $\tau \to 0$ and using Fatou's lemma, we obtain

$$\int_{\{h>0\}} f^r h^{1-r} d\mu \le 1, \qquad f \in W. \tag{7.8}$$

In particular (7.7) and (7.8) hold for $f = C^{-p}\|x\|^{-p}|Tx|^p$ when $0 \ne x \in X$. This immediately gives (7.2) and (7.1). □

**Theorem 7.1.4.** *Let $1 \le p < \infty$. Suppose that $T$ is an operator from a Banach space $X$ into $L_p(\mu)$. If $X$ has type 2, then there exists a constant $C = C(p)$ such that for every finite sequence $(x_k)_{k=1}^n$ in $X$ we have*

$$\Big\|\Big(\sum_{k=1}^n |Tx_k|^2\Big)^{1/2}\Big\|_p \le C\Big(\sum_{k=1}^n \|x_k\|^2\Big)^{1/2}.$$

*Proof.* By Theorem 6.2.13, for every $1 \le p < \infty$ there is a constant $c = c(p)$ such that for every finite set of vectors $(x_k)_{k=1}^n$ in $X$,

$$\Big\|\Big(\sum_{k=1}^n |Tx_k|^2\Big)^{\frac{1}{2}}\Big\|_p \le c\,\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k Tx_k\Big\|_p \le c\,\|T\|\,\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|.$$

Using Kahane's inequality and the type 2 of $X$ yields

$$\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\| \le \Big(\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|^2\Big)^{1/2} \le T_2(X)\Big(\sum_{k=1}^n \|x_k\|^2\Big)^{1/2}.$$

□

Since $L_r(\mu)$ for $r \ge 2$ are type-2 spaces, we immediately obtain the following corollary:

**Corollary 7.1.5.** *(a) Every operator from a subspace of $L_r(\mu)$ $(2 \le r < \infty)$ into $L_p(\mu)$ $(1 \le p < 2)$ factors through a Hilbert space.*

*(b) If a Banach space $X$ is isomorphic to a closed subspace of both $L_p(\mu)$ for some $1 \le p < 2$ and $L_r(\mu)$ for some $2 < r < \infty$, then $X$ is isomorphic to a Hilbert space.*

Corollary 7.1.5 follows immediately from Theorems 7.1.2 and 7.1.4. Curiously, the isometric version of $(b)$ does not hold. That is, if $X$ is isometric to a subspace of $L_p$ $(1 \le p < 2)$ and isometric to a subspace of $L_r$ $(2 < r < \infty)$, it is not true that

$X$ must be isometric to a Hilbert space. Finite-dimensional counterexamples were given by Koldobsky [172]; however, the following problem is still open (see [174]):

**Problem 7.1.6.** *If an infinite-dimensional Banach space $X$ is isometric to a closed subspace of both $L_p$ for some $1 \le p < 2$ and $L_r$ for some $2 < r < \infty$, must $X$ be isometric to a Hilbert space?*

To push our results further we need a replacement for Theorem 6.2.13 for exponents other than 2. If $1 \le q < 2$, then it turns out that the $q$-stable random variables constructed in the previous chapter do very nicely. Indeed, we could have used Gaussians in place of Rademachers in the preceding argument.

**Lemma 7.1.7.** *Let $1 \le p < q < 2$. Suppose that $\gamma = (\gamma_j)_{j=1}^{\infty}$ is a sequence of independent normalized $q$-stable random variables. Then for every finite sequence of functions $(f_j)_{j=1}^n$ in $L_p(\mu)$,*

$$\Big\| \Big( \sum_{j=1}^n |f_j|^q \Big)^{1/q} \Big\|_p = c \Big( \mathbb{E} \Big\| \sum_{j=1}^n \gamma_j f_j \Big\|_p^p \Big)^{1/p},$$

*where $c = c(p,q) > 0$.*

*Proof.* We recall from Theorem 6.4.17 that there is a constant $c = c(p,q)$ such that

$$\left( \mathbb{E} \Big| \sum_{j=1}^n a_j \gamma_j \Big|^p \right)^{1/p} = c^{-1} \left( \sum_{j=1}^n |a_j|^q \right)^{1/q}, \qquad (a_j)_{j=1}^n \subset \mathbb{R}.$$

Using Fubini's theorem, we obtain

$$\int \Big( \sum_{j=1}^n |f_j|^q \Big)^{\frac{p}{q}} d\mu = c^p \, \mathbb{E} \int \Big| \sum_{j=1}^n \gamma_j f_j \Big|^p d\mu,$$

and the lemma follows. □

**Theorem 7.1.8.** *Let $1 \le p < 2$. Suppose that $T$ is an operator from a Banach space $X$ into $L_p(\mu)$. If $X$ is of type $r$ for some $p < r < 2$, then for each $q \in (p,r)$ there exists a constant $C$ such that*

$$\Big\| \Big( \sum_{j=1}^n |Tx_j|^q \Big)^{1/q} \Big\|_p \le C \Big( \sum_{j=1}^n \|x_j\|^q \Big)^{1/q},$$

*for every finite sequence $(x_j)_{j=1}^n$ in $X$.*

*Proof.* In this proof we will require three mutually independent sequences of independent identically distributed random variables: a sequence $(\varepsilon_j)_{j=1}^{\infty}$ of Rademachers, a sequence $(\gamma_j)_{j=1}^{\infty}$ of normalized $q$-stable random variables, and a sequence $(\eta_j)_{j=1}^{\infty}$ of normalized $r$-stable random variables.

Let $(x_j)_{j=1}^n$ be a finite sequence in $X$. By the previous lemma, for a certain constant $c = c(p,q)$ we have

$$\Big\| \big(\sum_{j=1}^n |Tx_j|^q\big)^{1/q} \Big\|_p = c\,\Big(\mathbb{E}_\gamma \,\Big\| \sum_{j=1}^n \gamma_j T x_j \Big\|_p^p\Big)^{1/p} \le c\,\|T\|\,\Big(\mathbb{E}_\gamma \,\Big\| \sum_{j=1}^n \gamma_j x_j \Big\|^p\Big)^{1/p}.$$

Since the normalized $q$-stables are symmetric and $X$ is of type $r$,

$$\begin{aligned}
\Big(\mathbb{E}_\gamma \,\Big\| \sum_{j=1}^n \gamma_j x_j \Big\|^p\Big)^{1/p} &= \Big(\mathbb{E}_\gamma \mathbb{E}_\varepsilon \Big\| \sum_{j=1}^n \varepsilon_j \gamma_j x_j \Big\|^p\Big)^{1/p} \\
&\le \Bigg(\mathbb{E}_\gamma \Bigg(\mathbb{E}_\varepsilon \Big\| \sum_{j=1}^n \varepsilon_j \gamma_j x_j \Big\|^r\Bigg)^{p/r}\Bigg)^{1/p} \\
&\le T_r(X)\,\Big(\mathbb{E}_\gamma \big(\sum_{j=1}^n |\gamma_j|^r\,\|x_j\|^r\big)^{p/r}\Big)^{1/p}.
\end{aligned}$$

Now notice that

$$\mathbb{E}\Big| \sum_{j=1}^n a_j \eta_j \Big|^p = c_1 \Big(\sum_{j=1}^n |a_j|^r\Big)^{p/r},$$

for a certain constant $0 < c_1 = \mathbb{E}|\eta_1|^p$, which is finite, since $p < r$. Thus letting $c_2, c_3$ be positive constants depending only on $p$, $q$, and $r$ yields

$$\begin{aligned}
\mathbb{E}_\gamma \Big(\sum_{j=1}^n |\gamma_j|^r\,\|x_j\|^r\Big)^{p/r} &= c_1^{-1} \mathbb{E}_\gamma \mathbb{E}_\eta \Big| \sum_{j=1}^n \eta_j \gamma_j \|x_j\| \Big|^p \\
&= c_1^{-1} \mathbb{E}_\eta \mathbb{E}_\gamma \Big| \sum_{j=1}^n \eta_j \gamma_j \|x_j\| \Big|^p \\
&= c_2 \mathbb{E}_\eta \Big(\sum_{j=1}^n |\eta_j|^q \|x_j\|^q\Big)^{p/q} \\
&\le c_2 \Big(\mathbb{E}_\eta \sum_{j=1}^n |\eta_j|^q \|x_j\|^q\Big)^{p/q} \\
&= c_3 \Big(\sum_{j=1}^n \|x_j\|^q\Big)^{p/q}.
\end{aligned}$$

□

The next result now follows immediately from Theorem 7.1.2:

**Theorem 7.1.9.** *Let $X$ be a Banach space of type $r > 1$. Suppose that $1 \le p < r$ and that $T : X \to L_p(\mu)$ is an operator. Then $T$ factors through $L_q(\mu)$ for every $p < q < r$. More precisely, for each $p < q < r$ there is a strictly positive density function $h$ on $\Omega$ such that $Sx = h^{-1/p}Tx$ defines a bounded operator from $X$ into $L_q(\Omega, h\,d\mu)$.*

Note here that there is a fundamental difference between the cases of type $r < 2$ and type 2. In the former we obtain a factorization through $L_q(\mu)$ only when $q < r$. Can we do better and take $q = r$? The answer is no, and to see why, we must consider subspaces of $L_p$ for $1 \le p < 2$. This will be the topic of the next section, but let us mention that an improvement is possible: a later theorem of Nikishin [228] implies that $T$ actually factors through the space "weak $L_r$." See [258] and the problems.

*Remark 7.1.10.* An examination of the proofs of the theorems of this section shows that the main theorem (Theorem 7.1.9) will also hold if $0 < p < 1$, when $L_p$ is no longer a Banach space; in this case we can take $r = 1$, and every Banach space is of type one! Thus we conclude that if a Banach space isomorphically embeds in some $L_p$ where $0 < p < 1$, then it embeds in every $L_q$ for $p \le q < 1$.

The following problem, originally raised by Kwapień in 1969, is open:

**Problem 7.1.11.** *If $X$ is a Banach space that embeds in $L_p$ for some $0 < p < 1$, does $X$ embed in $L_1$?*

In the isometric setting the answer is negative: a Banach space that embeds isometrically in $L_p$ for some $0 < p < 1$ need not embed isometrically in $L_1$, as Koldobsky proved in 1996 [173]; see also [161]. In the isomorphic case the only known result is that $X$ embeds in $L_1$ if and only if $\ell_1(X)$ embeds in some $L_p$ when $0 < p < 1$ [153].

## 7.2 Subspaces of $L_p$ for $1 \le p < 2$

We start our discussion by showing, as promised, that Theorem 7.1.8 cannot be improved to allow factorization through $L_r$. We will need the following simple lemma:

**Lemma 7.2.1.** *Suppose $f, g \in L_p$ $(1 \le p < \infty)$. Then if $0 < \theta < 1$, we have $|f|^{1-\theta}|g|^{\theta} \in L_p$ and*

$$\| |f|^{1-\theta}|g|^{\theta} \|_p \le \|f\|_p^{1-\theta}\|g\|_p^{\theta}.$$

*Proof.* Just note that for $s, t \geq 0$ we have $s^{1-\theta}t^{\theta} \leq (1-\theta)s + \theta t$. Then, assuming $\|f\|_p, \|g\|_p > 0$, by convexity we have

$$\left\| \left( \frac{|f|}{\|f\|_p} \right)^{\theta} \left( \frac{|g|}{\|g\|_p} \right)^{1-\theta} \right\|_p \leq 1,$$

and the lemma follows. □

**Theorem 7.2.2.** *If* $1 \leq p < 2$, $\ell_p$ *cannot be strongly embedded in* $L_p$.

*Proof.* Let us suppose $(f_n)_{n=1}^{\infty}$ is a normalized basic sequence in $L_p$ equivalent to the $\ell_p$-basis and such that $X = [f_n]$ is strongly embedded.

Let us fix $q < p$ (in the case $p = 1$ this implies $q < 1$). Then, using Theorem 6.2.13 and Proposition 6.4.5, we can find a constant $C > 0$ such that

$$C^{-1}n^{1/p} \leq \left\| \left( \sum_{j \in \mathbb{A}} |f_j|^2 \right)^{\frac{1}{2}} \right\|_q \leq \left\| \left( \sum_{j \in \mathbb{A}} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p \leq Cn^{1/p},$$

for every $n$ and each $\mathbb{A} \subset \mathbb{N}$ with $|\mathbb{A}| = n$.

Let $N \in \mathbb{N}$ and $a > 0$. Note that since $\|f_j\|_p = 1$, estimating $\int |f_j|^p \, dt$ gives

$$\sum_{k=1}^{\infty} \lambda\Big(|f_j| > (ak)^{\frac{1}{p}}\Big) \leq a^{-1}, \qquad 1 \leq j \leq N,$$

where $\lambda$ denotes the Lebesgue measure on $[0, 1]$. Thus

$$\sum_{k=1}^{\infty} \sum_{j=1}^{N} \lambda\Big(|f_j| > (ak)^{\frac{1}{p}}\Big) \leq Na^{-1}.$$

It follows that there exists at least one $m \leq N$ such that

$$\sum_{j=1}^{N} \lambda\Big(|f_j| > (am)^{\frac{1}{p}}\Big) \leq a^{-1}m^{-1}N\Big(\sum_{k=1}^{N} \frac{1}{k}\Big)^{-1} \leq \frac{N}{am \log N}.$$

By an averaging argument over all subsets of size $m$ we can find a subset $\mathbb{A}$ of $\{1, 2, \ldots, N\}$ with $|\mathbb{A}| = m$ such that

$$\sum_{j \in \mathbb{A}} \lambda(|f_j| > (am)^{\frac{1}{p}}) \leq \frac{1}{a \log N}.$$

Let $g = \max_{j\in\mathbb{A}} |f_j|$ and $E = \{t : g(t) > (am)^{\frac{1}{p}}\}$. Then

$$\|g\chi_E\|_q \le \lambda(E)^{\frac{1}{q}-\frac{1}{p}}\|g\|_p$$

by Hölder's inequality, and

$$\|g\|_p \le \Big\|(\sum_{j\in\mathbb{A}} |f_j|^2)^{\frac{1}{2}}\Big\|_p \le Cm^{\frac{1}{p}}.$$

Thus

$$\|g\chi_E\|_q \le Cm^{\frac{1}{p}}(a\log N)^{\frac{1}{p}-\frac{1}{q}}.$$

Hence

$$\|\max_{j\in\mathbb{A}} |f_j|\|_q \le (am)^{\frac{1}{p}} + Cm^{\frac{1}{p}}(a\log N)^{\frac{1}{p}-\frac{1}{q}}.$$

It follows that given $\delta > 0$, we can pick $a$ and $N$ to ensure the existence of a subset $\mathbb{A}$ of $\mathbb{N}$ of cardinality $m$ such that

$$\|\max_{j\in\mathbb{A}} |f_j|\|_q \le \delta m^{\frac{1}{p}}.$$

On the other hand,

$$\Big\|\Big(\sum_{j\in\mathbb{A}} |f_j|^p\Big)^{1/p}\Big\|_q \le \Big\|\Big(\sum_{j\in\mathbb{A}} |f_j|^p\Big)^{1/p}\Big\|_p \le m^{\frac{1}{p}}.$$

Hence

$$\begin{aligned} C^{-1}m^{\frac{1}{p}} &\le \Big\|\Big(\sum_{j\in\mathbb{A}} |f_j|^2\Big)^{1/2}\Big\|_q \le \Big\|\Big(\sum_{j\in\mathbb{A}} |f_j|^p\Big)^{1/p}\Big\|_q^{p/2}\Big\|\max_{j\in\mathbb{A}} |f_j|\Big\|_q^{1-p/2} \\ &\le \delta^{1-\frac{p}{2}}m^{\frac{1}{p}}. \end{aligned}$$

By choosing $\delta > 0$ appropriately, we reach a contradiction. □

*Remark 7.2.3.* Let us observe that now it is clear that we cannot take $q = r$ in Theorem 7.1.9. Indeed, if $r < 2$, then $\ell_r$ is of type $r$ and does embed into $L_p$ for $1 \le p \le r$ by Theorem 6.4.17. However, if such a factorization of the embedding $J : \ell_r \to L_p$ were possible, we would deduce that $\ell_r$ strongly embeds into $L_r([0,1], h\,dt)$ for some strictly positive density function $h$, which contradicts Theorem 7.2.2.

We are now going to delve a little further into the structure of subspaces of $L_p$ for $1 \le p < 2$. We need some initial observations about type in general Banach spaces; we shall establish similar results for cotype for later use.

Let $X$ be an infinite-dimensional Banach space, and $(\varepsilon_i)_{i=1}^\infty$ a Rademacher sequence. For each $n \in \mathbb{N}$ define $\alpha_n(X)$ to be the least constant $\alpha$ such that

$$\Big(\mathbb{E}\Big\| \sum_{i=1}^n \varepsilon_i x_i \Big\|^2\Big)^{1/2} \le \alpha \Big( \sum_{i=1}^n \|x_i\|^2\Big)^{1/2}, \qquad \{x_i\}_{i=1}^n \subset X;$$

and define $\beta_n(X)$ to be the least constant $\beta$ such that

$$\Big( \sum_{i=1}^n \|x_i\|^2\Big)^{1/2} \le \beta \Big(\mathbb{E}\Big\| \sum_{i=1}^n \varepsilon_i x_i \Big\|^2\Big)^{1/2}, \qquad \{x_i\}_{i=1}^n \subset X.$$

Note that $1 \le \alpha_n(X), \beta_n(X) \le n^{\frac{1}{2}}$ for $n = 1, 2, \dots$.

**Lemma 7.2.4.** *Both parameters $\alpha_n(X)$ and $\beta_n(X)$ are submultiplicative, i.e.,*

$$\alpha_{mn}(X) \le \alpha_m(X)\alpha_n(X), \qquad m, n \in \mathbb{N}, \tag{7.9}$$

*and*

$$\beta_{mn}(X) \le \beta_m(X)\beta_n(X), \qquad m, n \in \mathbb{N}. \tag{7.10}$$

*Proof.* Let us take $m \times n$ vectors in the unit ball of $X$ and consider them as a matrix $(x_{ij})_{i,j=1}^{m,n}$. Let $(\varepsilon_{ij})_{i,j=1}^{m,n}$ be a Rademacher sequence, and $(\varepsilon_i')_{i=1}^n$ another Rademacher sequence, independent of $(\varepsilon_{ij})$. The independence of the Rademacher sequence $(\varepsilon_i'\varepsilon_{ij})$ yields

$$\mathbb{E} \Big\| \sum_{i=1}^m \sum_{j=1}^n \varepsilon_{ij} x_{ij} \Big\|^2 = \mathbb{E} \Big\| \sum_{i=1}^m \varepsilon_i' \sum_{j=1}^n \varepsilon_{ij} x_{ij} \Big\|^2.$$

Then,

$$\begin{aligned}\Big(\mathbb{E} \Big\| \sum_{i=1}^m \varepsilon_i' \sum_{j=1}^n \varepsilon_{ij} x_{ij} \Big\|^2\Big)^{1/2} &\le \alpha_m(X) \Big(\mathbb{E} \sum_{i=1}^m \Big\| \sum_{j=1}^n \varepsilon_{ij} x_{ij} \Big\|^2\Big)^{1/2} \\ &\le \alpha_m(X)\alpha_n(X) \Big( \sum_{i=1}^m \sum_{j=1}^n \|x_{ij}\|^2\Big)^{1/2}.\end{aligned}$$

Similarly,

$$\Big(\sum_{i=1}^m\sum_{j=1}^m \|x_{ij}\|^2\Big)^{1/2} \le \beta_n(X)\Big(\sum_{i=1}^m \mathbb{E}\Big\|\sum_{j=1}^n \varepsilon_{ij}x_{ij}\Big\|^2\Big)^{1/2}$$
$$\le \beta_m(X)\beta_n(X)\Big(\mathbb{E}\Big\|\sum_{i=1}^m\sum_{j=1}^n \varepsilon_{ij}x_{ij}\Big\|^2\Big)^{1/2}.$$

□

**Proposition 7.2.5.** *Suppose $p < 2 < q$.*

*(a) In order that $X$ have type $r$ for some $p < r$ it is necessary and sufficient that for some $N$, $\alpha_N(X) < N^{\frac{1}{p}-\frac{1}{2}}$.*
*(b) In order that $X$ have cotype $s$ for some $s < q$ it is necessary and sufficient that for some $N$, $\beta_N(X) < N^{\frac{1}{2}-\frac{1}{q}}$.*

*Proof.* One easily checks that if $X$ has type $r > p$ [respectively, cotype $s < q$], then $\alpha_N(X) < N^{\frac{1}{p}-\frac{1}{2}}$ [respectively, $\beta_N(X) < N^{\frac{1}{2}-\frac{1}{q}}$] for some $N$ by taking arbitrary sequences of vectors $\{x_i\}_{i=1}^n$ in $X$ all equal to some $x$ with $\|x\| = 1$.

Let us now complete the proof of ($a$). Assume $N$ is such that $\alpha_N(X) < N^{\frac{1}{p}-\frac{1}{2}}$. Then we can write $\alpha_N(X) = N^{\theta-\frac{1}{2}}$ for some $\frac{1}{2} < \theta < \frac{1}{p}$, and by (7.9),

$$\alpha_{N^k}(X) \le N^{k(\theta-\frac{1}{2})}, \quad k \in \mathbb{N}.$$

Given any $n$, if we take $k \in \mathbb{N}$ such that $N^{k-1} \le n \le N^k$, then

$$\alpha_n(X) \le \alpha_{N^k}(X) \le N^{k(\theta-\frac{1}{2})} = (N^{k-1})^{\theta-\frac{1}{2}}N^{\theta-\frac{1}{2}},$$

and so we have an estimate of the form

$$\alpha_n(X) \le Cn^{(\theta-\frac{1}{2})}, \tag{7.11}$$

for $C = N^{\theta-\frac{1}{2}}$.

Pick $r$ such that $p < r < \frac{1}{\theta}$. Given any sequence $(x_i)_{i=1}^n$ of vectors in $X$, without loss of generality we will suppose that $\|x_1\| \ge \|x_2\| \ge \cdots \ge \|x_n\|$. For notational convenience let $x_i = 0$ for $i > n$. Then for $k \in \mathbb{N}$, using (7.11), we obtain

$$\Big(\mathbb{E}\Big\|\sum_{i=2^{k-1}}^{2^k-1} \varepsilon_i x_i\Big\|^2\Big)^{1/2} \le C2^{k(\theta-\frac{1}{2})}\Big(\sum_{i=2^{k-1}}^{2^k-1} \|x_i\|^2\Big)^{1/2}$$
$$\le C2^{k\theta}\|x_{2^{k-1}}\|$$
$$\le C2^{k\theta}2^{-(k-1)/r}\Big(\sum_{i=1}^\infty \|x_i\|^r\Big)^{1/r}.$$

Summing over $k$ yields

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^{\infty}\varepsilon_i x_i\Big\|^2\Big)^{1/2} \le C2^{\frac{1}{r}}\sum_{k=1}^{\infty}2^{k(\theta-\frac{1}{r})}\Big(\sum_{i=1}^{\infty}\|x_i\|^r\Big)^{1/r}.$$

This implies, using the Kahane–Khintchine inequality (Theorem 6.2.5), that $X$ has type $r$.

The proof of ($b$) is similar: Assume $\beta_N(X) < N^{\frac{1}{2}-\frac{1}{q}}$ for some $N$. Then in place of (7.11) we find $\theta > \frac{1}{q}$ such that for some constant $C$, we have

$$\beta_n(X) \le Cn^{\frac{1}{2}-\theta}, \qquad n \in \mathbb{N}. \tag{7.12}$$

Pick $s$ such that $\frac{1}{\theta} < s < q$. For $(x_i)$ as in ($a$), for $k \in \mathbb{N}$ we have

$$\begin{aligned}\Big(\sum_{i=2^{k-1}}^{2^k-1}\|x_i\|^2\Big)^{1/2} &\le C2^{k(\frac{1}{2}-\theta)}\Big(\mathbb{E}\Big\|\sum_{i=2^{k-1}}^{2^k-1}\varepsilon_i x_i\Big\|^2\Big)^{1/2}\\ &\le C2^{k(\frac{1}{2}-\theta)}\Big(\mathbb{E}\Big\|\sum_{i=1}^{\infty}\varepsilon_i x_i\Big\|^2\Big)^{1/2}.\end{aligned}$$

Now

$$\sum_{i=2^{k-1}}^{2^k-1}\|x_i\|^s \le \|x_{2^{k-1}}\|^{s-2}\sum_{i=2^{k-1}}^{2^k-1}\|x_i\|^2.$$

Thus

$$\begin{aligned}\sum_{i=1}^{\infty}\|x_i\|^s &\le C^2\Big(\sum_{k=1}^{\infty}2^{k(1-2\theta)}\|x_{2^{k-1}}\|^{s-2}\Big)\mathbb{E}\Big\|\sum_{i=1}^{\infty}\varepsilon_i x_i\Big\|^2\\ &\le C^2\Big(\sum_{k=1}^{\infty}2^{k(1-2\theta)}2^{(1-k)(1-\frac{2}{s})}\Big)\Big(\sum_{i=1}^{\infty}\|x_i\|^s\Big)^{1-\frac{2}{s}}\mathbb{E}\Big\|\sum_{i=1}^{\infty}\varepsilon_i x_i\Big\|^2.\end{aligned}$$

Rearranging the last expression gives us an estimate

$$\Big(\sum_{i=1}^{\infty}\|x_i\|^s\Big)^{1/s} \le C'\Big(\mathbb{E}\Big\|\sum_{i=1}^{\infty}\varepsilon_i x_i\Big\|^2\Big)^{1/2},$$

for some constant $C'$; hence $X$ has cotype $s$ by Kahane's inequality. ◻

The following theorem was proved by Rosenthal in 1973 [272] using somewhat different techniques; it strongly influenced the development of factorization theory by Maurey.

**Theorem 7.2.6.** *Suppose $X$ is a closed linear subspace of $L_p$ $(1 \le p < 2)$. Then the following conditions are equivalent:*

*(i) $X$ does not contain any subspace isomorphic to $\ell_p$.*
*(ii) $X$ does not contain any complemented subspace isomorphic to $\ell_p$.*
*(iii) $X$ has type $r$ for some $r > p$.*
*(iv) The set $\{|f|^p : f \in B_X\} \subset L_1$ is equi-integrable.*
*(v) $X$ is strongly embedded in $L_p$.*

*Moreover, if $p = 1$, these conditions are equivalent to:*

*(vi) $X$ is reflexive.*

*Proof.* Notice that in the case $p = 1$ we already have the equivalence of $(i)$, $(iv)$, and $(vi)$ (see Theorem 5.2.8 and Proposition 5.7.2).

$(i) \Rightarrow (iv)$ We need only consider the case $1 < p < 2$.

If $\{|f|^p : f \in B_X\}$ is not equi-integrable, we can find a sequence $(g_n)_{n=1}^\infty$ in $B_X$ and a sequence of disjoint Borel sets $(A_n)_{n=1}^\infty$ such that $\|g_n \chi_{A_n}\|_p > 3\delta$ for some $\delta > 0$. Since $L_p$ is reflexive, by passing to a subsequence we can assume that $(g_n)_{n=1}^\infty$ is weakly convergent to some $g \in L_p$ (Corollary 1.6.4). Then, by the disjointedness of the sets $(A_n)$,

$$\sum_{n=1}^{\infty} \|g\chi_{A_n}\|_p^p < \infty.$$

Hence, by deleting finitely many terms, without loss of generality, we will assume that $\|g\chi_{A_n}\|_p < \delta$ for all $n$.

Let us consider the sequence of functions $(f_n)_{n=1}^\infty \subset B_X$ given by

$$f_n = \frac{1}{2}(g_n - g), \qquad n \in \mathbb{N}.$$

Then $\|f_n \chi_{A_n}\|_p > \delta$ for all $n$, and $(f_n)_{n=1}^\infty$ is weakly null. We can argue that a further subsequence (which we still label $(f_n)_{n=1}^\infty$) is a basic sequence equivalent to a block basis of the Haar basis in $L_p$, and thus is unconditional. This uses the Bessaga–Pełczyński selection principle (Proposition 1.3.10) and the unconditionality of the Haar basis in $L_p$ (Theorem 6.1.7). We will show that $(f_n)_{n=1}^\infty$ is equivalent to the canonical $\ell_p$-basis.

For any sequence of scalars $(a_n)_{n=1}^{\infty} \in c_{00}$, by unconditionality there is a constant $K$ such that

$$K^{-1}\mathbb{E}\Big\| \sum_{j=1}^{\infty} \epsilon_j a_j f_j \Big\|_p \le \Big\| \sum_{j=1}^{n} a_j f_j \Big\|_p \le K\mathbb{E}\Big\| \sum_{j=1}^{\infty} \epsilon_j a_j f_j \Big\|_p, \tag{7.13}$$

for any choice of signs $(\epsilon_j)$. Then, by the fact that $L_p$ has type $p$, we obtain an upper estimate

$$\Big\| \sum_{j=1}^{\infty} a_j f_j \Big\|_p \le C_p \Big( \sum_{j=1}^{\infty} |a_j|^p \Big)^{1/p}$$

for a suitable constant $C_p$.

To get a lower estimate, first we use equation (7.13) in combination with Theorem 6.2.13 and Kahane's inequality to obtain

$$\Big\| \sum_{j=1}^{n} a_j f_j \Big\|_p \ge K_p \Big\| (\sum_{j=1}^{\infty} |a_j|^2 |f_j|^2)^{\frac{1}{2}} \Big\|_p,$$

for some constant $K_p$; and now we argue that

$$\begin{aligned} \Big\| (\sum_{j=1}^{\infty} |a_j|^2 |f_j|^2)^{\frac{1}{2}} \Big\|_p &\ge \| \max_j |a_j f_j| \|_p \\ &\ge \| \max_j |a_j f_j| \chi_{A_j} \|_p \\ &= \Big\| \sum_{j=1}^{\infty} |a_j f_j| \chi_{A_j} \Big\|_p \\ &= \Big( \sum_{j=1}^{\infty} |a_j|^p \| f_j \chi_{A_j} \|_p^p \Big)^{1/p} \\ &\ge \delta \Big( \sum_{j=1}^{\infty} |a_j|^p \Big)^{1/p}. \end{aligned}$$

$(iv) \Rightarrow (iii)$ Since $\{|f|^p : f \in B_X\}$ is equi-integrable, using Lemma 5.2.5 show that there is a function $\theta(M)$ with $\lim_{M\to\infty} \theta(M) = 0$ such that

$$\| f \chi_{(|f|>M)} \|_p \le \theta(M), \qquad f \in B_X.$$

For each $N \in \mathbb{N}$ let $f_1, \dots, f_N$ be any sequence of norm-one functions in $X$. Combining Theorem 6.2.13 and Kahane's inequality we can see there is a constant

$C$ (depending only on $p$) such that

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^N \varepsilon_j a_j f_j\Big\|_p^2\Big)^{1/2} \le C\Big\|\Big(\sum_{j=1}^N |a_j|^2|f_j|^2\Big)^{1/2}\Big\|_p,$$

for any sequence of scalars $(a_j)$. Let us estimate the latter expression by splitting each $f_j$ in the form $f_j = g_j + h_j$, where $|g_j| \le M$ and $\|h_j\|_p \le \theta(M)$:

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^N |a_j|^2|f_j|^2\Big)^{1/2}\Big\|_p &\le \Big\|\Big(\sum_{j=1}^N |a_j|^2|g_j|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_{j=1}^N |a_j|^2|h_j|^2\Big)^{1/2}\Big\|_p \\ &\le M\Big(\sum_{j=1}^N |a_j|^2\Big)^{1/2} + \Big\|\Big(\sum_{j=1}^N |a_j|^p|h_j|^p\Big)^{1/p}\Big\|_p \\ &\le M\Big(\sum_{j=1}^N |a_j|^2\Big)^{1/2} + \theta(M)\Big(\sum_{j=1}^N |a_j|^p\Big)^{1/p} \\ &\le \Big(M + \theta(M)N^{\frac{1}{p}-\frac{1}{2}}\Big)\Big(\sum_{j=1}^N |a_j|^2\Big)^{1/2}.\end{aligned}$$

If we choose $M$ such that $\theta(M) < (2C)^{-1}$, we see that for large enough $N$ we have

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^N \varepsilon_j a_j f_j\Big\|_p^2\Big)^{1/2} \le \frac{1}{2}N^{\frac{1}{p}-\frac{1}{2}}\Big(\sum_{j=1}^N |a_j|^2\Big)^{1/2}.$$

Hence, for that $N$, whenever $(f_j)_{j=1}^N \subset X$, we have

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^N \varepsilon_j f_j\Big\|_p^2\Big)^{1/2} \le \frac{1}{2}N^{\frac{1}{p}-\frac{1}{2}}\Big(\sum_{j=1}^N \|f_j\|^2\Big)^{1/2},$$

and so $X$ has type $r$ for some $r > p$ (Proposition 7.2.5).

To prove $(iii) \Rightarrow (v)$ we use factorization theory. Consider the inclusion map $J : X \to L_p$. By Theorem 7.1.9, for $p < q < r$ we can find a strictly positive density function $h$ such that $h^{-\frac{1}{p}}J$ maps $X$ into $L_q([0,1], h\,dt)$. Since $h^{-\frac{1}{p}}J$ is also an isometry of $X$ into $L_p([0,1], h\,dt)$, this implies that $h^{-\frac{1}{p}}J$ strongly embeds $X$ into $L_p([0,1], h\,dt)$ by Proposition 6.4.5. But this means that convergence in measure is equivalent to norm convergence in $X$ for the original Lebesgue measure as well.

The implication $(v) \Rightarrow (i)$ is simply Theorem 7.2.2; this completes the equivalence of $(i)$, $(iii)$, $(iv)$, and $(v)$.

Finally we note that $(i) \Rightarrow (ii)$ is trivial and that Theorem 6.4.7 shows that $(ii) \Rightarrow (v)$. □

## 7.3 Factoring Through Hilbert Spaces

In the first section of this chapter we saw that if $X$ is of type 2 and $1 \leq p < 2$, then every operator $T : X \to L_p$ factors through a Hilbert space. In this section we shall give a characterization for an operator between Banach spaces to factor through a Hilbert space.

**Definition 7.3.1.** Suppose that $X$ and $Y$ are Banach spaces. We say that an operator $T$ from $X$ to $Y$ *factors through a Hilbert space* if there exist a Hilbert space $H$ and operators $S : X \longrightarrow H$ and $R : H \longrightarrow Y$ satisfying $T = RS$.

We will begin by making some remarks that will lead us to the necessary condition we are seeking. We will consider only real scalars, although at the appropriate moment we will discuss the alterations necessary to handle complex scalars. Throughout this section $H$ will denote a generic Hilbert space with a scalar product $\langle \cdot \rangle$.

Suppose we have $n$ arbitrary vectors $x_1, \ldots, x_n$ in $H$. Given a real orthogonal matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, let us consider the new vectors in $H$ defined from $A$ given by

$$z_i = \sum_{j=1}^{n} a_{ij} x_j, \quad 1 \leq i \leq n. \tag{7.14}$$

Then,

$$\begin{aligned}
\sum_{i=1}^{n} \|z_i\|^2 &= \sum_{i=1}^{n} \Big\| \sum_{j=1}^{n} a_{ij} x_j \Big\|^2 \\
&= \sum_{i=1}^{n} \langle \sum_{j=1}^{n} a_{ij} x_j, \sum_{k=1}^{n} a_{ik} x_k \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij} a_{ik} \langle x_j, x_k \rangle \\
&= \sum_{j=1}^{n} \langle x_j, x_j \rangle \\
&= \sum_{j=1}^{n} \|x_j\|^2 .
\end{aligned}$$

Every real $n \times n$ matrix $A = (a_{ij})$ defines a linear operator (which will be denoted in the same way) $A : \ell_2^n \longrightarrow \ell_2^n$ via

$$A\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}.$$

The matrix $(a_{ij})_{1 \le i,j \le n}$ is orthogonal if and only if the operator $A$ is an isometry. If $(a_{ij})_{i,j=1}^n$ is not orthogonal but $\|A\| \le 1$, it is an exercise of linear algebra to prove that $(a_{ij})$ can be written as a convex combination of orthogonal matrices. In fact, it is always possible to find an orthonormal basis $(e_j)_{j=1}^n$ and $(f_j)_{j=1}^n$ in $\ell_2^n$ such that $Ae_j = \lambda_j f_j$ with $\lambda_j \ge 0$: just find an orthonormal basis of eigenvectors $(e_j)_{j=1}^n$ for $A'A$, where $A'$ is the transpose. Then $A = DU$, where $Df_j = \lambda_j f_j$ and $Ue_j = f_j$. The matrix $U$ is orthogonal, and since $0 \le \lambda_j \le 1$, we can write $D$ as a convex combination of the orthogonal matrices $V_\epsilon f_j = \epsilon_j f_j$, where $\epsilon_j = \pm 1$.

Thus, if $x_1, \dots, x_n, z_1, \dots, z_n$ are arbitrary vectors in $H$ satisfying equation (7.14), where $\|(a_{jk})_{j,k=1}^n\|_{\ell_2^n \to \ell_2^n} \le 1$, we will have

$$\sum_{i=1}^n \|z_i\|^2 \le \sum_{j=1}^n \|x_j\|^2 .$$

This can easily be extended to the case of differing numbers of $x_j$'s and $z_i$'s by adding zeros to one of the two collections of vectors.

**Theorem 7.3.2.** *Let $T$ be an operator from a Banach space $X$ into a Banach space $Y$. Suppose that there exist operators $S : X \longrightarrow H$ and $R : H \longrightarrow Y$ satisfying $T = RS$. If $(x_j)_{j=1}^m$ and $(z_i)_{i=1}^n$ are vectors in $X$ related by the equation*

$$z_i = \sum_{j=1}^m a_{ij} x_j, \quad 1 \le i \le n, \tag{7.15}$$

*where $(a_{ij})$ is a real $n \times m$ matrix such that $\|A\|_{\ell_2^m \to \ell_2^n} \le 1$, then*

$$\Big(\sum_{i=1}^n \|Tz_i\|^2\Big)^{1/2} \le \|S\|\,\|R\|\,\Big(\sum_{j=1}^m \|x_j\|^2\Big)^{1/2}.$$

*Proof.* The proof easily follows from the comments we made. Indeed, given $x_1, \dots, x_m$ and $z_1, \dots, z_n$ in $X$ satisfying (7.15), since the collections of vectors $(Sx_j)_{j=1}^m$ and $(Sz_i)_{i=1}^n$ lie inside $H$, we have

$$
\begin{aligned}
\sum_{i=1}^{n} \|Tz_i\|^2 &= \sum_{i=1}^{n} \|RSz_i\|^2 \\
&\le \|R\|^2 \sum_{i=1}^{n} \|Sz_i\|^2 \\
&\le \|R\|^2 \sum_{j=1}^{m} \left\|Sx_j\right\|^2 \\
&\le \|R\|^2 \, \|S\|^2 \sum_{j=1}^{m} \left\|x_j\right\|^2 .
\end{aligned}
$$

□

In light of the previous theorem, we want to give an alternative formulation of the property that $(x_j)_{j=1}^m$ and $(z_i)_{i=1}^n$ are vectors in $X$ related by the equation

$$
z_i = \sum_{j=1}^{m} a_{ij} x_j, \quad 1 \le i \le n,
$$

where $A = (a_{ij})$ is a real $n \times m$ matrix such that $\|A\|_{\ell_2^m \to \ell_2^n} \le 1$.

**Proposition 7.3.3.** *Given $n, m \in \mathbb{N}$ and any two sets of vectors $(x_j)_{j=1}^m$ and $(z_i)_{i=1}^n$ in a Banach space $X$, the following are equivalent:*

$(a)$ *There is a real $n \times m$ matrix $A = (a_{ij})$ such that $\|A\|_{\ell_2^m \to \ell_2^n} \le 1$ and*

$$
z_i = \sum_{j=1}^{m} a_{ij} x_j, \quad 1 \le i \le n.
$$

$(b)$ $\displaystyle\sum_{j=1}^{m} |x^*(z_j)|^2 \le \sum_{i=1}^{n} |x^*(x_i)|^2$ *for all $x^* \in X^*$.*

*Proof.* Assume that $(a)$ holds. Then, since $\|A\|_{\ell_2^m \to \ell_2^n} \le 1$, it follows that

$$
\sum_{i=1}^{n} |x^*(z_i)|^2 = \sum_{i=1}^{n} \left| x^*\Big( \sum_{j=1}^{m} a_{ij} x_j \Big) \right|^2 = \sum_{i=1}^{n} \left| \sum_{j=1}^{m} a_{ij} x^*(x_j) \right|^2 \le \sum_{j=1}^{m} |x^*(x_j)|^2.
$$

For the reverse implication, $(b) \Rightarrow (a)$, consider the linear operators

$$
\alpha : X^* \longrightarrow \ell_2^m, \quad x^* \mapsto (x^*(x_j))_{j=1}^m,
$$

and

$$\beta : X^* \longrightarrow \ell_2^n, \quad x^* \mapsto (x^*(z_i))_{i=1}^n.$$

The hypothesis says that

$$\|\beta x^*\|_{\ell_2^m} \le \|\alpha x^*\|_{\ell_2^n}, \quad \forall x^* \in X^*.$$

Thus we can define an operator $A_0 : \alpha(X^*) \to \beta(X^*)$ with $\|A_0\| \le 1$ and $\beta = A_0 \circ \alpha$. Then $A_0$ can be extended to an operator $A : \ell_2^m \to \ell_2^n$ with $\|A\| \le 1$. Let $(a_{ij})$ be the matrix associated with $A$. Then

$$x^*(z_i) = \sum_{j=1}^m a_{ij} x^*(x_j) \quad \text{for all } \ x^* \in X^*,$$

which implies

$$z_i = \sum_{j=1}^m a_{ij} x_j, \quad i = 1, \ldots, n.$$

□

The main result of this section is the following criterion:

**Theorem 7.3.4.** *Let $X$ and $Y$ be Banach spaces. Suppose $E$ is a closed linear subspace of $X$ and $T : E \to Y$ is an operator. In order that there exist a Hilbert space $H$ and operators $R : X \to H$, $S : H \to Y$ with $\|R\|\|S\| \le C$ such that $T = RS|_E$, it is necessary and sufficient that for all sets of vectors $(x_j)_{j=1}^m \subset X$ and $(z_i)_{i=1}^n \subset E$ such that*

$$\sum_{i=1}^n |x^*(z_i)|^2 \le \sum_{j=1}^m |x^*(x_j)|^2, \qquad x^* \in X^*,$$

*we have*

$$\Big(\sum_{i=1}^n \|Tz_i\|^2\Big)^{1/2} \le C\Big(\sum_{j=1}^m \|x_j\|^2\Big)^{1/2}.$$

In the proof of this result and other proofs in the next chapter, we will make use of the following lemma. If $\mathcal{A}$ is a subset of a real vector space, we define

$$\text{cone}\,(\mathcal{A}) = \Big\{\sum_{j=1}^n \alpha_j a_j : \ a_1, \ldots, a_n \in A, \ \alpha_1, \ldots, \alpha_n \ge 0, \ n = 1, 2, \ldots\Big\}.$$

**Lemma 7.3.5.** *Let $\mathcal{V}$ be a real vector space. Given two subsets $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{V}$ such that $\mathcal{V} = \text{cone}(\mathcal{B}) - \text{cone}(\mathcal{A})$, and two functions $\phi : \mathcal{A} \to \mathbb{R}$, $\psi : \mathcal{B} \to \mathbb{R}$, the following are equivalent:*

($i$) *There is a linear functional $\mathcal{L}$ on $\mathcal{V}$ satisfying*

$$\phi(a) \leq \mathcal{L}(a), \qquad a \in \mathcal{A},$$

*and*

$$\psi(b) \geq \mathcal{L}(b), \qquad b \in \mathcal{B}.$$

($ii$) *If $(\alpha_i)_{i=1}^m$, $(\beta_j)_{j=1}^n$ are two finite sequences of nonnegative scalars such that*

$$\sum_{i=1}^m \alpha_i a_i = \sum_{j=1}^n \beta_j b_j$$

*for some $(a_i)_{i=1}^m \subset \mathcal{A}$, $(b_j)_{j=1}^n \subset \mathcal{B}$, then*

$$\sum_{i=1}^m \alpha_i \phi(a_i) \leq \sum_{j=1}^n \beta_j \psi(b_j).$$

*Proof.* The implication ($i$) $\Rightarrow$ ($ii$) is immediate.

($ii$) $\Rightarrow$ ($i$) Let us define $p \colon \mathcal{V} \to [-\infty, \infty)$ as

$$p(v) = \inf \Big\{ \sum_{j=1}^n \beta_j \psi(b_j) - \sum_{i=1}^m \alpha_i \phi(a_i) \Big\},$$

the infimum being taken over all representations of $v$ in the form $v = \sum_{j=1}^n \beta_j b_j - \sum_{i=1}^m \alpha_i a_i$, where $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \geq 0$, $a_1, \ldots, a_m \in \mathcal{A}$, and $b_1, \ldots, b_m \in \mathcal{B}$.

The map $p$ is well defined, since $\mathcal{V} = \text{cone}(\mathcal{B}) - \text{cone}(\mathcal{A})$. Besides, one easily checks that $p$ is positive-homogeneous and satisfies $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ for all $v_1, v_2$ in $\mathcal{V}$. In order to prove that $p$ is a sublinear functional, we need to show that $p(v) > -\infty$ for every $v \in \mathcal{V}$. This will follow if $p(0) = 0$. Indeed, $p(v) + p(-v) \geq p(0)$, so neither $p(v)$ nor $p(-v)$ could be $-\infty$ if $p(0) = 0$.

For each representation of 0 in the form $0 = \sum_{j=1}^n \beta_j b_j - \sum_{i=1}^m \alpha_i a_i$, by the hypothesis it follows that $\sum_{j=1}^n \beta_j \psi(b_j) \geq \sum_{i=1}^m \alpha_i \phi(a_i)$. Therefore, by the definition, $p(0) \geq 0$, whence $p(0) = 0$.

As an consequence of the Hahn–Banach theorem, there is a linear functional $\mathcal{L}$ on $\mathcal{V}$ such that $\mathcal{L}(v) \leq p(v)$ for every $v \in \mathcal{V}$, and so $\phi(a) \leq \mathcal{L}(a)$ for all $a \in \mathcal{A}$ and $\mathcal{L}(b) \leq \psi(b)$ for all $b \in \mathcal{B}$. $\square$

*Proof of Theorem 7.3.4.* We need only show that the condition is sufficient. Let $\mathcal{F}(X^*)$ denote the set of all functions from $X^*$ to $\mathbb{R}$, and consider the natural map $X \to \mathcal{F}(X^*)$, $x \mapsto \hat{x}$, where

$$\hat{x}(x^*) = x^*(x), \qquad x^* \in X^*.$$

Let $\mathcal{V}$ be the linear subspace of $\mathcal{F}(X^*)$ of all finite linear combinations of functions of the form $\hat{x}\hat{z}$, with $x$, $z$ in $X$. That is,

$$\mathcal{V} = \Big\{ \sum_{k=1}^{N} \lambda_k \hat{x}_k \hat{z}_k : (\lambda_k)_{k=1}^N \text{ in } \mathbb{R}, \ (x_k)_{k=1}^N \text{ and } (z_k)_{k=1}^N \text{ in } X, \text{ and } N \in \mathbb{N} \Big\}.$$

Clearly, the set $\{\hat{x}^2 : x \in X\}$ spans $\mathcal{V}$, since each product $\hat{x}\hat{z}$ with $x$ and $z$ in $X$ can be written in the form

$$\hat{x}\hat{z} = \frac{1}{4}\big((\hat{x} + \hat{z})^2 - (\hat{x} - \hat{z})^2\big).$$

We want to construct a linear functional $\mathcal{L}$ on $\mathcal{V}$ with the following properties:

$$0 \le \mathcal{L}(\hat{x}^2) \le C^2\|x\|^2, \quad x \in X, \tag{7.16}$$

and

$$\|Tx\|^2 \le \mathcal{L}(\hat{x}^2), \quad x \in E. \tag{7.17}$$

To this end, let us apply Lemma 7.3.5 in the case $\mathcal{A} = \mathcal{B} = \{\hat{x}^2 : x \in X\}$ by putting

$$\phi(\hat{x}^2) = \begin{cases} 0 & \text{if } x \in X \setminus E, \\ \|Tx\|^2 & \text{if } x \in E, \end{cases}$$

and

$$\psi(\hat{x}^2)^2 = C^2\|x\|^2.$$

Suppose that

$$\sum_{i=1}^{n} \beta_i^2 \hat{z}_i^2 = \sum_{j=1}^{m} \alpha_j^2 \hat{x}_j^2$$

for some vectors $(\hat{x}_j)_{j=1}^m$, $(\hat{z}_i)_{i=1}^n$ in $X$ and some nonnegative scalars $(\alpha_j^2)_{j=1}^m$, $(\beta_j^2)_{j=1}^n$. Let us suppose $z_1, \ldots, z_l \in E$ and $z_{l+1}, \ldots, z_n \in X \setminus E$. Then

$$\sum_{i=1}^{l} \beta_i^2 \hat{z}_i^2 \le \sum_{j=1}^{m} \alpha_j^2 \hat{x}_j^2,$$

whence

$$\sum_{i=1}^{l} \|T(\beta_j z_i)\|^2 \le C^2 \sum_{j=1}^{m} \|\alpha_j x_j\|^2.$$

Thus

$$\sum_{i=1}^{n} \beta_i^2 \phi(\hat{z}_i^2) \le \sum_{j=1}^{m} \alpha_j^2 \psi(\hat{x}_j^2).$$

Lemma 7.3.5 yields a linear functional $\mathcal{L}$ on $\mathcal{V}$ with

$$\phi(\hat{x}^2) \le \mathcal{L}(\hat{x}^2) \le \psi(\hat{x}^2), \qquad x \in X.$$

The map $\mathcal{L}$, in turn, induces a symmetric bilinear form $\langle \cdot \rangle$ on $X$ given by

$$\langle x, z \rangle = \mathcal{L}(\hat{x}\hat{z}),$$

so that the mapping

$$X \longrightarrow [0, \infty), \qquad x \mapsto \sqrt{\langle x, x \rangle} = \sqrt{\mathcal{L}(\hat{x}^2)},$$

defines a seminorm on $X$.

Thus, $X$ (modulo the subspace $\{x\,;\,\langle x, x\rangle = 0\}$) endowed with the (now) inner product $\langle\ ,\ \rangle$ is an inner product space, and $\|x\|_0 = \sqrt{\langle x, x\rangle}$ is a norm on $X$. Let $H$ be the completion of $X_0$ under this norm. Then $H$ is a Hilbert space.

Take $S$ to be the induced operator $S : X \to H$ mapping $x$ to its equivalence class in $X_0$. Then we have

$$\|Sx\| \le C\|x\|, \qquad x \in X.$$

The operator $S$ has norm one and dense range. By construction, if $x \in E$, we have

$$\|Tx\| \le \|Sx\|;$$

therefore, we can find an operator $R_0 : S(E) \to Y$ with $\|R_0\| \le 1$ and $T = R_0 S|_E$. Compose $R_0$ with the orthogonal projection of $H$ onto $\overline{S(E)}$ to create $R$.

If $X$ and $Y$ are complex Banach spaces, we proceed first by *forgetting* their complex structure and treating them as real spaces. The same argument creates a

real symmetric bilinear form $\langle \cdot \rangle$ on $X$ that is continuous for the original norm. We can then define a complex inner product by *recalling* the complex structure of $X$ and setting

$$(x, z) = \frac{1}{2\pi} \int_0^{2\pi} \langle e^{i\theta} x, e^{i\theta} z \rangle - i \langle i e^{i\theta} x, e^{i\theta} z \rangle \, d\theta.$$

We leave it to the reader to check that this induces a complex inner product and that using this to define $H$ gives the same conclusion. □

## 7.4 The Kwapień–Maurey Theorems for Type-2 Spaces

We saw in Proposition 6.2.9 that if $H$ is a Hilbert space, then $H$ is of type 2 and cotype 2. More generally, since the type and cotype are isomorphic invariants, every Banach space isomorphic to a Hilbert space is of type 2 and cotype 2. In 1972, Kwapień [184] showed that the converse is also true:

**Theorem 7.4.1.** *A Banach space $X$ is of type 2 and cotype 2 if and only if $X$ is isomorphic to a Hilbert space.*

As Maurey noticed soon after Kwapień obtained Theorem 7.4.1, this is also a factorization theorem, which follows from Theorem 7.4.2 by taking $T$ the identity on $X$:

**Theorem 7.4.2 (Kwapień–Maurey).** *Let $X$ and $Y$ be Banach spaces and $T$ an operator from $X$ to $Y$. If $X$ is of type 2 and $Y$ is of cotype 2, then $T$ factors through a Hilbert space.*

Shortly afterward, Maurey [211] discovered a beautiful Hahn–Banach result for operators from type-2 spaces into a Hilbert space, which we now combine with Theorem 7.4.2 to give the following composite statement (which of course implies both Theorem 7.4.1 and Theorem 7.4.2 by taking $E = X$). In its proof, this lemma will be needed:

**Lemma 7.4.3.** *Let $X$ be a Banach space. Assume that the sets of vectors $\{z_i\}_{i=1}^n$ and $\{x_j\}_{j=1}^m$ of $X$ satisfy the condition*

$$\sum_{i=1}^n |x^*(z_i)|^2 \leq \sum_{j=1}^m |x^*(x_j)|^2, \qquad x^* \in X^*.$$

*Then, if $(\gamma_i)_{i=1}^\infty$ is a sequence of independent Gaussians, we have*

$$\Big(\mathbb{E}\Big\| \sum_{i=1}^n \gamma_i z_i \Big\|^2\Big)^{1/2} \leq \Big(\mathbb{E}\Big\| \sum_{j=1}^m \gamma_j x_j \Big\|^2\Big)^{1/2}.$$

*Proof.* Let $F$ be the linear span of $\{x_1, \ldots, x_m, z_1, \ldots, z_n\}$ in $X$. By hypothesis, the quadratic form $Q$ defined on $F^*$ by

$$Q(f^*) = \sum_{j=1}^{m} |f^*(x_j)|^2 - \sum_{i=1}^{n} |f^*(z_i)|^2$$

is positive definite. Hence we can find $z_{n+1}, \ldots, z_{n+l} \in F$ such that

$$Q(f^*) = \sum_{i=1}^{l} |f^*(z_{n+i})|^2, \qquad f^* \in F^*.$$

This implies that

$$\sum_{i=1}^{n+l} |x^*(z_i)|^2 = \sum_{j=1}^{m} |x^*(x_j)|^2, \qquad x^* \in X^*.$$

Then the vector-valued random variables $\sum_{i=1}^{n+l} \gamma_i z_i$ and $\sum_{j=1}^{m} \gamma_j x_j$ have the same distributions on $X$. As a consequence,

$$\mathbb{E}\Big\| \sum_{i=1}^{n+l} \gamma_i z_i \Big\|^2 = \mathbb{E}\Big\| \sum_{j=1}^{m} \gamma_j x_j \Big\|^2. \tag{7.18}$$

Now,

$$\begin{aligned}
&\Big(\mathbb{E}\Big\| \sum_{i=1}^{n} \gamma_i z_i \Big\|^2\Big)^{1/2} \\
&\leq \frac{1}{2}\Big(\mathbb{E}\Big\| \sum_{i=1}^{n} \gamma_i z_i + \sum_{i=n+1}^{n+l} \gamma_i z_i \Big\|^2\Big)^{1/2} + \frac{1}{2}\Big(\mathbb{E}\Big\| \sum_{i=1}^{n} \gamma_i z_i - \sum_{i=n+1}^{n+l} \gamma_i z_i \Big\|^2\Big)^{1/2} \\
&= \Big(\mathbb{E}\Big\| \sum_{i=1}^{n+l} \gamma_i z_i \Big\|^2\Big)^{1/2} \\
&= \Big(\mathbb{E}\Big\| \sum_{j=1}^{m} \gamma_j x_j \Big\|^2\Big)^{1/2},
\end{aligned}$$

which completes the proof. □

**Theorem 7.4.4.** *Let $X$ and $Y$ be Banach spaces and $E$ a closed subspace of $X$. Suppose $T : E \to Y$ is an operator. If $X$ is of type 2 and $Y$ is of cotype 2, then there exist a Hilbert space $H$ and operators $S : X \to H$, $R : H \to Y$ so that $\|R\|\|S\| \leq T_2(X)C_2(Y)\|T\|$ and $RS|_E = T$.*

*Proof.* We shall prove that for all sequences $(z_i)_{i=1}^n$ in $E$ and $(x_j)_{j=1}^m$ in $X$ such that

$$\sum_{i=1}^n |x^*(z_i)|^2 \leq \sum_{j=1}^m |x^*(x_j)|^2, \qquad x^* \in X^*, \tag{7.19}$$

we have

$$\Big(\sum_{i=1}^n \|Tz_i\|^2\Big)^{1/2} \leq T_2(X)C_2(Y)\|T\|\Big(\sum_{j=1}^m \|x_j\|^2\Big)^{1/2},$$

and then we will appeal to the factorization criterion given by Theorem 7.3.4. The key to the argument is to replace the Rademacher functions in the definition of type and cotype by Gaussian random variables.

On the one hand, for every $(z_i)_{i=1}^n \subset E$, using the cotype-2 property of $Y$, we have

$$\sum_{i=1}^n \|Tz_i\|^2 \leq C_2(Y)^2\,\mathbb{E}\,\Big\|\sum_{i=1}^n \varepsilon_i Tz_i\Big\|^2.$$

Then, if for each $N \in \mathbb{N}$ we consider $(\varepsilon_{ki})_{1\leq i,k\leq N}$, a sequence of $N\times N$ Rademachers, we obtain

$$\sum_{i=1}^n \|Tz_i\|^2 \leq \frac{C_2(Y)^2}{N}\,\mathbb{E}\,\Big\|\sum_{k=1}^N\sum_{i=1}^n \varepsilon_{ki}Tz_i\Big\|^2 = C_2(Y)^2\,\mathbb{E}\,\Big\|\sum_{i=1}^n\sum_{k=1}^N \frac{\varepsilon_{ki}}{\sqrt{N}}Tz_i\Big\|^2.$$

Notice that for each $1 \leq i \leq n$, the random variables $\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{iN}$ are independent and identically distributed, so by the central limit theorem, for each $i$ the sequence $(\sum_{k=1}^N \varepsilon_{ik}/\sqrt{N})_{N=1}^\infty$ converges in distribution to a Gaussian, $\gamma_i$. Thus,

$$\lim_{N\to\infty}\mathbb{E}\,\Big\|\sum_{i=1}^n\sum_{k=1}^N \frac{\varepsilon_{ki}}{\sqrt{N}}Tz_i\Big\|^2 = \mathbb{E}\,\Big\|\sum_{i=1}^n \gamma_i Tz_i\Big\|^2,$$

and therefore,

$$\sum_{i=1}^n \|Tz_i\|^2 \leq C_2(Y)^2\,\mathbb{E}\,\Big\|\sum_{i=1}^n \gamma_i Tz_i\Big\|^2. \tag{7.20}$$

On the other hand, if we let $(\varepsilon_i)_{i=1}^\infty$ be a sequence of Rademachers independent of $(\gamma_i)_{i=1}^\infty$, then for every sequence $(x_j)_j^m \subset X$, the symmetry of the Gaussians yields

$$\begin{aligned}\mathbb{E}\Big\|\sum_{j=1}^{m}\gamma_i x_j\Big\|^2 &= \mathbb{E}\mathbb{E}_\varepsilon\Big\|\sum_{j=1}^{m}\varepsilon_j\gamma_j x_j\Big\|^2\\ &\le T_2(X)^2\mathbb{E}\sum_{j=1}^{m}|\gamma_j|^2\|x_j\|^2\\ &= T_2(X)^2\sum_{j=1}^{m}\|x_j\|^2\mathbb{E}|\gamma_j|^2\\ &= T_2(X)^2\sum_{j=1}^{m}\|x_j\|^2. \end{aligned}\tag{7.21}$$

Suppose that the vectors $(z_i)_{i=1}^n$ in $E$ and $(x_j)_{j=1}^m$ in $X$ satisfy equation (7.19). Using Lemma 7.4.3 in combination with (7.18), (7.20), and (7.21), we obtain the inequality we need to apply Theorem 7.3.4:

$$\begin{aligned}\sum_{i=1}^{n}\|Tz_i\|^2 &\le C_2(Y)^2\mathbb{E}\,\Big\|\sum_{i=1}^{n}\gamma_i z_i\Big\|^2\\ &\le C_2(Y)^2\,\|T\|^2\,\mathbb{E}\,\Big\|\sum_{i=1}^{n}\gamma_i z_i\Big\|^2\\ &\le C_2(Y)^2\,\|T\|^2\,\mathbb{E}\,\Big\|\sum_{j=1}^{m}\gamma_j x_j\Big\|^2\\ &\le C_2(Y)^2 T_2(X)^2\,\|T\|^2\,\sum_{j=1}^{m}\|x_j\|^2.\end{aligned}$$

□

There is a quantitative estimate here that we would like to emphasize:

**Definition 7.4.5.** If $X$ and $Y$ are two isomorphic Banach spaces, the *Banach–Mazur distance* between $X$ and $Y$, denoted by $d(X,Y)$, is defined by the formula

$$d(X,Y) = \inf\Big\{\|T\|\|T^{-1}\| \,:\, T : X \to Y \text{ is an isomorphism}\Big\}.\tag{7.22}$$

The Banach–Mazur distance is not a distance in the real sense of the term, since the triangle inequality does not hold, but $d$ satisfies a submultiplicative triangle inequality; that is,

$$d(X,Z) \le d(X,Y)d(Y,Z)$$

when $X, Y, Z$ are all isomorphic. Note that if $X$ and $Y$ are isomorphic, then by a straightforward scaling argument, the Banach–Mazur distance between $X$ and $Y$ is equivalently given by

$$d(X, Y) = \inf\Big\{\|T\|: T : X \to Y \text{ is an isomorphism with } \|T^{-1}\| = 1\Big\}.$$

If $X$ is isomorphic to a Hilbert space, we put

$$d_X := d(X, H), \tag{7.23}$$

where $H$ is a Hilbert space of the same density character as $X$.

If $X$ and $Y$ are infinite-dimensional Banach spaces, the infimum in (7.22) need not be attained. In particular, it is possible that $d(X, Y) = 1$ and $X$ and $Y$ are not isometric (see the problems). Of course, if $X$ and $Y$ are isometrically isomorphic, then $d(X, Y) = 1$, and the converse holds for finite-dimensional spaces. For this kind of spaces the infimum in (7.22) is always attained:

**Lemma 7.4.6.** *If $X$ and $Y$ are two isomorphic finite-dimensional Banach spaces, then*

$$d(X, Y) = \min\Big\{\|T\|\|T^{-1}\| : T : X \to Y \textit{ is an isomorphism}\Big\}.$$

*Proof.* Pick out $C > d(X, Y)$ and consider

$$\mathcal{K} = \{T \in \mathcal{B}(X, Y): \|x\| \le \|T(x)\| \le C\|x\|, \ \forall x \in X\}.$$

The set $\mathcal{K}$ is a nonempty compact subset of $\mathcal{B}(X, Y)$. Let $S \in \mathcal{K}$ be such that $\|S\|$ is the minimum of the set $\{\|T\|: T \in \mathcal{K}\}$. The operator $S$ satisfies

$$\|x\| \le \|S(x)\| \le \|S\|\|x\|, \quad \forall x \in X.$$

This yields that $S$ is injective, hence invertible given the hypothesis. That is it: $S$ is the operator we were looking for. □

We can take advantage of the language just introduced to state the following quantitative version of Kwapień's theorem:

**Theorem 7.4.7.** *If $X$ is a Banach space of type 2 and cotype 2 then there exist a Hilbert space $H$ and an isomorphism $T: X \to H$ such that $\|T\|\|T^{-1}\| \le T_2(X)C_2(X)$. In particular,*

$$d_X \le T_2(X)C_2(X).$$

We have seen (Theorem 6.4.8) that if $p > 2$, every subspace of $L_p$ that is isomorphic to a Hilbert space is necessarily complemented. Theorem 7.4.4 shows that this phenomenon is simply a consequence of the type-2 property:

**Theorem 7.4.8 (Maurey).** *Let $X$ be a Banach space of type 2. Let $Y$ be a closed subspace of $X$ that is isomorphic to a Hilbert space. Then $Y$ is complemented in $X$.*

*Proof.* Since $Y$ is of cotype 2, the identity map on $Y$ can be extended to a projection of $X$ onto $Y$. □

As we mentioned above, if we specialize the range space in Theorem 7.4.4 to be a Hilbert space, then the assertion is a form of the Hahn–Banach theorem for Hilbert-space-valued operators defined on a type-2 space. An interesting question is whether the extension property in Theorem 7.4.4 actually characterizes type-2 spaces:

**Problem 7.4.9.** *Suppose $X$ is a Banach space with the property that for every closed subspace $E$ of $X$ and every operator $T_0 : E \to H$ ($H$ a Hilbert space) there is a bounded extension $T : X \to H$. Must $X$ be a space of type 2?*

For a partial positive solution of this problem we refer to [44].

Up to now, the only spaces that we have considered in the context of type and cotype are the $L_p$-spaces (and their subspaces and quotients). It is worth pointing out that there are many other Banach spaces to which this theory can be applied. Perhaps the simplest examples are the *noncommutative $\ell_p$-spaces* or *Schatten ideals*. These are ideals of operators on a separable Hilbert space that were originally introduced in 1946 by Schatten and studied in several papers by Schatten and von Neumann; an account is given in [278].

If $H$ is a separable (complex) Hilbert space, we define $\mathcal{S}_p$ to be the set of compact operators $A : H \to H$ such that the positive operator $(A^*A)^{p/2}$ has finite trace, and we impose the norm

$$\|A\|_{\mathcal{S}_p} = \operatorname{tr}(A^*A)^{p/2}.$$

It is not entirely obvious, but it is true, that this is a norm and that the class of such operators forms a Banach space.

In many ways, the structure of $\mathcal{S}_p$ resembles that of $\ell_p$. Thus if $1 \le p \le 2$, then $\mathcal{S}_p$ is of type $p$ and cotype 2, while if $2 \le p < \infty$, then $\mathcal{S}_p$ is of cotype $p$ and type 2 (see [100, 294]). See [13] for the structure of subspaces of $\mathcal{S}_p$.

Recently there has been considerable interest in noncommutative $L_p$-spaces, but even to formulate the definition would take us too far afield.

## Problems

**7.1.** For $1 \le r, p < \infty$, prove that the space $\ell_r(\ell_p)$ embeds in $L_p$ if and only if $r = p$.

**7.2.** Let $p_n = 1 + \frac{1}{n}$. Consider the Banach space $X = \ell_2(\ell_{p_n}^2)$. Show that $\ell_1^2$ does not embed isometrically into $X$ but that $d(X, X \oplus_2 \ell_1^2) = 1$.

**7.3.** Show that every reflexive quotient of a $\mathcal{C}(K)$ space is of type two.

**7.4 (The Weak $L_p$-Spaces, $L_{p,\infty}$).** Let $(\Omega, \mu)$ be a probability measure space and $0 < p < \infty$. A measurable function $f$ is said to belong to *weak* $L_p$, denoted by $L_{p,\infty}$, if

$$\|f\|_{p,\infty} = \sup_{t>0} t\mu(|f| > t)^{1/p} < \infty.$$

(a) Show that $L_{p,\infty}$ is a linear space and that $\|\cdot\|_{p,\infty}$ is a *quasi-norm* on $L_{p,\infty}$, i.e., $\|\cdot\|_{p,\infty}$ satisfies the properties of a norm except the triangle inequality which is replaced by

$$\|f + g\|_{p,\infty} \leq C(\|f\|_{p,\infty} + \|g\|_{p,\infty}), \qquad f, g \in L_{p,\infty},$$

where $C \geq 1$ is a constant independent of $f$, $g$.

(b) Show that $L_{p,\infty}$ is complete for this quasi-norm and hence becomes a *quasi-Banach space*.

(c) Show that if $p > 1$, then $\|\cdot\|_{p,\infty}$ is equivalent to the norm

$$\|f\|_{p,\infty,c} = \sup_{t>0} \sup_{\mu(A)=t} t^{1/p-1} \int_A |f| d\mu.$$

Thus $L_{p,\infty}$ can be regarded as a Banach space.

(d) Show that $L_{p,\infty} \subset L_r$ whenever $0 < r < p$.

**7.5 (Nikishin [228]).** (Continuation.) Suppose $X$ is a Banach space of type $p$ for some $1 \leq p < 2$. Suppose $1 \leq r < p$ and $T : X \to L_r(\mu)$ is a bounded linear operator.

(a) Show that for some suitable constant $C$ we have the following estimate:

$$\mu\Big(\bigcup_{j=1}^{m} \{|Tx_j| \geq 1\}\Big)^{1/r} \leq C\Big(\sum_{j=1}^{m} \|x_j\|^p\Big)^{1/p}, \qquad x_1, \ldots, x_m \in X.$$

(b) For any constant $K > C$ consider a maximal family of disjoint sets of positive measure $(E_i)_{i\in I}$ such that we can find $x_i \in X$ with $\|x_i\| \leq 1$ and $|Tx_i| \geq K(\mu(E_i)^{-1/p})$ on $E_i$. Show that this collection is countable and that

$$\sum_{i\in I} \mu(E_i) \leq \Big(\frac{C}{K}\Big)^{\frac{rp}{p-r}}.$$

(c) Show that given $\epsilon > 0$, there is a set $E$ with $\mu(E) > 1 - \epsilon$ such that the map $T_E f = \chi_E T f$ is a bounded operator from $X$ into $L_{p,\infty}(\mu)$.

This gives a *factorization* through weak $L_p$; it is possible to obtain a more elegant *change of density* formulation (see [258]). Note that if $X$ is an arbitrary Banach space and $r < 1$, we get boundedness of $T_E$ into weak $L_1$.

**7.6 (Jordan–von Neumann [144]).** Show, without appealing to Kwapień's theorem, that if a Banach space $X$ is of type 2 with $T_2(X) = 1$ then $X$ is isometrically a Hilbert space.
[*Hint:* For real scalars, define an inner product by $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$.]

**7.7.** Let $\mu, \nu$ be $\sigma$-finite measures. A linear operator $T : L_p(\mu) \to L_r(\nu)$, $0 < r, p < \infty$, is said to be a *positive* operator if $f \geq 0$ implies $Tf \geq 0$.

(a) Show that if $1 \leq s \leq \infty$, then for every sequence $(f_j)_{j=1}^n \in L_p(\mu)$, we have

$$\Big\|\big(\sum_{j=1}^n |Tf_j|^s\big)^{1/s}\Big\|_r \leq \|T\| \Big\|\big(\sum_{j=1}^n |f_j|^s\big)^{1/s}\Big\|_p.$$

(b) Deduce that if $r < p$ and $p \geq 1$, then $T$ factorizes through $L_p(h\nu)$ for some density function $h$.

**7.8.** Let $T : \ell_p \to L_r$, $r < p < 2$, be the linear operator defined by

$$T(\xi) = \sum_{j=1}^\infty \xi(j)\eta_j,$$

where $(\eta_j)_{j=1}^\infty$ is a sequence of independent normalized $p$-stable random variables.

(a) Using the boundedness of $T$, show that the operator $S : \ell_{p/2} \to L_{r/2}$ defined by

$$S(\xi) = \sum_{j=1}^\infty \xi(j)|\eta_j|^2$$

is a bounded positive linear operator.

(b) Show that, however,

$$\Big\|\big(n^{-1}\sum_{j=1}^n |Se_j|^{p/2}\big)^{2/p}\Big\|_{r/2} \to \infty$$

and deduce that $S$ cannot be factored via a change of density through $L_{p/2}$. Thus the conclusion of Problem 7.7 fails when $p < 1$. [*Hint:* You need to show that

$$\lim_{n\to\infty} \Big\|n^{-1}\sum_{j=1}^n |\eta_j|^p\Big\|_{r/p} = \infty.$$

Consider $\min(|\eta_j|^p, M)$ for any fixed $M$.]

# Chapter 8
# Absolutely Summing Operators

The theory of absolutely summing operators was one of the most profound developments in Banach space theory between 1950 and 1970. It originates in a fundamental paper of Grothendieck [121] (which actually appeared in 1956). However, some time passed before Grothendieck's remarkable work really became well known among specialists. There are several reasons for this. One major point is that Grothendieck stopped working in the field at just about this time and moved into algebraic geometry (his work in algebraic geometry earned the Fields Medal in 1966). Thus he played no role in the dissemination of his own ideas. He also chose to publish in a relatively obscure journal that was not widely circulated; before the advent of the Internet it was much more difficult to track down copies of articles. Thus it was not until the 1968 paper of Lindenstrauss and Pełczyński [196] that Grothendieck's ideas became widely known. Since 1968, the theory of absolutely summing operators has become a cornerstone of modern Banach space theory.

In fact, most (but not all) of this chapter was known to Grothendieck, although his presentation would have been different. We will utilize the more modern concepts of type and cotype and use the factorization theory from Chapter 7 in our exposition. Although Grothendieck's work predates the material in Chapter 7, it can be considered a development of it. In Chapter 7 we considered conditions on an operator $T : X \to Y$ that would ensure factorization through a Hilbert space; this culminated in the Kwapień–Maurey theorem (Theorem 7.4.2), which says that the conditions that $X$ be of type 2 and $Y$ be of cotype 2 are sufficient. Grothendieck's inequality yields the fact that every operator $T : \mathcal{C}(K) \to L_1$ also factors through a Hilbert space, even though $\mathcal{C}(K)$ is very far from type 2. This seemed quite mysterious until the work of Pisier showed that the condition that $X$ be of type 2 can in certain cases be relaxed to $X^*$ being of cotype 2.

Two good references for further developments of Grothendieck theory are Pisier's CBMS conference lectures [257] and the monograph of Diestel, Jarchow, and Tonge [63].



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## 8.1 Grothendieck's Inequality

Let us state and prove the *fundamental theorem of the metric theory of tensor products* of Grothendieck as was reinterpreted by Lindenstrauss and Pełczyński in their seminal work [196].

**Theorem 8.1.1 (Grothendieck's Inequality).** *There exists a universal constant $K_G$ such that whenever $(a_{jk})_{j,k=1}^{m,n}$ is a real matrix such that*

$$\Big|\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}s_j t_k\Big| \leq \max_j |s_j| \max_k |t_k|$$

*for any two sequences of scalars $(s_j)_{j=1}^m$ and $(t_k)_{k=1}^n$, then*

$$\Big|\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}\langle u_j, v_k\rangle\Big| \leq K_G \max_j \left\|u_j\right\| \max_k \left\|v_k\right\|,$$

*for all sequences of vectors $(u_j)_{j=1}^m$ and $(v_k)_{k=1}^n$ in an arbitrary real Hilbert space $H$.*

*Proof.* Since all Hilbert spaces are linearly isometric, we can choose any Hilbert space to prove the theorem, but it is most convenient to consider the closed subspace $H$ of $L_2$ spanned by a sequence of independent Gaussians $(g_k)_{k=1}^\infty$, equipped with the $L_2$-norm. Notice that if $f = \sum_{k=1}^\infty a_k g_k \in H$ with $\|f\|_2 = \sum_{k=1}^\infty |a_k|^2 = 1$, then $f$ is also a Gaussian, and so

$$\|f\|_4^4 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^4 e^{-\frac{1}{2}x^2}\,dx = 3.$$

Thus for $f \in H$ we have

$$\|f\|_2 \leq \|f\|_4 = 3^{\frac{1}{4}}\|f\|_2. \tag{8.1}$$

This shows that the subspace $(H, \|\cdot\|_2)$ is strongly embedded in $L_4$.

Obviously, for each matrix $A = (a_{jk})_{j,k=1}^{m,n}$, using Schwarz's inequality there is a best constant $\Gamma = \Gamma(A)$ such that for any two finite sequences of functions $(u_j)_{j=1}^m$ and $(v_k)_{k=1}^n$ in $H$,

$$\Big|\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}\langle u_j, v_k\rangle\Big| \leq \Gamma \max_j \left\|u_j\right\|_2 \max_k \left\|v_k\right\|_2. \tag{8.2}$$

Let us assume that $\|u_j\|_2 \leq 1$ for $1 \leq j \leq m$ and $\|v_k\|_2 \leq 1$ for $1 \leq k \leq n$. For fixed $M$, we consider the truncations of the functions $(u_j)_{j=1}^m$ and $(v_k)_{k=1}^n$ at $M$:

$$u_j^M = \begin{cases} u_j & \text{if } |u_j| \leq M, \\ M \operatorname{sgn} u_j & \text{if } |u_j| > M, \end{cases} \qquad v_k^M = \begin{cases} v_k & \text{if } |v_k| \leq M. \\ M \operatorname{sgn} v_k & \text{if } |v_k| > M. \end{cases}$$

Taking into account that $4(x-1) \le x^2$ for $x > 1$, we deduce that if $x > M$, then $16M^2(x-M)^2 \le x^4$. Combining this inequality with (8.1), we obtain

$$16M^2 \int_0^1 |u_j(t) - u_j^M(t)|^2 \, dt \le \int_0^1 |u_j(t)|^4 \, dt \le 3,$$

hence

$$\left\| u_j - u_j^M \right\|_2^2 \le \frac{3}{16M^2}, \quad j = 1, \ldots, n. \tag{8.3}$$

Analogously,

$$\left\| v_k - v_k^M \right\|_2^2 \le \frac{3}{16M^2}, \quad k = 1, \ldots, n. \tag{8.4}$$

Now,

$$\begin{aligned} \Big| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \langle u_j, v_k \rangle \Big| &= \Big| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \int_0^1 u_j v_k \, dt \Big| \\ &\le \int_0^1 \Big| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} u_j^M v_k^M \Big| \, dt + \Big| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \int_0^1 (u_j - u_j^M) v_k^M \, dt \Big| \\ &\quad + \Big| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \int_0^1 u_j (v_k - v_k^M) \, dt \Big|. \end{aligned}$$

By the hypothesis on the matrix $(a_{jk})$, for each $t \in [0, 1]$ we have

$$\Big| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} u_j^M(t) v_k^M(t) \Big| \, dt \le M^2.$$

On the other hand, the equations (8.2), (8.3), and (8.4) yield

$$\Big| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \int_0^1 (u_j - u_j^M) v_k^M \, dt \Big| = \Big| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \langle u_j - u_j^M, v_k^M \rangle \Big| \le \Gamma \frac{\sqrt{3}}{4M}$$

and

$$\Big| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \int_0^1 u_j (v_k - v_k^M) \, dt \Big| = \Big| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \langle u_j, v_k - v_k^M \rangle \Big| \le \Gamma \frac{\sqrt{3}}{4M}.$$

Combining gives

$$\Big|\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}\langle u_j, v_k\rangle\Big| \leq M^2 + \Gamma\frac{\sqrt{3}}{2M}.$$

By our assumption on $\Gamma$, the following inequality must hold:

$$\Gamma \leq M^2 + \Gamma\frac{\sqrt{3}}{2M}.$$

To minimize the right-hand side, we take $M = \left(\frac{\sqrt{3}}{4}\Gamma\right)^{1/3}$ and thus $\Gamma \leq 3(\frac{\sqrt{3}\Gamma}{4})^{2/3}$, which gives $\Gamma \leq \frac{81}{16}$. Thus Grothendieck's inequality is proved with constant $K_G \leq \frac{81}{16}$. □

While the proof given above is, we feel, the most transparent, it is far from being effective in determining the *Grothendieck constant* $K_G$. Grothendieck's original argument gave $K_G \leq \sinh(\pi/2)$ (see the problems). The best estimate known is that of Krivine [182] that $K_G \leq 2(\sinh^{-1} 1)^{-1} < 2$. The corresponding constant for complex scalars is known to be smaller than $K_G$. See [63] for a full discussion on Grothendieck's inequality.

*Remark 8.1.2.* Suppose $(a_{jk})$ is a real $m \times n$ matrix such that the bilinear form $B : \ell_\infty^m \times \ell_\infty^n \to \mathbb{R}$ given by

$$B((s_j)_{j=1}^m, (t_k)_{k=1}^n) = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}s_jt_k$$

has norm

$$\|B\| = \sup\left\{\Big|\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}s_jt_k\Big| : \max_j |s_j| \leq 1, \max_k |t_k| \leq 1\right\} \leq 1.$$

Suppose $(f_l)_{l=1}^N$ and $(g_l)_{l=1}^N$ are finite sequences in $\ell_\infty^m$ and $\ell_\infty^n$, respectively. For each $1 \leq l \leq N$ let $f_l = (f_l(j))_{j=1}^m$ and $g_l = (g_l(k))_{k=1}^n$. Let us also consider the following two sets of vectors in the Hilbert space $\ell_2^N$:

$$u_j = (f_l(j))_{l=1}^N, \qquad 1 \leq j \leq m,$$

and

$$v_k = (g_l(k))_{k=1}^N, \qquad 1 \leq k \leq n.$$

Then Grothendieck's inequality yields

$$\begin{aligned}
\Big|\sum_{l=1}^{N} B(f_l, g_l)\Big| &= \Big|\sum_{l=1}^{N}\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk} f_l(j) g_l(k)\Big| \\
&= \Big|\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk} \sum_{l=1}^{N} f_l(j) g_l(k)\Big| \\
&= \Big|\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}\langle u_j, v_k\rangle\Big| \\
&\le K_G \max_{1\le j\le m}\|u_j\| \max_{1\le k\le n}\|v_k\| \\
&= K_G \max_{1\le j\le m}\Big(\sum_{l=1}^{N}|f_l(j)|^2\Big)^{1/2} \max_{1\le k\le n}\Big(\sum_{l=1}^{N}|g_l(k)|^2\Big)^{1/2}.
\end{aligned}$$

If we put

$$\max_{1\le j\le m}\Big(\sum_{l=1}^{N}|f_l(j)|^2\Big)^{1/2} = \Big\|\Big(\sum_{l=1}^{N}|f_l|^2\Big)^{\frac{1}{2}}\Big\|_\infty$$

and

$$\max_{1\le k\le n}\Big(\sum_{l=1}^{N}|g_l(k)|^2\Big)^{1/2} = \Big\|\Big(\sum_{l=1}^{N}|g_l|^2\Big)^{\frac{1}{2}}\Big\|_\infty,$$

we obtain an equivalent way of stating Grothendieck's inequality: *Suppose that the bilinear form* $B\colon \ell_\infty^m \times \ell_\infty^n \to \mathbb{R}$ *has norm* $\le 1$. *Then for every* $(f_l)_{l=1}^N$ *in* $\ell_\infty^m$ *and* $(g_l)_{l=1}^N$ *in* $\ell_\infty^n$,

$$\Big|\sum_{l=1}^{N} B(f_l, g_l)\Big| \le K_G \Big\|\Big(\sum_{l=1}^{N}|f_l|^2\Big)^{\frac{1}{2}}\Big\|_\infty \Big\|\Big(\sum_{l=1}^{N}|g_l|^2\Big)^{\frac{1}{2}}\Big\|_\infty.$$

The space $\ell_\infty^m \times \ell_\infty^n$ can be regarded as the space of continuous functions $\mathcal{C}(K_{(m)}) \times \mathcal{C}(L_{(n)})$, where $K_{(m)}$ and $L_{(n)}$ are finite sets of cardinality $m$ and $n$, respectively, equipped with the discrete topology. Our next result extends the previous remark about Grothendieck's inequality to general $\mathcal{C}(K)$-spaces.

**Theorem 8.1.3.** *Let* $K$ *and* $L$ *be two compact Hausdorff spaces and let* $B : \mathcal{C}(K) \times \mathcal{C}(L) \to \mathbb{R}$ *be a bounded bilinear form. Then for every* $(f_k)_{k=1}^n$ *in* $\mathcal{C}(K)$ *and* $(g_k)_{k=1}^n$ *in* $\mathcal{C}(L)$ *we have*

$$\Big|\sum_{k=1}^{n} B(f_k, g_k)\Big| \le K_G\|B\| \Big\|\Big(\sum_{k=1}^{n}|f_k|^2\Big)^{\frac{1}{2}}\Big\|_\infty \Big\|\Big(\sum_{k=1}^{n}|g_k|^2\Big)^{\frac{1}{2}}\Big\|_\infty,$$

*where*

$$\|B\| = \sup\left\{|B(f,g)| : f \in B_{\mathcal{C}(K)}, g \in B_{\mathcal{C}(L)}\right\}.$$

*Proof.* The proof relies on a partition of unity argument. Let $(f_k)_{k=1}^n$ a sequence in $\mathcal{C}(K)$ and $(g_k)_{k=1}^n$ be a sequence in $\mathcal{C}(L)$. Given $\delta > 0$, one can find a finite open cover $(U_i)_{i=1}^N$ of $K$ such that $|f_k(x) - f_k(x')| < \delta$ for each $1 \le k \le n$, whenever $x$ and $x'$ belong to some $U_i$ in the cover. Pick a partition of unity $(\varphi_j)_{j=1}^l$ subordinate to the cover $(U_i)_{i=1}^N$. Thus each $\varphi_j$ satisfies $0 \le \varphi_j \le 1$. Furthermore, supp $\varphi_j = \overline{\{\varphi_j > 0\}}$ lies inside a set $U_{i(j)}$ in the partition, and for all $x \in K$,

$$\sum_{j=1}^{l} \varphi_j(x) = 1.$$

For each $1 \le j \le l$ pick $x_j \in U_{i(j)}$ and put

$$f_k' = \sum_{j=1}^{l} f_k(x_j)\varphi_j, \qquad 1 \le k \le n.$$

Then, for every $x \in K$ with $\varphi_j(x) \neq 0$ we have $|f_k(x_j) - f_k(x)| < \delta$. Hence,

$$|f_k'(x) - f_k(x)| < \delta, \qquad x \in K,\ 1 \le k \le n.$$

That is, $\|f_k' - f_k\|_\infty < \delta$ for $1 \le k \le n$. Note also that $\|f_k'\|_\infty \le \|f_k\|_\infty$ by construction.

Similarly, for any $\delta > 0$ we may find a partition of unity $(\psi_j)_{j=1}^m$ on $L$ with associated points $(y_j)_{j=1}^m$ such that if

$$g_k' = \sum_{j=1}^{m} g_k(y_j)\psi_j, \qquad 1 \le k \le n,$$

then $\|g_k'\|_\infty \le \|g_k\|_\infty$ and $\|g_k' - g_k\|_\infty < \delta$ for $1 \le k \le n$.

Let $(a_{jk})_{j,k=1}^{l,m}$ be the $l \times m$ matrix defined by

$$a_{jk} = B(\varphi_j, \psi_k).$$

For every $(s_j)_{j=1}^l$ and $(t_k)_{k=1}^m$ we have

$$\left|\sum_{j=1}^{l}\sum_{k=1}^{m} a_{jk}s_jt_k\right| \le \|B\| \max_j |s_j| \max_k |t_k|.$$

We select $(u_j)_{j=1}^l$ and $(v_k)_{k=1}^m$ in $\ell_2^n$ by

$$u_j = (f_i(x_j))_{i=1}^n, \qquad v_k = (g_i(y_k))_{i=1}^n.$$

Then

$$\sum_{i=1}^n B(f_i', g_i') = \sum_{i=1}^n \sum_{j=1}^l \sum_{k=1}^m a_{jk} f_i(x_j) g_i(y_k) = \sum_{j=1}^l \sum_{k=1}^m a_{jk} \langle u_j, v_k \rangle,$$

so by Grothendieck's inequality,

$$\Big| \sum_{i=1}^n B(f_i', g_i') \Big| \le K_G \|B\| \sup_j \Big( \sum_{i=1}^n |f_i(x_j)|^2 \Big)^{1/2} \sup_k \Big( \sum_{i=1}^n |g_i(y_k)|^2 \Big)^{1/2}.$$

Now for $1 \le i \le n$,

$$B(f_i, g_i) - B(f_i', g_i') = B(f_i - f_i', g_i) + B(f_i', g_i - g_i'),$$

and so

$$|B(f_i, g_i) - B(f_i', g_i')| \le \delta \|B\| \Big( \|f_i\|_\infty + \|g_i\|_\infty \Big).$$

Putting everything together, we obtain

$$\begin{aligned} \Big| \sum_{i=1}^n B(f_i, g_i) \Big| &\le \Big| \sum_{i=1}^n B(f_i', g_i') \Big| + \delta \|B\| \sum_{i=1}^n \Big( \|f_i\|_\infty + \|g_i\|_\infty \Big) \\ &\le \|B\| \left( K_G \Big\| \Big( \sum_{i=1}^n |f_i|^2 \Big)^{\frac{1}{2}} \Big\|_\infty \Big\| \Big( \sum_{i=1}^n |g_i|^2 \Big)^{\frac{1}{2}} \Big\|_\infty + \delta \sum_{i=1}^n \Big( \|f_i\|_\infty + \|g_i\|_\infty \Big) \right). \end{aligned}$$

Letting $\delta \to 0$ gives the theorem. □

Theorem 8.1.3 also holds for complex scalars if we replace $K_G$ by the complex Grothendieck constant.

*Remark 8.1.4 (Square-function estimates in $\mathcal{C}(K)$-spaces).* In Chapter 6 we saw that in the $L_p$-spaces $(1 \le p < \infty)$ the following square-function estimates hold:

$$\Big\| \Big( \sum_{i=1}^n |f_i|^2 \Big)^{\frac{1}{2}} \Big\|_p \approx \Big( \mathbb{E} \Big\| \sum_{i=1}^n \varepsilon_i f_i \Big\|_p^2 \Big)^{1/2},$$

for every sequence $(f_i)_{i=1}^n$ in $L_p$. Now, in $\mathcal{C}(K)$-spaces, we clearly have

$$\Big\| \Big( \sum_{i=1}^n |f_i|^2 \Big)^{1/2} \Big\|_\infty \le \Big( \mathbb{E} \Big\| \sum_{i=1}^n \varepsilon_i f_i \Big\|_\infty^2 \Big)^{1/2}$$

whenever $(f_i)_{i=1}^n \subset \mathcal{C}(K)$, but the converse estimate does not hold in general. Take, for instance, $\mathcal{C}(\Delta)$, the space of continuous functions on the Cantor set $\Delta$, which we identify here with the topological product space $\{-1,1\}^{\mathbb{N}}$. For each $i$, let $f_i$ be the $i$th projection from $\{-1,1\}^{\mathbb{N}}$ onto $\{-1,1\}$. Then for each $n$ and any choice of signs $(\epsilon_i)_{i=1}^n$, we have

$$\Big\| \sum_{i=1}^n \epsilon_i f_i \Big\|_{\mathcal{C}(\Delta)} = \sup_{x\in\Delta} \Big| \sum_{i=1}^n \epsilon_i f_i(x) \Big| = n,$$

whence

$$\Big( \mathbb{E} \Big\| \sum_{i=1}^n \varepsilon_i f_i \Big\|_{\mathcal{C}(K)}^2 \Big)^{1/2} = n,$$

whereas on the other hand,

$$\Big\| \Big( \sum_{i=1}^n |f_i|^2 \Big)^{1/2} \Big\|_{\mathcal{C}(\Delta)} = \sqrt{n}.$$

**Theorem 8.1.5.** *Suppose $K$ is a compact Hausdorff space, that $(\Omega,\mu)$ is a $\sigma$-finite measure space, and that $T:\mathcal{C}(K)\to L_1(\mu)$ is a continuous operator. Then for every finite sequence $(f_k)_{k=1}^n$ in $\mathcal{C}(K)$ we have*

$$\Big\| \Big( \sum_{k=1}^n |Tf_k|^2 \Big)^{1/2} \Big\|_1 \le K_G \, \|T\| \, \Big\| \Big( \sum_{k=1}^n |f_k|^2 \Big)^{1/2} \Big\|_\infty .$$

*Proof.* Let us define a bilinear form $B:\mathcal{C}(K)\times L_\infty(\mu)\to\mathbb{R}$ by

$$B(f,g) = \int_\Omega g\cdot T(f)\, d\mu.$$

Given a sequence $(f_k)_{k=1}^n$ in $\mathcal{C}(K)$, put $G = (\sum_{k=1}^n |Tf_k|^2)^{1/2}$, and then define

$$g_k(\omega) = \begin{cases} G(\omega)^{-1}(Tf_k)(\omega) & \text{if } G(\omega)\neq 0, \\ 0 & \text{if } G(\omega)=0, \end{cases} \qquad 1\le k\le n.$$

In Chapter 4 we saw that $L_\infty(\mu)$ is isometrically isomorphic to a space of continuous functions $\mathcal{C}(L)$ for some compact Hausdorff space $L$. With that identification we can apply Theorem 8.1.3, which combined with the fact that $\sum_{k=1}^n |g_k|^2 \le 1$ everywhere and $\|B\| = \|T\|$ gives

$$\left\| \Big( \sum_{k=1}^{n} |Tf_k|^2 \Big)^{1/2} \right\|_1 = \sum_{k=1}^{n} \int_{\Omega} g_k T(f_k)\, d\mu$$
$$= \sum_{k=1}^{n} B(f_k, g_k)$$
$$\le K_G \|T\| \left\| \Big( \sum_{k=1}^{n} |f_k|^2 \Big)^{1/2} \right\|_{\infty},$$

as desired. □

We are now in a position to apply Theorem 7.1.2.

**Theorem 8.1.6.** *Suppose $K$ is a compact Hausdorff space, that $(\Omega, \mu)$ is a probability measure space, and that $T : \mathcal{C}(K) \to L_1(\mu)$ is a continuous operator. Then there exists a density function $h$ on $\Omega$ such that for all $f \in \mathcal{C}(K)$,*

$$\left( \int_{\Omega} |h^{-1} Tf|^2 h \, d\mu \right)^{1/2} \le K_G \|T\| \, \|f\|.$$

*In particular, $T$ factors through a Hilbert space.*

*Proof.* It is enough to note that Theorem 8.1.5 implies that

$$\left\| \Big( \sum_{i=1}^{n} |Tf_i|^2 \Big)^{1/2} \right\|_1 \le K_G \, \|T\| \left( \sum_{i=1}^{n} \|f_i\|_{\infty}^2 \right)^{1/2}.$$

Now Theorem 7.1.2 applies. □

Let us recall that Kwapień's theorem (Theorem 7.4.1), or more precisely the Kwapień–Maurey theorem (Theorem 7.4.2), allows us to factorize an arbitrary operator $T : X \to Y$, where $X$ is of type 2 and $Y$ is of cotype 2, through a Hilbert space. However, in the above theorem we achieved the same result when $X = \mathcal{C}(K)$ (which fails to have any nontrivial type) and $Y = L_1(\mu)$. This is rather strange and needs explanation. If $\mathcal{C}(K)$ fails to be of type 2, what is the substitute? Might the answer be that $\mathcal{C}(K)^* = \mathcal{M}(K)$ is of cotype 2? Although type and cotype are not in duality, one is led to wonder whether the optimal hypothesis in the Kwapień–Maurey theorem is that $X^*$ is of cotype 2. Let us prove a result in this direction:

**Theorem 8.1.7.** *Let $X$ be a Banach space whose dual $X^*$ is of cotype 2. Let $T : X \to L_1$ be a bounded operator. Then $T$ factors through a Hilbert space.*

*Proof.* The key here is to obtain an estimate of the form

$$\left\| \Big( \sum_{j=1}^{n} |Tx_j|^2 \Big)^{1/2} \right\|_1 \le C \Big( \sum_{j=1}^{n} \|x_j\|^2 \Big)^{1/2}, \tag{8.5}$$

for some constant $C$ and for all finite sequences $(x_j)_{j=1}^n$ in $X$, so that we can appeal to Theorem 7.1.2.

Assume first that $T$ is a finite-rank operator. In this case, we are guaranteed the existence of a constant such that (8.5) holds. Let the least such constant be denoted by $\Theta = \Theta(T)$. Theorem 7.1.2 yields a density function $h$ on $[0, 1]$ such that for all $x \in X$,

$$\left(\int |Tx(t)|^2 h^{-1}(t)\,dt\right)^{1/2} \le \Theta \|x\|.$$

By Hölder's inequality,

$$\begin{aligned}\int |Tx|^{\frac{4}{3}} h^{-\frac{1}{3}}\,dt &= \int |Tx|^{\frac{2}{3}} (|Tx|^2 h^{-1})^{\frac{1}{3}}\,dt \\ &\le \left(\int |Tx|\,dt\right)^{2/3} \left(\int |Tx|^2 h^{-1}\,dt\right)^{1/3} \\ &\le \|T\|^{2/3} \Theta^{2/3} \|x\|^{4/3}.\end{aligned}$$

Thus if we define $S : X \to L_{4/3}([0, 1], h\,dt)$ by $Sx = h^{-1}Tx$, and $R : L_{4/3}([0, 1], h\,dt) \to L_1$ by $Rf = hf$, we have $\|R\| = 1$, and

$$\|Sx\| \le \|T\|^{\frac{1}{2}} \Theta^{\frac{1}{2}} \|x\|, \qquad x \in X;$$

that is, $\|S\| \le \|T\|^{\frac{1}{2}} \Theta^{\frac{1}{2}}$. Let us consider the adjoint $S^* : L_4([0, 1], h\,dt) \to X^*$. Since $L_4$ is of type 2 and $X^*$ is of cotype 2, we can apply Theorem 7.4.4 to deduce the existence of a Hilbert space $H$, and operators $U : L_4 \to H$ and $V : H \to X^*$ such that $S^* = VU$ and

$$\|V\|\|U\| \le T_2(L_4) C_2(X^*) \|S^*\| \le T_2(L_4) C_2(X^*) \|T\|^{\frac{1}{2}} \Theta^{\frac{1}{2}}.$$

It follows that we can factor $S^{**} = U^* V^* : X^{**} \to L_{4/3}([0, 1], h\,dt)$ through the Hilbert space $H^*$. The restriction to $X$ is a factorization of $S$. For every sequence $(x_k)_{k=1}^n$ in $X$ we have

$$\begin{aligned}\left\| \left(\sum_{k=1}^n |Tx_k|^2\right)^{1/2} \right\|_1 &\le \left(\mathbb{E}\left\| \sum_{k=1}^n \varepsilon_k T x_k \right\|_1^2\right)^{1/2} \\ &\le \left(\mathbb{E}\left\| \sum_{k=1}^n \varepsilon_k S x_k \right\|^2\right)^{1/2} \\ &\le \|U\| \left(\mathbb{E}\left\| \sum_{k=1}^n \varepsilon_k V^* x_k \right\|^2\right)^{1/2} \\ &= \|U\| \left(\sum_{k=1}^n \|V^* x_k\|^2\right)^{1/2} \\ &\le \|V\|\|U\| \left(\sum_{k=1}^n \|x_k\|^2\right)^{1/2},\end{aligned}$$

and so from the definition of $\Theta$,

$$\Theta \le \|U\|\|V\| \le T_2(L_4)C_2(X^*)\|T\|^{\frac{1}{2}}\Theta^{\frac{1}{2}},$$

which implies

$$\Theta(T) \le \big(T_2(L_4)C_2(X^*)\big)^2\|T\|.$$

Now suppose that $T$ is not necessarily of finite rank. Let $(S_k)_{k=1}^\infty$ be the partial sum projections for the Haar basis in $L_1$. Then each $S_kT$ is of finite rank, and $\|S_kT\| \le \|T\|$, since the Haar basis is monotone. Thus

$$\Theta(S_kT) \le \big(T_2(L_4)C_2(X^*)\big)^2\|T\|.$$

By passing to the limit in (8.5) we obtain that $T$ satisfies such an estimate with constant $\Theta(T) \le \big(T_2(L_4)C_2(X^*)\big)^2\|T\|$, and the result follows. □

Notice how we needed to use finite-rank operators and a bootstrap method to obtain this result. This argument is the basis for Pisier's abstract Grothendieck theorem [255]:

**Theorem 8.1.8 (Pisier's Abstract Grothendieck Theorem).** *Let $X$ and $Y$ be Banach spaces such that $X^*$ is of cotype 2, $Y$ is of cotype 2, and either $X$ or $Y$ has the approximation property. Then every operator $T : X \to Y$ factors through a Hilbert space.*

The appearance of the approximation property here is at first unexpected, but recall that we must use finite-rank approximations to our operator. Is the approximation property necessary? In a remarkable paper in 1983, Pisier [256] constructed a Banach space $X$ such that both $X$ and $X^*$ are of cotype 2 but $X$ is not a Hilbert space. Applying Theorem 8.1.8 to the identity operator on this space shows that $X$ must fail the approximation property.

## 8.2 Absolutely Summing Operators

We now introduce an important definition that goes back to the work of Grothendieck.

**Definition 8.2.1.** Let $X$, $Y$ be Banach spaces. An operator $T : X \to Y$ is said to be *absolutely summing* if there is a constant $C$ such that for all choices of $(x_k)_{k=1}^n$ in $X$,

$$\sum_{k=1}^n \|Tx_k\| \le C \sup\left\{\sum_{k=1}^n |x^*(x_k)| : x^* \in X^*, \|x^*\| \le 1\right\}.$$

The least such constant $C$ is denoted by $\pi_1(T)$ and is called the *absolutely summing norm* of $T$.

If $T : X \to Y$ is absolutely summing, in particular, $T$ is bounded and $\|T\| \le \pi_1(T)$, since by definition, for each $x \in X$, then

$$\|Tx\| \le \pi_1(T) \sup \{|x^*(x)| : x^* \in B_{X^*}\} = \pi_1(T)\|x\|.$$

Notice also that for every sequence $(x_k)_{k=1}^n$ in $X$ we have

$$\sup \left\{ \sum_{k=1}^n |x^*(x_k)| : x^* \in B_{X^*} \right\} = \sup \left\{ \Big\| \sum_{k=1}^n \varepsilon_k x_k \Big\| : (\varepsilon_k) \in \{-1, 1\}^n \right\},$$

and so we can equivalently rewrite the definition of absolutely summing operator in terms of the right-hand-side expression.

The next result identifies absolutely summing operators as exactly those operators that transform unconditionally convergent series into absolutely convergent series. We omit the routine proof (see the problems).

**Proposition 8.2.2.** *An operator $T : X \longrightarrow Y$ is absolutely summing if and only if $\sum_{n=1}^\infty \|Tx_n\| < \infty$ whenever $\sum_{n=1}^\infty x_n$ is unconditionally convergent (or simply a (WUC) series).*

Recall that a classical theorem of Riemann asserts that if $\sum x_n$ is a series of real numbers, then $\sum |x_n| < \infty$ if and only if $\sum x_n$ converges unconditionally. This easily extends to every finite-dimensional Banach space. During the late 1940s there was a flurry of interest in the problem of whether the same phenomenon could occur in any *infinite-dimensional* Banach space. In our language this asks whether the identity operator $I_X$ can ever be absolutely summing if $X$ is infinite-dimensional. Note, for example, that if $X$ is a Hilbert space and $(e_n)_{n=1}^\infty$ is an orthonormal sequence, then $\sum \frac{1}{n} e_n$ converges unconditionally but $\sum \frac{1}{n} = \infty$. Before addressing this problem, let us introduce a more general definition:

**Definition 8.2.3.** Let $X$, $Y$ be Banach spaces. An operator $T : X \to Y$ is called *p-absolutely summing* ($1 \le p < \infty$) if there exists a constant $C$ such that for all choices of $(x_k)_{k=1}^n$ in $X$ we have

$$\Big( \sum_{k=1}^n \|Tx_k\|^p \Big)^{1/p} \le C \sup \left\{ \Big( \sum_{k=1}^n |x^*(x_k)|^p \Big)^{1/p} : x^* \in B_{X^*} \right\}. \tag{8.6}$$

The least such constant $C$ is denoted by $\pi_p(T)$ and is called the *p-absolutely summing norm* of $T$.

Let us point out that in practice, we will use only the most important cases, namely $p = 1$ and $p = 2$. In fact, 2-absolutely summing operators play a very important role in further developments.

**Theorem 8.2.4.** *Let $T$ be an operator between the Banach spaces $X$ and $Y$. If $1 \leq r < p < \infty$ and $T$ is $r$-absolutely summing, then $T$ is $p$-absolutely summing with $\pi_p(T) \leq \pi_r(T)$.*

*Proof.* Given $p > r$, let us pick $q$ such that $1/p + 1/q = 1/r$. Suppose $(x_i)_{i=1}^n$ in $X$ satisfy

$$\Big(\sum_{i=1}^n |x^*(x_i)|^p\Big)^{1/p} \leq 1, \qquad \forall\, x^* \in B_{X^*}.$$

Then for all scalars $(c_i)_{i=1}^n$ such that $(\sum_{i=1}^n |c_i|^q)^{1/q} \leq 1$, using Hölder's inequality with the conjugate indices $q/r$ and $p/r$ gives

$$\Big(\sum_{i=1}^n |c_i|^r |x^*(x_i)|^r\Big)^{1/r} \leq \Big(\sum_{i=1}^n |x^*(x_i)|^p\Big)^{1/p} \leq 1,$$

for all $x^* \in B_{X^*}$. Hence

$$\Big(\sum_{i=1}^n |c_i|^r \|Tx_i\|^r\Big)^{1/r} \leq \pi_r(T),$$

and by Hölder's inequality,

$$\Big(\sum_{i=1}^n \|Tx_i\|^p\Big)^{1/p} \leq \pi_r(T).$$

Finally, a standard homogeneity argument immediately yields

$$\Big(\sum_{i=1}^n \|Tx_i\|^p\Big)^{1/p} \leq \pi_r(T) \sup_{\|x^*\|\leq 1} \Big(\sum_{i=1}^n |x^*(x_i)|^p\Big)^{1/p}$$

for every choice of vectors $(x_i)_{i=1}^n$ in $X$. That is, $T$ is $p$-absolutely summing and $\pi_p(T) \leq \pi_r(T)$. □

Before proceeding, let us note the obvious *ideal* properties of the absolutely summing norms, whose proof we leave for the problems.

**Proposition 8.2.5.** *Suppose $1 \leq p < \infty$.*

*(i) If $S, T : X \to Y$ are $p$-absolutely summing operators, then $S + T$ is also $p$-absolutely summing and $\pi_p(S + T) \leq \pi_p(S) + \pi_p(T)$.*
*(ii) Suppose $T : X \to Y$, $S : Y \to Z$, and $R : Z \to W$ are operators. If $S$ is $p$-absolutely summing, then so is $RST$, and $\pi_p(RST) \leq \|R\|\pi_p(S)\|T\|$.*

There is an extensive theory of operator ideals primarily developed by Pietsch and his school; we refer the reader to the survey [64].

Next we will recast the results of the previous section in the language of absolutely summing operators, but first let us make the following useful remark:

*Remark 8.2.6.* Suppose $X$ is a subspace of $\mathcal{C}(K)$, where $K$ is a compact Hausdorff space. Using Jensen's inequality, and the fact that $\nu \in \mathcal{C}(K)^* = \mathcal{M}(K)$ is an extreme point of the unit ball of $\mathcal{C}(K)^*$ if and only if $\nu = \pm\delta_s$, where $\delta_s(f) = f(s)$ for $f \in \mathcal{C}(K)$, we have

$$\begin{aligned}\sup_{x^* \in B_{X^*}} \sum_{j=1}^n |x^*(f_j)|^p &= \sup\left\{\sum_{j=1}^n \left|\int_K f_j \, d\nu\right|^p : \nu \in B_{\mathcal{C}(K)^*}\right\} \\ &\le \sup\left\{\sum_{j=1}^n \int_K |f_j|^p \, d|\nu| : \nu \in B_{\mathcal{M}(K)}\right\} \\ &= \max_{s\in K} \sum_{j=1}^n |f_j(s)|^p\end{aligned}$$

for all $(f_j)_{j=1}^n$ in $X$.

**Theorem 8.2.7.** *Let $K$ be a compact Hausdorff space and let $\mu$ be a $\sigma$-finite measure. Then every bounded operator $T : \mathcal{C}(K) \to L_1(\mu)$ is 2-absolutely summing with $\pi_2(T) \le K_G\|T\|$.*

*Proof.* Using Lemma 6.2.16 in combination with Theorem 8.1.5, we obtain

$$\begin{aligned}\Big(\sum_{i=1}^n \|Tf_i\|_1^2\Big)^{1/2} &= \Big(\sum_{i=1}^n \big\||Tf_i|^2\big\|_{1/2}\Big)^{1/2} \\ &\le \Big\|\sum_{i=1}^n |Tf_i|^2\Big\|_{1/2}^{1/2} \\ &= \Big\|\Big(\sum_{i=1}^n |Tf_i|^2\Big)^{\frac{1}{2}}\Big\|_1 \\ &\le K_G\|T\|\Big\|\Big(\sum_{i=1}^n |f_i|^2\Big)^{1/2}\Big\|_\infty\end{aligned}$$

for all $(f_i)_{i=1}^n$ in $\mathcal{C}(K)$. To complete the proof we need only observe that

$$\Big\|\Big(\sum_{i=1}^n |f_i|^2\Big)^{1/2}\Big\|_\infty = \max_{s\in K}\Big(\sum_{i=1}^n |f_i(s)|^2\Big)^{1/2}$$

$$= \max_{s\in K}\Big(\sum_{i=1}^{n}|\delta_s(f_i)|^2\Big)^{1/2}$$

$$= \sup\Big\{\Big(\sum_{i=1}^{n}|x^*(f_i)|^2\Big)^{1/2} : x^* \in B_{\mathcal{C}(K)^*}\Big\}.$$

□

The next theorem of Pietsch [253] is a fundamental link with factorization theory.

**Theorem 8.2.8.** *Suppose $X$ is a closed subspace of $\mathcal{C}(K)$ ($K$ compact Hausdorff). An operator $T$ from $X$ into a Banach space $Y$ is $p$-absolutely summing for some $1 \le p < \infty$ with $\pi_p(T) \le C$ if and only if there is a regular Borel probability measure $\nu$ on $K$ such that for all $f \in X$,*

$$\|Tf\| \le C\Big(\int_K |f|^p \, d\nu\Big)^{1/p}. \tag{8.7}$$

*Proof.* Assume first that $0 \ne T$ is a $p$-absolutely summing operator. We will use Lemma 7.3.5 to find a linear functional $\mathcal{L}$ on $\mathcal{C}(K)$ satisfying

$$\mathcal{L}(f) \le \max_{s\in K} f(s), \quad \forall f \in \mathcal{C}(K) \tag{8.8}$$

and

$$\pi_p(T)^{-p}\|Tf\|^p \le \mathcal{L}(|f|^p), \quad \forall f \in X. \tag{8.9}$$

To this end, suppose we have functions $f_1, \ldots, f_n \in \mathcal{C}(K)$, $g_1, \ldots, g_m \in X$, and nonnegative scalars $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ such that

$$\sum_{i=1}^{n}\alpha_i f_i = \sum_{j=1}^{m}\beta_j|g_j|^p.$$

Then

$$\pi_p(T)^{-p}\sum_{j=1}^{m}\beta_j\|Tg_j\|^p \le \max_{s\in K}\sum_{j=1}^{m}\beta_j|g_j(s)|^p$$

$$= \max_{s\in K}\sum_{i=1}^{n}\alpha_i f_i(s)$$

$$\le \sum_{i=1}^{n}\alpha_i \max_{s\in K} f_i(s).$$

This guarantees the existence of a linear functional $\mathcal{L}$ on $\mathcal{C}(K)$ satisfying both (8.8) and (8.9). In particular, $\mathcal{L}$ is a positive functional, since $\mathcal{L}(f) \le 0$ whenever $f < 0$, and $\mathcal{L}(-1) \le -1$. By the Riesz representation theorem there is a regular Borel probability measure $\nu$ on $K$ such that

$$\mathcal{L}f = \int_K f\,d\nu$$

for all $f \in \mathcal{C}(K)$. It is then clear that $\nu$ solves our problem.

Suppose conversely that there is a regular Borel probability measure $\nu$ on $K$ such that for all $f \in X$,

$$\|Tf\|^p \le C^p \int_K |f|^p\,d\nu.$$

Then for all $f_1, \dots, f_n \in X$ we have

$$\begin{aligned}\sum_{j=1}^n \|Tf_j\|^p &\le C^p \sum_{j=1}^n \int_K |f_j|^p\,d\nu \\ &\le C^p \max_{s\in K} \sum_{j=1}^n |f_j(s)|^p \\ &= C^p \sup\Big\{\sum_{j=1}^n \Big|\int_K f_j\,d\nu\Big|^p : \nu \in \mathcal{M}(K), \|\nu\| = 1\Big\}.\end{aligned}$$

□

*Remark 8.2.9.* Notice that we just showed that if $\nu$ is a probability measure on some compact Hausdorff space $K$, then for $1 \le p < \infty$ the inclusion maps

$$j_p\colon \mathcal{C}(K) \to L_p(K,\nu)$$

and

$$\iota_p\colon L_\infty(K,\nu) \to L_p(K,\nu)$$

are canonical examples of $p$-absolutely summing operators.

Since every Banach space $X$ can be considered a closed subspace of $\mathcal{C}(B_{X^*})$ (where $B_{X^*}$ has the weak* topology), one usually states Theorem 8.2.8 in the following form:

**Theorem 8.2.10 (Pietsch Factorization Theorem).** *An operator $T : X \to Y$ is $p$-absolutely summing if and only if there is a regular Borel probability measure $\nu$ on $B_{X^*}$ (in its weak* topology) such that for each $x \in X$,*

$$\|Tx\| \le \pi_p(T)\left(\int_{B_{X^*}} |x^*(x)|^p\,d\nu(x^*)\right)^{1/p}. \tag{8.10}$$

**Interpretation** Let us denote by $j_p : \mathcal{C}(B_{X^*}) \to L_p(B_{X^*},\nu)$ the canonical inclusion map and by $X_p$ the closure in $L_p(B_{X^*},\nu)$ of the natural copy of $X$ in $\mathcal{C}(B_{X^*})$. Then

we can induce an operator $S : X_p \to Y$ with $\|S\| = \pi_p(T)$ such that $T = S \circ j_p$. We thus have the following picture:

$$\begin{array}{ccccc} X & \xrightarrow{j_p} & X_p & \xrightarrow{S} & Y \\ \downarrow & & \downarrow & & \\ \mathcal{C}(B_{X^*}) & \xrightarrow{j_p} & L_p(B_{X^*},\nu) & & \end{array} \qquad (T : X \to Y)$$

*Remark 8.2.11.* The case $p = 2$ is special. Suppose $T : X \to Y$ is 2-absolutely summing. Then, since there is an orthogonal projection from $L_2(B_{X^*}, \nu)$ onto the subspace $X_2$, we can factor $T$ in the following manner:

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow & & \uparrow \tilde{S} \\ \mathcal{C}(B_{X^*}) & \xrightarrow{j_2} & L_2(B_{X^*},\nu) \end{array}$$

An immediate consequence is the following theorem.

**Theorem 8.2.12.** *If an operator $T : X \to Y$ is 2-absolutely summing, then it factors through a Hilbert space.*

**Theorem 8.2.13.** *Suppose that $X$, $Y$ are Banach spaces and that $E$ is a closed subspace of $X$. Suppose the operator $T : E \to Y$ is 2-absolutely summing. Then there exists a 2-absolutely summing operator $\tilde{T} : X \to Y$ such that $\tilde{T}|_E = T$ and $\pi_2(\tilde{T}) = \pi_2(T)$.*

*Proof.* We can factor the operator $T : E \to Y$ using Remark 8.2.11:

$$E \xhookrightarrow{\iota_E} \mathcal{C}(B_{E^*}) \xrightarrow{j_2} L_2(B_{E^*},\nu) \xrightarrow{\tilde{S}} Y \qquad (T : E \to Y)$$

On the other hand, the natural inclusion $j_2 : \mathcal{C}(B_{E^*}) \to L_2(B_{E^*}, \nu)$ admits a factorization through $L_\infty(B_{E^*}, \nu)$:

$$\begin{array}{ccccc} \mathcal{C}(B_{E^*}) & & \xrightarrow{j_2} & & L_2(B_{E^*},\nu) \\ & \searrow^{\iota_\infty} & & \nearrow^{\iota_2} & \\ & & L_\infty(B_{E^*},\nu) & & \end{array}$$

If we combine these two diagrams, we see that the operator $\iota_\infty \circ i_E$ maps $E$ continuously into $L_\infty(B_{E^*}, \nu)$, which is an isometrically injective space. Thus $\iota_\infty \circ i_E$ can be extended with preservation of norm to an operator $R$ defined on $X$:

$$\begin{array}{ccccccc} X & & \overset{R}{\cdots\cdots\rightarrow} & & L_\infty(B_{E^*},\nu) & & \\ \uparrow & & & \nearrow^{\iota_\infty} & \downarrow \iota_2 & & \\ E & \xhookrightarrow{i_E} & \mathcal{C}(B_{E^*}) & \xrightarrow{j_2} & L_2\ B_{E^*},\nu) & \xrightarrow{\tilde{S}} & Y \end{array}$$

Clearly, the operator $\tilde{T} = \tilde{S}\iota_2 R$ is an extension of $T$ to $X$. From Proposition 8.2.5 (*ii*) and Remark 8.2.9 we deduce that $\tilde{T}$ is 2-absolutely summing, and since $\pi_2(\iota_2) = 1$, $\|R\| = 1$, and $\|\tilde{S}\| = \pi_2(T)$, it follows that $\pi_2(\tilde{T}) = \pi_2(T)$. □

We can now answer the question we raised on the converse of the Riemann theorem. This result was proved by Dvoretzky and Rogers [81] in 1950, which predates the entire theory of absolutely summing operators. In fact, the proof of Dvoretzky and Rogers that we will touch on later (see Proposition 13.3.5 and Problem 13.9) is quite different and relies on geometric ideas. With the passage of time, the theorem looks a lot easier today than it did in 1950!

**Theorem 8.2.14 (Dvoretzky–Rogers Theorem).** *Let $X$ be a Banach space such that every unconditionally convergent series in $X$ is absolutely convergent. Then $X$ is finite-dimensional.*

*Proof.* By Proposition 8.2.2, our hypothesis is equivalent to saying that the identity operator $I_X : X \to X$ is absolutely summing; hence it is also 2-absolutely summing by Theorem 8.2.4. Now by Theorem 8.2.12 we deduce that $X$ is isomorphic to a Hilbert space. But we have already seen that every infinite-dimensional Hilbert space contains an unconditionally convergent series that is not absolutely convergent, namely, $\sum_{n=1}^{\infty} \frac{1}{n} e_n$, where $(e_n)_{n=1}^{\infty}$ is an orthonormal sequence. □

If we combine Theorem 8.2.7 and Theorem 8.2.8, we obtain an alternative way to see that every operator $T : \mathcal{C}(K) \to L_1(\mu)$ factors through a Hilbert space. This approach is dual to the methods of the previous section, such as those in Theorem 8.1.6. We are in effect introducing a *density* on $K$ rather than on $\Omega$.

**Corollary 8.2.15.** *If $X$ and $Y$ are Banach spaces and $T : X \to Y$ is a $p$-absolutely summing operator for some $1 \leq p < \infty$, then $T$ is Dunford–Pettis and weakly compact. In particular, if $T : X \to Y$ is $p$-absolutely summing and $X$ is reflexive, then $T$ is compact.*

*Proof.* Without loss of generality we can assume $p > 1$. The Pietsch factorization theorem tells us that $T$ factors through a subspace $X_p$ of $L_p(B_{X^*}, \nu)$ for some probability measure $\nu$; hence $T$ must be weakly compact by the reflexivity of $X_p$.

Assume now that $(x_n)_{n=1}^{\infty}$ is a weakly null sequence in $X$. By equation (8.10), for each $x_n$ we have

$$\|Tx_n\| \leq \pi_p(T) \left( \int_{B_{X^*}} |x^*(x_n)|^p \, d\nu(x^*) \right)^{1/p}.$$

The Lebesgue dominated convergence theorem easily yields that $\lim_{n\to\infty} \|Tx_n\| = 0$; hence $T$ is Dunford–Pettis. □

We conclude this section by identifying the 2-absolutely summing operators on a Hilbert space with the well known class of Hilbert–Schmidt operators. In a certain sense we can regard the class of 2-absolutely summing operators as the natural generalization to arbitrary Banach spaces of this class.

**Definition 8.2.16.** Suppose $H_1, H_2$ are separable Hilbert spaces. We assume $H_1, H_2$ infinite-dimensional for notational convenience. An operator $T : H_1 \to H_2$ is said to be *Hilbert–Schmidt* if $\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$ for some orthonormal basis $(e_n)_{n=1}^{\infty}$ of $H_1$.

Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis of $H_1$, and $(f_n)_{n=1}^{\infty}$ an orthonormal basis of $H_2$. Then, by Parseval's identity,

$$\sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle Te_n, f_k\rangle|^2 = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |\langle e_n, T^*f_k\rangle|^2 = \sum_{k=1}^{\infty} \|T^*f_k\|^2. \tag{8.11}$$

This implies that the expression $\sum_{n=1}^{\infty} \|Te_n\|^2$ is independent of the choice of orthonormal basis in $H_1$. The quantity

$$\|T\|_{HS} = \left(\sum_{n=1}^{\infty} \|Te_n\|^2\right)^{1/2}$$

is called the *Hilbert–Schmidt norm of* $T$.

Notice that equation (8.11) also shows that

$$\|T\|_{HS} = \|T^*\|_{HS},$$

so $T : H_1 \to H_2$ is Hilbert–Schmidt if and only if $T^* : H_2 \to H_1$ is.

*Remark 8.2.17.* (a) If $T : H \to H$ is Hilbert–Schmidt, then $\|T\| \le \|T\|_{HS}$.

(b) If $T : H \to H$ is Hilbert–Schmidt, then $T$ is compact. Indeed, take $(P_m)_{m=1}^{\infty}$ the partial sum projections associated to an orthonormal basis $(e_n)_{n=1}^{\infty}$ of $H$ and let $I_H$ be the identity operator on $H$. Then,

$$\|T - TP_m\|_{HS} = \|T(I_H - P_m)\|_{HS} = \|T|_{[e_j; j>m+1]}\|_{HS} \to 0.$$

Therefore $\|T - TP_m\| \to 0$. Since $(TP_m)_{m=1}^{\infty}$ are finite-rank operators, it follows that $T$ is compact.

**Theorem 8.2.18.** *An operator $T : H_1 \to H_2$ is Hilbert–Schmidt if and only if $T$ is 2-absolutely summing. Furthermore, $\|T\|_{HS} = \pi_2(T)$.*

*Proof.* Suppose first that $T$ is 2-absolutely summing. If $(e_j)_{j=1}^{\infty}$ is an orthonormal basis of $H_1$, then for each $n \in N$ we have

$$\sup\left\{\Big(\sum_{j=1}^{n} |\langle e_j, x\rangle|^2\Big)^{1/2} : x \in H_1, \|x\| \le 1\right\} = 1,$$

and so

$$\Big(\sum_{j=1}^{n}\|Te_j\|^2\Big)^{1/2} \le \pi_2(T).$$

Hence $T$ is Hilbert–Schmidt and $\|T\|_{HS} \le \pi_2(T)$.

Suppose conversely that $T$ is Hilbert–Schmidt. Let $(x_j)_{j=1}^n$ in $H_1$ have the property that

$$\sup\left\{\Big(\sum_{j=1}^{n}|\langle x_j, x\rangle|^2\Big)^{1/2} : x \in H_1, \|x\| \le 1\right\} \le 1.$$

Then the operator $S : H_1 \to H_1$ defined by $Se_j = x_j$ for $1 \le j \le n$ and $Se_j = 0$ for $j > n$ satisfies $\|S\| \le 1$. Hence

$$\|TS\|_{HS} = \|S^*T^*\|_{HS} \le \|T^*\|_{HS} = \|T\|_{HS}.$$

Thus

$$\sum_{j=1}^{n}\|Tx_j\|^2 = \sum_{j=1}^{n}\|TSe_j\|^2 \le \|T\|_{HS}^2,$$

which implies that $T$ is 2-absolutely summing with $\pi_2(T) \le \|T\|_{HS}$. □

## 8.3 Absolutely Summing Operators on $L_1(\mu)$-Spaces and an Application to Uniqueness of Unconditional Bases

We now revisit Grothendieck's inequality to obtain another rather startling application from Grothendieck's *Résumé* [121].

**Theorem 8.3.1.** *Suppose $T : L_1(\mu) \to \ell_2$ is a bounded operator. Then $T$ is absolutely summing and $\pi_1(T) \le K_G\|T\|$.*

*Proof.* Suppose $(f_i)_{i=1}^n$ in $L_1(\mu)$ are such that

$$\sup\left\{\sum_{i=1}^{n}\Big|\int_\Omega f_i g\,d\mu\Big| : g \in L_\infty(\mu), \|g\|_\infty \le 1\right\} \le 1.$$

We must show that $\sum_{i=1}^n \|Tf_i\| \le K_G\|T\|$. Notice that it is enough to prove the latter inequality when $(f_i)_{i=1}^n$ are simple functions, so that there is decomposition of $\Omega$ into finitely many measurable sets $A_1, \dots, A_m$ of positive measure such that each $f_i$ is a linear combination of $\{\chi_{A_j}\}_{j=1}^m$. Thus it suffices to prove the result for an operator $T : \ell_1^m \to \ell_2^m$.

Let $T : \ell_1^m \to \ell_2^m$ with $\|T\| \le 1$. Suppose $(x_i)_{i=1}^n$ in $\ell_1^m$ satisfy

$$\sup\left\{\sum_{i=1}^n |\langle x_i, \eta\rangle| : \eta \in \ell_\infty^n, \|\eta\|_\infty \le 1\right\} \le 1.$$

If for each $1 \le i \le n$ we let $x_i = (x_{ik})_{k=1}^m$, it is easy to see that

$$\left|\sum_{i=1}^n \sum_{k=1}^n x_{ik} s_i t_k\right| \le 1$$

whenever $\max |s_i| \le 1$ and $\max |t_k| \le 1$.

Let $(e_k)_{k=1}^m$ denote the canonical basis of $\ell_1^m$ and put $u_k = Te_k \in \ell_2^m$. By our assumption on $T$, $\|u_j\|_2 \le 1$. For $1 \le j \le n$ pick $v_j$ with $\|v_j\|_2 = 1$ such that $\langle T\xi_j, v_j\rangle = \|T\xi_j\|_2$. Then by Grothendieck's inequality (Theorem 8.1.1),

$$\sum_{j=1}^n \|T\xi_j\|_2 = \sum_{j=1}^n \langle T\xi_j, v_j\rangle = \sum_{j=1}^n \sum_{k=1}^m \xi_{jk}\langle u_k, v_j\rangle \le K_G.$$

This establishes the result. □

*Remark 8.3.2.* (a) Since $\ell_1$ is an $L_1(\mu)$-space for a suitable $\mu$, Theorem 8.3.1 holds for operators $T: \ell_1 \to \ell_2$. In particular, it also holds for a quotient map of $\ell_1$ onto $\ell_2$. This is in sharp contrast to the fact that every absolutely $p$-summing operator (for any $p$) on a reflexive space is compact.

(b) Theorem 8.3.1 is actually equivalent to Grothendieck's inequality in the sense that Grothendieck's inequality could equally be derived from this theorem. It is also equivalent to either Theorem 8.1.5 or Theorem 8.2.7.

Lindenstrauss and Pełczyński [196] discovered a very neat application of Theorem 8.3.1 to the isomorphic theory of Banach spaces by showing that the spaces $c_0$ and $\ell_1$ have essentially (i.e., up to equivalence) only one unconditional basis, namely, the unit vector basis. It is almost unfortunate that later, Johnson (cf. [137]) found an *elementary* proof that completely circumvents the use of Grothendieck's inequality!

**Theorem 8.3.3.** *Every normalized unconditional basis in $\ell_1$ [respectively, $c_0$] is equivalent to the canonical basis of $\ell_1$ [respectively, $c_0$].*

*Proof.* Assume that $(u_n)_{n=1}^\infty$ is a normalized unconditional basis in $\ell_1$. For every sequence of scalars $(a_i)$ we have

$$\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2} \le C_2(\ell_1)\Big(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i a_i u_i\Big\|^2\Big)^{1/2} \le C_2(\ell_1)\mathsf{K_u}\Big\|\sum_{i=1}^n a_i u_i\Big\|,$$

where $C_2(\ell_1)$ is the cotype-2 constant of $\ell_1$. From here it follows that the operator $T : \ell_1 \to \ell_2$ defined by

$$T\Big(\sum_{i=1}^{\infty} a_i u_i\Big) = (u_i^*(x))_{i=1}^{\infty} = (a_1, a_2, \dots, a_i, \dots)$$

is bounded with $\|T\| \le C_2(\ell_1)\mathsf{K_u}$. Therefore, by Theorem 8.3.1, $T$ is absolutely summing and $\pi_1(T) \le K_G C_2(\ell_1)\mathsf{K_u}$. Thus

$$\begin{aligned}\sum_{i=1}^{n} |a_i| &= \sum_{i=1}^{n} \|T(a_i u_i)\| \\ &\le K_G C_2(\ell_1)\mathsf{K_u} \sup_{\varepsilon_i=\pm1} \Big\| \sum_{i=1}^{n} \varepsilon_i a_i u_i \Big\| \\ &\le K_G C_2(\ell_1)\mathsf{K_u^2} \Big\| \sum_{i=1}^{n} a_i u_i \Big\|,\end{aligned}$$

which shows that $(u_n)_{n=1}^{\infty}$ is equivalent to the canonical basis of $\ell_1$.

Suppose now that $(u_n)_{n=1}^{\infty}$ is a normalized unconditional basis of $c_0$. We know that every unconditional basis of $c_0$ is shrinking by James's theorem (Theorem 3.3.1); hence the biorthogonal functionals $(u_n^*)_{n=1}^{\infty}$ form an unconditional basis of $\ell_1$. By the first part of the proof, $(u_n^*/\|u_n^*\|)_{n=1}^{\infty}$ is equivalent to the canonical basis of $\ell_1$, and since $1 \le \|u_n^*\| \le \mathsf{K_u}$, $(u_n^*)_{n=1}^{\infty}$ is equivalent to the canonical $\ell_1$-basis. Hence the sequence $(u_n^{**})_{n=1}^{\infty}$ of coordinate functionals associated to $(u_n^*)_{n=1}^{\infty}$ is equivalent to the unit vector basis of $c_0$, and so is $(u_n)_{n=1}^{\infty}$ by Corollary 3.2.4. □

*Remark 8.3.4.* Notice that the argument of Theorem 8.3.3 could be applied to an unconditional basis of $L_1$; the conclusion would be that every normalized unconditional basis of $L_1$ is equivalent to the canonical basis of $\ell_1$. Since $L_1$ is not isomorphic to $\ell_1$, this provides yet another proof that $L_1$ has no unconditional basis. Similarly, every $\mathcal{C}(K)$-space with unconditional basis must be isomorphic to $c_0$ (we have already seen this for quite different reasons in Chapter 4, Theorem 4.5.2).

Let us observe that unconditional bases of Hilbert spaces also share this uniqueness property:

**Theorem 8.3.5.** *If $(u_n)_{n=1}^{\infty}$ is a normalized unconditional basis of a Hilbert space, then $(u_n)_{n=1}^{\infty}$ is equivalent to the canonical basis of $\ell_2$.*

*Proof.* Let $\mathsf{K_u}$ be the unconditional basis constant of $(u_n)_{n=1}^{\infty}$. The unconditionality of the basis and the generalized parallelogram law yield

$$\Big\| \sum_{i=1}^{n} a_i u_i \Big\| \le \mathsf{K_u}\Big(\mathbb{E}\Big\| \sum_{i=1}^{n} \varepsilon_i a_i u_i \Big\|^2\Big)^{1/2} = \mathsf{K_u}\Big(\sum_{i=1}^{n} |a_i|^2\Big)^{1/2},$$

for all scalars $(a_i)_{i=1}^{n}$. The reverse estimate follows in the same way. □

**Definition 8.3.6.** If $X$ is a Banach space with a normalized unconditional basis $(e_n)_{n=1}^{\infty}$, we say that $X$ has a *unique unconditional basis* if whenever $(u_n)_{n=1}^{\infty}$ is another normalized unconditional basis of $X$, then $(u_n)_{n=1}^{\infty}$ is equivalent to $(e_n)_{n=1}^{\infty}$. That is, there is a constant $D$ such that

$$D^{-1}\Big\|\sum_{n=1}^{\infty} a_n u_n\Big\| \le \Big\|\sum_{n=1}^{\infty} a_i e_i\Big\| \le D\Big\|\sum_{n=1}^{\infty} a_n u_n\Big\|,$$

for all $(a_n)_{n=1}^{\infty} \in c_{00}$.

The fact that the three spaces $\ell_1, \ell_2$, and $c_0$ have the property of uniqueness of unconditional basis leads us to consider what other spaces might have the same property. We will resolve this problem later, but let us first show how to construct essentially distinct unconditional bases in $\ell_p$ when $1 < p < \infty$ and $p \neq 2$. This is due to Pełczyński [241], and it beautifully illustrates the use of $L_p$*-methods* to deduce properties about their relatives, the spaces $\ell_p$.

**Proposition 8.3.7.** *If $1 < p < \infty$, $p \neq 2$, then $\ell_p$ has at least two non-equivalent unconditional bases.*

*Proof.* Let $1 < p < \infty$, $p \neq 2$. We saw in Proposition 6.4.2 that the operator $P$ defined in $L_p$ by

$$P(f) = \sum_{k=1}^{\infty}\Big(\int_0^1 f(t) r_k(t)\, dt\Big) r_k$$

is a projection onto $R_p$, the closed subspace spanned in $L_p$ by the Rademacher functions. For each $n$ let $F_p^{(n)}$ denote the subspace of $L_p$ spanned by the characteristic functions on the dyadic intervals of the family $[\frac{k}{2^n}, \frac{k+1}{2^n}]$, $k = 0, 1, \dots, 2^{n-1}$, and let $R_p^{(n)} = [r_k]_{k=1}^n$. Clearly the space $F_p^{(n)}$ is isometric to $\ell_p^{2^n}$, and $R_p^{(n)}$ is isometric to $\ell_2^n$. Moreover, $P|_{F_p^{(n)}}$ is a projection from $F_p^{(n)}$ onto its subspace $R_p^{(n)}$ (with projection constant independent of $n$). It is easy to see that this defines (coordinatewise) a projection from $\ell_p(F_p^{(n)})$ onto $\ell_p(R_p^{(n)})$. Obviously $\ell_p(F_p^{(n)})$ is isometric to $\ell_p(\ell_p^{2^n}) = \ell_p$, and $\ell_p(R_p^{(n)})$ is isometric to $\ell_p(\ell_2^n)$. Since $\ell_p$ is prime and $\ell_p(\ell_2^n)$ is complemented in $\ell_p$, it follows that $\ell_p(\ell_2^n)$ is isomorphic to $\ell_p$.

Then, if $\ell_p$ had a unique unconditional basis, then in particular, the canonical basis of $\ell_p$ and the canonical basis of $\ell_p(\ell_2^n)$ would be equivalent, which is not true. $\square$

## Problems

**8.1 (Grothendieck's Original Proof of Grothendieck's Inequality).**

(a) Let $g_1, g_2$ be (normalized) Gaussians. Show that

$$\mathbb{E}(\operatorname{sgn} g_1)(\operatorname{sgn}(g_1 \cos\theta + g_2 \sin\theta)) = 1 - \frac{2}{\pi}\theta, \qquad 0 \le \theta \le \pi.$$

Now let $X$ be the space of $m \times n$ real matrices with the norm

$$\|A\|_X = \sup_{|s_i|\le 1} \sup_{|t_j|\le 1} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} s_i t_j \right|$$

and define the *multiplier norm* of an $m \times n$ matrix $B$ by

$$\|B\|_{\mathcal{M}} = \sup_{\|A\|_X \le 1} \|B \cdot A\|_X,$$

where $B \cdot A$ is the matrix $(b_{ij}a_{ij})_{i,j=1}^{m,n}$.

(b) Let $u_i, v_j \in \ell_2^N$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Suppose $\|u_i\|_2 = \|v_j\|_2 = 1$ for all $i, j$. By considering $\sum_{k=1}^{N} u_i(k) g_k$ and $\sum_{k=1}^{N} v_j(k) g_k$, where $g_1, \ldots, g_N$ are normalized Gaussians, show that

$$\left\| \left(1 - \frac{2}{\pi}\theta_{ij}\right)_{i,j=1}^{m,n} \right\|_{\mathcal{M}} \le 1,$$

where $\theta_{ij}$ is the unique solution of $0 \le \theta_{ij} \le \pi$ and $\cos\theta_{ij} = \langle u_i, v_j \rangle$.

(c) Using the fact that $\cos\theta_{ij} = \sin(\pi/2 - \theta_{ij})$, show that

$$\|(\cos\theta_{ij})_{i,j=1}^{m,n}\|_m \le \sinh\frac{\pi}{2}.$$

(d) Deduce Grothendieck's inequality with the estimate for the constant $K_G \le \sinh\frac{\pi}{2}$.

**8.2.** (a) Show that Grothendieck's inequality is equivalent to the statement that every bounded operator $T : \ell_1 \to \ell_2$ is absolutely summing (Theorem 8.3.1).

(b) Deduce that Grothendieck's inequality is equivalent to the statement that there is a quotient map $Q : X \to \ell_2$ that is absolutely summing for some separable Banach space $X$.

Using (a) and (b), Pełczyński and Wojtaszczyk proved that Grothendieck's inequality follows from a classical inequality of Paley (if rather indirectly) [248].

**8.3.** Prove Proposition 8.2.2.

**8.4.** Prove Proposition 8.2.5.

**8.5.** Prove that the identity operator $I_X$ on an infinite-dimensional Banach space $X$ is never $p$-absolutely summing for any $p < \infty$.

**8.6.** Prove the dual form of Theorem 8.1.7: *Suppose $X$ is a Banach space of cotype 2. Then every operator $T : \mathcal{C}(K) \to X$ factors through a Hilbert space, and hence $T$ is 2-absolutely summing.*

Deduce that given an operator $T : c_0 \to X$, then there exist $a_n \geq 0$ with $\sum_{n=1}^{\infty} a_n = 1$ and

$$\|T(\xi)\| \leq C\Big(\sum_{j=1}^{\infty} |\xi(j)|^2 a_j\Big)^{1/2}.$$

**8.7.** (a) Show if $T : c_0 \to \ell_2$ is a bounded operator and $S : \ell_2 \to \ell_2$ is Hilbert–Schmidt, then (if $(e_n)_{n=1}^{\infty}$ is the canonical basis)

$$\sum_{n=1}^{\infty} \|STe_n\| < \infty.$$

(b) Deduce (using Problem 8.6) that if $X$ is of cotype 2, then every 2-absolutely summing operator $R : X \to \ell_2$ is absolutely summing.

**8.8.** (a) Let $T : \ell_2 \to \ell_2$ be a $p$-absolutely summing operator, where $p > 2$. Show that $T$ is Hilbert–Schmidt.

(b) Conversely, if $T$ is Hilbert–Schmidt, show that $T$ is absolutely summing.

These results are due to Pietsch [253] and Pełczyński [245]. The best constants involved were found by Garling [99].

**8.9.** (a) Let $X$ be a Banach space. Show that an operator $T : X \to \ell_2$ is 2-absolutely summing if and only if for every operator $S : \ell_2 \to X$, the composition $TS$ is Hilbert–Schmidt.

(b) Show that if every operator $T : X^* \to \ell_2$ is 2-absolutely summing, then every operator $T : X \to \ell_2$ is also 2-absolutely summing.

# Chapter 9
# Perfectly Homogeneous Bases and Their Applications

In this chapter we first prove a characterization of the canonical bases of the spaces $\ell_p$ $(1 \leq p < \infty)$ and $c_0$ due to Zippin [308]. In the remainder of the chapter we show how this is used in several different contexts to prove general theorems by reduction to the $\ell_p$ case. For example, we show that the Lindenstrauss–Pełczyński theorem on the uniqueness of the unconditional basis in $c_0, \ell_1$, and $\ell_2$ (Theorem 8.3.3) has a converse due to Lindenstrauss and Zippin; these are the only three such spaces. We also deduce a characterization of $c_0$ and $\ell_p$ in terms of complementation of block basic sequences due to Lindenstrauss and Tzafriri [200] and apply it to prove a result of Pełczyński and Singer [249] on the existence of conditional bases in any Banach space with a basis.

## 9.1 Perfectly Homogeneous Bases

The canonical bases of $\ell_p$ and $c_0$ have a very special property in that *every* normalized block basic sequence is equivalent to the original basis (Lemma 2.1.1). This property was given the name *perfect homogeneity*.

In the 1960s several papers appeared that proved results for a Banach space with a perfectly homogeneous basis mimicking known results for the $\ell_p$-spaces. However, it turns out that this property actually characterizes the canonical bases of the $\ell_p$-spaces! This is a very useful result, proved in 1966 by Zippin [308]. Thus the concept is quite redundant.

We shall define perfectly homogeneous bases in a slightly different way, which is, with hindsight, equivalent.



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**Definition 9.1.1.** A block basic sequence $(u_n)_{n=1}^{\infty}$ of a basis $(e_n)_{n=1}^{\infty}$,

$$u_n = \sum_{p_{n-1}+1}^{p_n} a_i e_i,$$

is a *constant-coefficient block basic sequence* if for each $n$ there is a constant $c_n$ such that $a_i = c_n$ or $a_i = 0$ for $p_{n-1} + 1 \leq i \leq p_n$; that is,

$$u_n = c_n \sum_{i \in A_n} e_i,$$

where $A_n$ is a subset of integers contained in $(p_{n-1}, p_n]$.

**Definition 9.1.2.** A basis $(e_n)_{n=1}^{\infty}$ of a Banach space $X$ is *perfectly homogeneous* if every normalized constant-coefficient block basic sequence of $(e_n)_{n=1}^{\infty}$ is equivalent to $(e_n)_{n=1}^{\infty}$.

This definition is enough to force every perfectly homogeneous basis to be unconditional, since $(e_n)_{n=1}^{\infty}$ must be equivalent to $(\epsilon_n e_n)_{n=1}^{\infty}$ for every choice of signs $\epsilon_n = \pm 1$.

**Lemma 9.1.3.** *Let $(e_n)_{n=1}^{\infty}$ be a normalized perfectly homogeneous basis of a Banach space $X$. Then $(e_n)_{n=1}^{\infty}$ is uniformly equivalent to all its normalized constant-coefficient block basic sequences. That is, there is a constant $\mathsf{K} \geq 1$ such that for every normalized constant-coefficient block basic sequences $(u_n)_{n=1}^{\infty}$ and $(v_n)_{n=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$ we have*

$$\mathsf{K}^{-1} \Big\| \sum_{k=1}^{n} a_k u_k \Big\| \leq \Big\| \sum_{k=1}^{n} a_k v_k \Big\| \leq \mathsf{K} \Big\| \sum_{k=1}^{n} a_k u_k \Big\|, \tag{9.1}$$

*for any choice of scalars $(a_i)_{i=1}^{n}$ and every $n \in \mathbb{N}$.*

*Proof.* It suffices to prove such an inequality for the basic sequence $(e_n)_{n=n_0+1}^{\infty}$ for some $n_0$. If the lemma fails, we can inductively build constant-coefficient block basic sequences $(u_n)_{n=1}^{\infty}$ and $(v_n)_{n=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$ such that for some increasing sequence of integers $(p_n)_{n=0}^{\infty}$ with $p_0 = 0$ and some scalars $(a_i)_{i=1}^{\infty}$ we have

$$\Big\| \sum_{i=p_{n-1}+1}^{p_n} a_i u_i \Big\| < 2^{-n},$$

but

$$\Big\| \sum_{i=p_{n-1}+1}^{p_n} a_i v_i \Big\| > 2^{-n},$$

which contradicts the assumption of perfect homogeneity. □

Equation (9.1) also yields that for every increasing sequence of integers $(n_k)_{k=1}^\infty$,

$$\mathsf{K}^{-1}\Big\|\sum_{k=1}^n e_k\Big\| \le \Big\|\sum_{k=1}^n e_{n_k}\Big\| \le \mathsf{K}\Big\|\sum_{k=1}^n e_k\Big\|. \tag{9.2}$$

Let us suppose that $(e_n)_{n=1}^\infty$ is a normalized basis for a Banach space $X$. For each $N \in \mathbb{N}$ put

$$\lambda(N) = \Big\|\sum_{n=1}^N e_n\Big\|.$$

Obviously,

$$\mathsf{K_b}^{-1} \le \lambda(N) \le N, \qquad N \in \mathbb{N}, \tag{9.3}$$

where $\mathsf{K_b} \ge 1$ is the basis constant. Notice that if $(e_n)_{n=1}^\infty$ is 1-unconditional, then the sequence $(\lambda(N))_{N=1}^\infty$ is nondecreasing.

**Lemma 9.1.4.** *Suppose that $(e_n)_{n=1}^\infty$ is a normalized unconditional basis of a Banach space $X$. If $\sup_N \lambda(N) < \infty$, then $(e_n)_{n=1}^\infty$ is equivalent to the canonical basis of $c_0$.*

*Proof.* For every $N$ and scalars $(a_n)_{n=1}^N$ we have

$$\frac{1}{\mathsf{K_u}} \sup_n |a_n| \le \Big\|\sum_{n=1}^N a_n e_n\Big\| \le \mathsf{K_u} \sup_n |a_n| \sup_N \lambda(N),$$

i.e., $(e_n)_{n=1}^\infty$ is equivalent to the unit vector basis of $c_0$. □

**Lemma 9.1.5.** *Let $(e_i)_{i=1}^\infty$ be a normalized perfectly homogeneous basis of a Banach space $X$. Then, if $\mathsf{K}$ is the constant given by Lemma 9.1.3, we have*

$$\frac{1}{\mathsf{K}^3}\lambda(n)\lambda(m) \le \lambda(nm) \le \mathsf{K}^3\lambda(n)\lambda(m) \tag{9.4}$$

*for all $m$, $n$ in $\mathbb{N}$.*

*Proof.* Consider a family $(f_j)_{j=1}^m$ of $m$ disjoint blocks of length $n$ of the basis $(e_i)_{i=1}^\infty$,

$$f_j = \sum_{i=(j-1)n+1}^{jn} e_i, \quad j = 1, \dots, m.$$

Let $c_j = \|f_j\|$ for $j = 1, \dots, m$. By hypothesis,

$$\mathsf{K}^{-1}\lambda(n) \le c_j \le \mathsf{K}\lambda(n), \quad j = 1, 2, \dots, m.$$

Note that $\mathsf{K}$ can also serve as an unconditional constant (of course, not necessarily the optimal) for $(e_n)_{n=1}^{\infty}$, so that

$$\frac{1}{\mathsf{K}^2\lambda(n)}\Big\|\sum_{j=1}^{m} f_j\Big\| \leq \Big\|\sum_{j=1}^{m} c_j^{-1} f_j\Big\| \leq \frac{\mathsf{K}^2}{\lambda(n)}\Big\|\sum_{j=1}^{m} f_j\Big\|.$$

Now, again by Lemma 9.1.3,

$$\mathsf{K}^{-1}\lambda(m) \leq \Big\|\sum_{j=1}^{m} c_j^{-1} f_j\Big\| \leq \mathsf{K}\lambda(m).$$

Hence,

$$\frac{\lambda(mn)}{\mathsf{K}^3\lambda(n)} \leq \lambda(m) \leq \frac{\mathsf{K}^3\lambda(mn)}{\lambda(n)}.$$

□

Before continuing, we need the following lemma, which is very useful in many different contexts.

**Lemma 9.1.6.** *Let $(s_n)_{n=1}^{\infty}$ be a sequence of real numbers.*

*(i) Suppose that $s_{m+n} \leq s_m + s_n$ for all $m, n \in \mathbb{N}$. Then $\lim_n s_n/n$ exists (possibly equal to $-\infty$) and*

$$\lim_{n\to\infty} \frac{s_n}{n} = \inf_n \frac{s_n}{n}.$$

*(ii) Suppose that $|s_{m+n} - s_m - s_n| \leq 1$ for all $m, n \in \mathbb{N}$. Then there is a constant $c$ such that*

$$|s_n - cn| \leq 1, \qquad n = 1, 2, \ldots.$$

*Proof.* $(i)$ Fix $n \in \mathbb{N}$. Then, each $m \in \mathbb{N}$ can be written as $m = ln + r$ for some $0 \leq l$ and $0 \leq r < n$. The hypothesis implies that

$$s_{ln} \leq l s_n, \qquad s_{ln+r} \leq l s_n + s_r.$$

Thus

$$\frac{s_m}{m} = \frac{s_{ln+r}}{ln+r} \leq \frac{l}{ln+r} s_n + \frac{s_r}{ln+r} \leq \frac{s_n}{n} + \frac{\max_{0\leq r<n} s_r}{m},$$

and so

$$\limsup_{m\to\infty} \frac{s_m}{m} \le \frac{s_n}{n}, \qquad n \in \mathbb{N}. \tag{9.5}$$

Hence,

$$\limsup_{m\to\infty} \frac{s_m}{m} \le \inf_n \frac{s_n}{n}.$$

$(ii)$ Let $t_n = s_n + 1$ and $u_n = s_n - 1$. Then $(t_n)_{n=1}^\infty$ and $(-u_n)_{n=1}^\infty$ both obey the conditions of $(i)$. Hence $\lim t_n/n = \lim u_n/n$ both exist and are finite; let $c$ be their common value. By $(i)$ we have

$$\frac{u_n}{n} \le c \le \frac{t_n}{n}, \qquad n = 1, 2, \ldots,$$

and the conclusion follows. □

**Lemma 9.1.7.** *Let $(e_n)_{n=1}^\infty$ be a normalized perfectly homogeneous basis of a Banach space $X$. Then, either $(e_n)_{n=1}^\infty$ is equivalent to the canonical basis of $c_0$ or there exist a constant $C$ and $1 \le p < \infty$ such that*

$$C^{-1}|A|^{\frac{1}{p}} \le \Big\| \sum_{n\in A} e_n \Big\| \le C|A|^{\frac{1}{p}},$$

*for every finite subset $A$ of $\mathbb{N}$.*

*Proof.* If we plug $m = 2^k$ and $n = 2^j$ in equation (9.4), we obtain

$$\frac{1}{\mathsf{K}^3}\lambda(2^k)\lambda(2^j) \le \lambda(2^{j+k}) \le \mathsf{K}^3\lambda(2^k)\lambda(2^j). \tag{9.6}$$

Let

$$h(k) = \log_2 \lambda(2^k), \quad k = 0, 1, 2, \ldots.$$

From (9.6) we get

$$|h(j) + h(k) - h(j+k)| \le 3\log_2 \mathsf{K}.$$

By $(ii)$ of Lemma 9.1.6 there is a constant $c$ such that

$$|h(j) - cj| \le 3\log_2 \mathsf{K}, \quad j = 1, 2, \ldots. \tag{9.7}$$

By equation (9.3),

$$\mathsf{K_b}^{-1} \le \lambda(2^k) \le 2^k, \quad k = 0, 1, 2, \ldots,$$

which implies

$$\log_2 \mathsf{K_b}^{-1} \le h(k) \le k, \quad k = 0, 1, 2, \dots,$$

and so $0 \le c \le 1$.

If $c = 0$, we would have $\lambda(2^j) \le \mathsf{K}^3$ for all $j \in \mathbb{N}$. Hence $(\lambda(N))_{N=1}^{\infty}$ would be bounded, and so $(e_n)_{n=1}^{\infty}$ would be equivalent to the canonical basis of $c_0$ by Lemma 9.1.4.

Otherwise, if $0 < c \le 1$, there is $p \in [1, \infty)$ such that $c = 1/p$. Thus we can rewrite equation (9.7) in the form

$$\frac{1}{\mathsf{K}^3} 2^{\frac{j}{p}} \le \lambda(2^j) \le \mathsf{K}^3 2^{\frac{j}{p}}, \quad j \in \mathbb{N}. \tag{9.8}$$

Now, given $n \in \mathbb{N}$, we pick the only integer $j$ such that $2^{j-1} \le n \le 2^j$. By unconditionality we have

$$\mathsf{K}^{-1}\lambda(2^{j-1}) \le \lambda(n) \le \mathsf{K}\lambda(2^j),$$

and combining with (9.8) gives

$$\mathsf{K}^{-4} 2^{-1/p} n^{1/p} \le \lambda(n) \le \mathsf{K}^4 2^{1/p} n^{1/p}.$$

Finally, if $A$ is any finite subset of $\mathbb{N}$, by (9.2) we have

$$\mathsf{K}^{-1}\lambda(|A|) \le \Big\| \sum_{j \in A} e_j \Big\| \le \mathsf{K}\lambda(|A|),$$

hence

$$C^{-1}|A|^{1/p} \le \Big\| \sum_{j \in A} e_j \Big\| \le C|A|^{1/p},$$

where $C = \mathsf{K}^5 2^{1/p}$. □

We now come to Zippin's theorem [308].

**Theorem 9.1.8 (Zippin).** *Let $X$ be a Banach space with normalized basis $(e_n)_{n=1}^{\infty}$. Suppose that $(e_n)_{n=1}^{\infty}$ is perfectly homogeneous. Then $(e_n)_{n=1}^{\infty}$ is equivalent to either the canonical basis of $c_0$ or the canonical basis of $\ell_p$ for some $1 \le p < \infty$.*

*Proof.* If the sequence $(\lambda(N))_{N=1}^{\infty}$ is bounded above, then $(e_n)_{n=1}^{\infty}$ is equivalent to the standard unit vector basis of $c_0$. If $(\lambda(N))_{N=1}^{\infty}$ is unbounded, we can use the preceding lemma to deduce the existence of a constant $C$ and $1 \le p < \infty$ such that

$$C^{-1}|A|^{\frac{1}{p}} \le \Big\| \sum_{k \in A} e_k \Big\| \le C|A|^{\frac{1}{p}},$$

for any finite subset $A$ of $\mathbb{N}$.

To show the equivalence of $(e_n)_{n=1}^{\infty}$ and the canonical basis of $\ell_p$, suppose $(a_i)_{i=1}^n$ is any finite sequence of scalars such that $\sum_{i=1}^n |a_i|^p = 1$. We will suppose that $(a_i)_{i=1}^n$ are such that $|a_i|^p \in \mathbb{Q}$ for all $i = 1, \dots, n$. Hence each $|a_i|^p$ can be written in the form $|a_i|^p = m_i/m$, where $m_i \in \mathbb{N}$, $m$ is the common denominator of the numbers $|a_i|^p$ for $1 \le i \le n$, and $\sum_{i=1}^n m_i = m$.

Let $E_1$ be the interval of natural numbers $[1, m_1]$ and for $i = 2, \dots, n$, let $E_i = [m_1 + \dots + m_{i-1} + 1, m_1 + \dots + m_i]$. Then $E_1, \dots, E_n$ are disjoint intervals of $\mathbb{N}$ such that $|E_i| = m_i$ for each $i = 1, \dots, n$. Consider the normalized constant-coefficient block basic sequence defined for every $i = 1, \dots, n$ as

$$u_i = c_i^{-1} \sum_{k \in E_i} e_k,$$

where $c_i = \| \sum_{k \in E_i} e_k \|$. Since $(e_n)_{n=1}^{\infty}$ is perfectly homogeneous, Lemma 9.1.3 yields

$$\mathsf{K}^{-1} \lambda(m_i) \le c_i \le \mathsf{K} \lambda(m_i)$$

for all $1 \le i \le n$, and so by Lemma 9.1.7,

$$C^{-1} \mathsf{K}^{-1} m_i^{1/p} \le c_i \le C \mathsf{K} m_i^{1/p}.$$

Therefore,

$$\frac{1}{C\mathsf{K}^2 m^{1/p}} \Big\| \sum_{i=1}^n \sum_{j \in E_i} e_j \Big\| \le \Big\| \sum_{i=1}^n a_i u_i \Big\| \le \frac{C\mathsf{K}^2}{m^{1/p}} \Big\| \sum_{i=1}^n \sum_{j \in E_i} e_j \Big\|.$$

This reduces to

$$\frac{\lambda(m)}{C\mathsf{K}^2 m^{1/p}} \le \Big\| \sum_{i=1}^n a_i u_i \Big\| \le \frac{C\mathsf{K}^2 \lambda(m)}{m^{1/p}},$$

and hence

$$\frac{1}{C^2 \mathsf{K}^2} \le \Big\| \sum_{i=1}^n a_i u_i \Big\| \le C^2 \mathsf{K}^2.$$

Using perfect homogeneity again, we have

$$\frac{1}{C^2 \mathsf{K}^3} \le \Big\| \sum_{i=1}^n a_i e_i \Big\| \le C^2 \mathsf{K}^3. \tag{9.9}$$

To finish the proof we note that a simple density argument shows that equation (9.9) holds whenever $\sum_{i=1}^{n} |a_i|^p = 1$ (i.e., without the assumption that $|a_i|^p$ is rational). □

## 9.2 Symmetric Bases

We next study a special class of bases that include the canonical bases of the spaces $\ell_p$ and $c_0$.

**Definition 9.2.1.** A basis $(e_n)_{n=1}^{\infty}$ of a Banach space $X$ is *symmetric* if $(e_n)_{n=1}^{\infty}$ is equivalent to $(e_{\pi(n)})_{n=1}^{\infty}$ for every permutation $\pi$ of $\mathbb{N}$.

Symmetric bases are in particular unconditional. They also have the property of being equivalent to all their (infinite) subsequences, as the next lemma states.

**Lemma 9.2.2.** *Suppose $(e_n)_{n=1}^{\infty}$ is a symmetric basis of a Banach space $X$. Then there exists a constant $D$ such that*

$$D^{-1}\Big\|\sum_{i=1}^{N} a_i e_{j_i}\Big\| \leq \Big\|\sum_{i=1}^{N} a_i e_{k_i}\Big\| \leq D\Big\|\sum_{i=1}^{N} a_i e_{j_i}\Big\|$$

*for every $N \in \mathbb{N}$, every choice of scalars $(a_i)_{i=1}^{N}$, and every two families of distinct natural numbers $\{j_1, \dots, j_N\}$ and $\{k_1, \dots, k_N\}$.*

*Proof.* It is enough to prove the lemma for the basic sequence $(e_n)_{n\geq n_0}$ for some $n_0$. If it is false, then for every $n_0$ we can build a strictly increasing sequence of natural numbers $(p_n)_{n=0}^{\infty}$ with $p_0 = 0$, natural numbers $m_n \leq p_n - p_{n-1}$, scalars $(a_{n,i})_{n=1,i=1}^{\infty,m_n}$, and families $\{j_{n,1}, \dots, j_{n,m_n}\}$, $\{k_{n,1}, \dots, k_{n,m_n}\}$ such that for all $n = 1, 2, \dots$ we have

$$p_{n-1} + 1 \leq j_{n,i}, k_{n,i} \leq p_n, \qquad 1 \leq i \leq m_n,$$

$$\Big\|\sum_{i=1}^{m_n} a_{n,i} e_{j_{n,i}}\Big\| < 2^{-n},$$

and

$$\Big\|\sum_{i=1}^{m_n} a_{n,i} e_{k_{n,i}}\Big\| > 2^{n}.$$

Now one can make a permutation $\pi$ of $\mathbb{N}$ such that $\pi[p_{n-1}+1, p_n] = [p_{n-1}+1, p_n]$ and $\pi(j_{n,i}) = k_{n,i}$, and this will contradict the equivalence of $(e_n)_{n=1}^{\infty}$ and $(e_{\pi(n)})_{n=1}^{\infty}$. □

*Remark 9.2.3.* The converse of Lemma 9.2.2 need not be true. In fact, the summing basis of $c_0$ is equivalent to all its subsequences and is not even unconditional.

**Definition 9.2.4.** A basis $(e_n)_{n=1}^{\infty}$ of a Banach space $X$ is *subsymmetric* provided it is unconditional and for every increasing sequence of integers $\{n_i\}_{i=1}^{\infty}$, the subbasis $(e_{n_i})_{i=1}^{\infty}$ is equivalent to $(e_n)_{n=1}^{\infty}$.

Lemma 9.2.2 yields that symmetric bases are subsymmetric. However, these two concepts do not coincide, as shown by the following example, due to Garling [98].

*Example 9.2.5.* A subsymmetric basis that is not symmetric.

Let $X$ be the space of all sequences of scalars $\xi = (\xi_n)_{n=1}^{\infty}$ for which

$$\|\xi\| = \sup \sum_{k=1}^{\infty} \frac{|\xi_{n_k}|}{\sqrt{k}} < \infty,$$

the supremum being taken over all increasing sequences of integers $(n_k)_{k=1}^{\infty}$. We leave for the reader the task to check that $X$, endowed with the norm defined above, is a Banach space whose unit vectors $(e_n)_{n=1}^{\infty}$ form a subsymmetric basis that is not symmetric.

Let $(e_n)_{n=1}^{\infty}$ be a symmetric basis in a Banach space $X$. For every permutation $\pi$ of $\mathbb{N}$ and every sequence of signs $\epsilon = (\epsilon_n)_{n=1}^{\infty}$, there is an automorphism

$$T_{\pi,\epsilon}\colon X \to X, \qquad x = \sum_{n=1} a_n e_n \mapsto T_{\pi,\epsilon}(x) = \sum_{n=1}^{\infty} \epsilon_n a_n e_{\pi(n)}.$$

The uniform boundedness principle yields a number $K$ such that

$$\sup_{\pi,\epsilon} \|T_{\pi,\epsilon}\| \le K,$$

i.e., the estimate

$$\Big\| \sum_{n=1}^{\infty} \epsilon_n a_n e_{\pi(n)} \Big\| \le K \Big\| \sum_{n=1}^{\infty} a_n e_n \Big\| \tag{9.10}$$

holds for all choices of signs $(\epsilon_n)$ and all permutations $\pi$.

The smallest constant $1 \le K$ in (9.10) is called the *symmetric constant* of $(e_n)_{n=1}^{\infty}$ and will be denoted by $\mathsf{K_s}$. We then say that $(e_n)_{n=1}^{\infty}$ is $K$-*symmetric* whenever $\mathsf{K_s} \le K$.

For every $x = \sum_{n=1}^{\infty} a_n e_n \in X$, put

$$|||x||| = \sup \Big\| \sum_{n=1}^{\infty} \epsilon_n a_n e_{\pi(n)} \Big\|, \tag{9.11}$$

the supremum being taken over all choices of scalars $(\epsilon_n)$ of signs and all permutations of the natural numbers. Equation (9.11) defines a new norm on $X$ equivalent to $\|\cdot\|$, since $\|x\| \le |||x||| \le K\|x\|$ for all $x \in X$. With respect to this norm, $(e_n)_{n=1}^{\infty}$ is a 1-symmetric basis of $X$.

**Theorem 9.2.6.** *Let $X$ be a Banach space with normalized 1-symmetric basis $(e_n)_{n=1}^{\infty}$. Suppose that $(u_n)_{n=1}^{\infty}$ is a normalized constant-coefficient block basic sequence. Then the subspace $[u_n]$ is complemented in $X$ by a norm-one projection.*

*Proof.* For each $k = 1, 2, \dots$, let $u_k = c_k \sum_{j\in A_k} e_k$, where $(A_k)_{k=1}^{\infty}$ is a sequence of mutually disjoint subsets of $\mathbb{N}$ (notice that since $(e_n)_{n=1}^{\infty}$ is 1-symmetric, the blocks of the basis need not be in increasing order). For every fixed $n \in \mathbb{N}$, let $\Pi_n$ denote the set of all permutations $\pi$ of $\mathbb{N}$ such that for each $1 \le k \le n$, $\pi$ restricted to $A_k$ acts as a cyclic permutation of the elements of $A_k$ (in particular, $\pi(A_k) = A_k$)), and $\pi(j) = j$ for all $j \notin \cup_{k=1}^{n} A_k$. Every $\pi \in \Pi_n$ has associated an operator on $X$ defined for $x = \sum_{j=1}^{\infty} a_j e_j$ as

$$T_{n,\pi}\left(\sum_{j=1}^{\infty} a_j e_j\right) = \sum_{j=1}^{\infty} a_j e_{\pi(j)}.$$

Notice that due to the 1-symmetry of $(e_n)_{n=1}^{\infty}$, we have $\|T_{n,\pi}(x)\| = \|x\|$.

Let us define an operator on $X$ by averaging over all possible choices of permutations $\pi \in \Pi_n$. Given $x = \sum_{j=1}^{\infty} a_j e_j$,

$$T_n(x) = \frac{1}{|\Pi_n|}\sum_{\pi\in\Pi_n} T_{n,\pi}(x) = \sum_{k=1}^{n}\left(\frac{1}{|A_k|}\sum_{j\in A_k} a_j\right)\sum_{j\in A_k} e_j + \sum_{j\notin\cup_{k=1}^{n}A_k} a_j e_j.$$

Then,

$$\|T_n(x)\| = \left\|\frac{1}{|\Pi_n|}\sum_{\pi\in\Pi_n} T_{n,\pi}(x)\right\| \le \frac{1}{|\Pi_n|}\sum_{\pi\in\Pi_n}\|T_{n,\pi}(x)\| = \|x\|.$$

Therefore, for each $n \in \mathbb{N}$ the operator

$$P_n(x) = \sum_{k=1}^{n}\left((\frac{1}{|A_k|}\sum_{j\in A_k} a_j\right)\sum_{j\in A_k} e_j, \qquad x \in X,$$

is a norm-one projection onto $[u_k]_{k=1}^{n}$. Now it readily follows that

$$P(x) = \sum_{k=1}^{\infty}\left(\frac{1}{|A_k|}\sum_{j\in A_k} a_j\right)\underbrace{\sum_{j\in A_k} e_j}_{c_k^{-1}u_k}$$

is a well-defined projection from $X$ onto $[u_k]$ with $\|P\| = 1$. $\square$

## 9.3 Uniqueness of Unconditional Basis

Zippin's theorem (Theorem 9.1.8) has a number of very elegant applications. We give a couple in this section. The first relates to the theorem of Lindenstrauss and Pełczyński proved in Section 8.3. There we saw that the normalized unconditional bases of the three spaces $c_0, \ell_1$, and $\ell_2$ are unique (up to equivalence); we also saw that in contrast, the spaces $\ell_p$ for $p \neq 1, 2$ have at least two nonequivalent normalized unconditional bases.

In 1969, Lindenstrauss and Zippin [205] completed the story by showing that the list ends with these three spaces!

**Theorem 9.3.1 (Lindenstrauss–Zippin).** *A Banach space $X$ has a unique unconditional basis (up to equivalence) if and only if $X$ is isomorphic to one of the following three spaces: $c_0$, $\ell_1$, $\ell_2$.*

*Proof.* Suppose that $X$ has a unique normalized unconditional basis, $(e_n)_{n=1}^{\infty}$. Then, in particular, the basis $(e_{\pi(n)})_{n=1}^{\infty}$ is equivalent to $(e_n)_{n=1}^{\infty}$ for each permutation $\pi$ of $\mathbb{N}$. That is, $(e_n)_{n=1}^{\infty}$ is a symmetric basis of $X$. Without loss of generality we can assume that its symmetric constant is 1.

Let $(u_n)_{n=1}^{\infty}$ be a normalized constant-coefficient block basic sequence with respect to $(e_n)_{n=1}^{\infty}$ such that there are infinitely many blocks of size $k$ for all $k \in \mathbb{N}$. That is, $|\{u_n \colon |\text{supp } u_n| = k\}| = \infty$ for each $k \in \mathbb{N}$. Let us call $Y$ the closed linear span of the sequence $(u_n)_{n=1}^{\infty}$.

The subspace $Y$ is complemented in $X$ by Theorem 9.2.6.

On the other hand, the subsequence of $(u_n)_{n=1}^{\infty}$ consisting of the blocks whose supports have size 1 spans a subspace isometrically isomorphic to $X$, which is complemented in $Y$ because of the unconditionality of $(u_n)_{n=1}^{\infty}$.

By the symmetry of the basis $(e_n)_{n=1}^{\infty}$, $X$ is isomorphic to $X^2$.

Analogously, if we split the natural numbers into two subsets $S_1, S_2$ such that

$$\left|\{n \in S_1 \colon |\text{supp} u_n| = k\}\right| = \left|\{n \in S_2 \colon |\text{supp} u_n| = k\}\right| = \infty$$

for all $k \in \mathbb{N}$, we see that

$$[u_n]_{n=1}^{\infty} \approx [u_n]_{n \in S_1} \oplus [u_n]_{n \in S_2} \approx [u_n]_{n=1}^{\infty} \oplus [u_n]_{n=1}^{\infty}.$$

Hence $Y \approx Y^2$.

Using Pełczyński's decomposition trick (Theorem 2.2.3), we deduce that $X \approx Y$.

Since $(u_n)_{n=1}^{\infty}$ is an unconditional basis of $Y$, by the hypothesis it must be equivalent to $(e_n)_{n=1}^{\infty}$. In particular $(u_n)_{n=1}^{\infty}$ is symmetric and, therefore, equivalent to all of its subsequences. Hence $(e_n)_{n=1}^{\infty}$ is perfectly homogeneous. Theorem 9.1.8 implies that $(e_n)_{n=1}^{\infty}$ is equivalent to the canonical basis of either $c_0$ or $\ell_p$ for some $1 \leq p < \infty$. But we saw in the previous chapter (Proposition 8.3.7) that if $p \in (1, \infty) \setminus \{2\}$, then $\ell_p$ has an unconditional basis that is not equivalent to the standard unit vector basis. The only remaining possibilities for the space $X$ are $c_0$, $\ell_1$, and $\ell_2$. □

Lindenstrauss–Zippin's theorem thus completes the classification of those Banach spaces with a unique unconditional basis. The elegance of this result encouraged further work in this direction. One obvious modification is to require uniqueness of unconditional basis up to a permutation (UTAP). In many ways this is a more natural concept for unconditional bases, whose order is irrelevant.

**Definition 9.3.2.** Two unconditional bases $(e_n)_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$ of a Banach space $X$ are said to be *permutatively equivalent* if there is a permutation $\pi$ of $\mathbb{N}$ such that $(e_{\pi(n)})_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$ are equivalent. Then we say that a Banach space $X$ has a (UTAP) unconditional basis $(e_n)_{n=1}^{\infty}$ if every normalized unconditional basis in $X$ is permutatively equivalent to $(e_n)_{n=1}^{\infty}$.

Classifying spaces with (UTAP) bases is more difficult, because the initial step (reduction to symmetric bases) is no longer available.

The first step toward this classification was taken in 1976 by Edelstein and Wojtaszczyk [83], who showed that the finite direct sums of the spaces $c_0, \ell_1$, and $\ell_2$ have (UTAP) bases (thus adding four new spaces to the already known ones). After their work, Bourgain, Casazza, Lindenstrauss, and Tzafriri embarked on a comprehensive study, completed in 1985 [32]. They added the spaces $c_0(\ell_1), \ell_1(c_0)$ and $\ell_1(\ell_2)$ to the list, but showed, remarkably, that $\ell_2(\ell_1)$ fails to have a (UTAP) basis! However, all hopes of a really satisfactory classification of Banach spaces having a (UTAP) basis were dashed when they also found a *nonclassical* Banach space that also has a (UTAP) basis. This space was a modification of Tsirelson space, to be constructed in the next chapter, which contains no copy of any space isomorphic to an $\ell_p$ $(1 \leq p < \infty)$ or $c_0$. The subject was revisited in [42, 43], and several other "pathological" spaces with (UTAP) bases have been discovered, including the original Tsirelson space. For an account of this topic see [298].

For the classification of symmetric basic sequences in $L_p$ spaces we refer to [34, 143, 267].

## 9.4 Complementation of Block Basic Sequences

We now turn our attention to the study of complementation of subspaces of a Banach space. Starting with the example of $c_0$ in $\ell_\infty$ we saw that a subspace of a Banach space need not be complemented. Using Zippin's theorem, we will now study the complementation in a Banach space of the span of block basic sequences of unconditional bases.

**Lemma 9.4.1.** *Let $(e_n)_{n=1}^{\infty}$ be an unconditional basis of a Banach space $X$. Suppose that $(u_k)_{k=1}^{\infty}$ is a normalized block basic sequence of $(e_n)_{n=1}^{\infty}$ such that the subspace $[u_k]$ is complemented in $X$. Then there is a projection $Q$ from $X$ onto $[u_k]$ of the form*

$$Q(x) = \sum_{k=1}^{\infty} u_k^*(x) u_k,$$

*where* $\operatorname{supp} u_k^* \subseteq \operatorname{supp} u_k$ *for all* $k \in \mathbb{N}$.

*Proof.* Suppose

$$u_k = \sum_{j \in A_k} a_j e_j,$$

where $A_k = \operatorname{supp}\ u_k$, and that $P$ is a bounded projection onto $[u_k]$. For each $k$ let $Q_k$ be the projection onto $[e_j]_{j \in A_k}$ given by

$$Q_k x = \sum_{j \in A_k} e_j^*(x) e_j.$$

We will show that the formula

$$Qx = \sum_{k=1}^{\infty} Q_k P Q_k x, \qquad x \in X,$$

defines a bounded projection onto $[u_k]$ (and it is clearly of the prescribed form).

Suppose $x = \sum_{j=1}^{m} e_j^*(x) e_j$ for some $m$. Then for a suitable $N$ such that $\operatorname{supp}\ x \subset A_1 \cup \cdots \cup A_N$, we have

$$\begin{aligned} Qx &= \sum_{k=1}^{N} Q_k P Q_k x \\ &= \operatorname*{Average}_{\epsilon_k = \pm 1} \sum_{j=1}^{N} \sum_{k=1}^{N} \epsilon_j \epsilon_k Q_j P Q_k x \\ &= \operatorname*{Average}_{\epsilon_k = \pm 1} \left( \sum_{j=1}^{N} \epsilon_j Q_j \right) P \left( \sum_{k=1}^{N} \epsilon_k Q_k \right) x. \end{aligned}$$

By the unconditionality of the original basis,

$$\|Qx\| \leq \mathsf{K_u}^2 \|P\| \|x\|.$$

It is now easy to check that $Q$ extends to a bounded operator and has the required properties. □

The following characterization of the canonical bases of the $\ell_p$-spaces and $c_0$ is due to Lindenstrauss and Tzafriri [200].

**Theorem 9.4.2.** *Let $(e_n)_{n=1}^{\infty}$ be an unconditional basis of a Banach space $X$. Suppose that for every block basic sequence $(u_n)_{n=1}^{\infty}$ of a permutation of $(e_n)_{n=1}^{\infty}$, the subspace $[u_n]$ is complemented in $X$. Then $(e_n)_{n=1}^{\infty}$ is equivalent to the canonical basis of $c_0$ or $\ell_p$ for some $1 \leq p < \infty$.*

*Proof.* Without loss of generality we may assume that the constant of unconditionality of the basis $(e_n)_{n=1}^\infty$ is 1. Our first goal is to show that whenever we have that

$$u_n = \sum_{k\in A_n} \alpha_k e_k, \quad v_n = \sum_{k\in B_n} \beta_k e_k, \quad n \in \mathbb{N},$$

are two normalized block basic sequences of $(e_n)_{n=1}^\infty$ such that $A_n \cap B_m = \emptyset$ for all $n, m$, then $(u_n)_{n=1}^\infty \sim (v_n)_{n=1}^\infty$.

First we will prove that if $(a_n)_{n=1}^\infty$ is a sequence of scalars for which $\sum_{n=1}^\infty a_n u_n$ converges, then the series $\sum_{n=1}^\infty s_n a_n v_n$ converges for every sequence of scalars $(s_n)_{n=1}^\infty$ tending to 0. For each $n \in \mathbb{N}$ consider

$$w_n = u_n + s_n v_n, \quad n \in \mathbb{N}.$$

Then $(w_n)_{n=1}^\infty$ is a seminormalized block basic sequence with respect to a permutation of $(e_n)_{n=1}^\infty$. To be precise, supp $w_n = A_n \cup B_n$ for each $n$ and $1 \le \|w_n\| \le 2$ (for $n$ big enough that $|s_n| \le 1$). By the hypothesis, the subspace $[w_n]$ is complemented in $X$. Lemma 9.4.1 yields a projection $Q : X \to X$ of the form

$$Q(x) = \sum_{n=1}^\infty w_n^*(x) w_n,$$

where the elements of the sequence $(w_n^*)_{n=1}^\infty \subset X^*$ satisfy supp $w_n^* \subseteq A_n \cup B_n$. Moreover, it is easy to see that $\|w_n^*\| \le \|Q\|$ for all $n$.

The series

$$\sum_{n=1}^\infty a_n Q(u_n) = \sum_{n=1}^\infty a_n w_n^*(u_n) w_n = \sum_{n=1}^\infty a_n w_n^*(u_n)(u_n + s_n v_n)$$

converges because $\sum_{n=1}^\infty a_n u_n$ does. Therefore, by unconditionality, it follows that $\sum_{n=1}^\infty a_n w_n^*(u_n) s_n v_n$ converges as well. From here we deduce the convergence of the series $\sum_{n=1}^\infty a_n s_n v_n$ by noticing that $w_n^*(u_n) \to 1$, since

$$w_n^*(u_n) = 1 - s_n w_n^*(v_n)$$

and

$$0 \le |s_n w_n^*(v_n)| \le |s_n| \left\|w_n^*\right\| \le \|Q\| \, |s_n| \to 0.$$

Now, if $(a_n)_{n=1}^\infty$ is a sequence of scalars for which $\sum_{n=1}^\infty a_n u_n$ converges, we can find a sequence of scalars $(t_n)_{n=1}^\infty$ tending to $\infty$ such that $\sum_{n=1}^\infty t_n a_n u_n$ converges. Since $(1/t_n)_{n=1}^\infty$ tends to 0, the previous argument applies, so $\sum_{n=1}^\infty a_n v_n$ converges.

Reversing the roles of $(u_n)$ and $(v_n)$, we get the equivalence of these two block basic sequences.

This argument applies not only to block basic sequences of $(e_n)_{n=1}^\infty$ but to block basic sequences of a permutation of $(e_n)_{n=1}^\infty$. Thus $(u_n)_{n=1}^\infty$ is equivalent to every permutation of $(v_n)_{n=1}^\infty$. This implies that $(e_{2n})_{n=1}^\infty$ and $(e_{2n-1})_{n=1}^\infty$ are both perfectly homogeneous and equivalent to each other. We conclude the proof by applying Zippin's theorem (Theorem 9.1.8). □

*Remark 9.4.3.* In the above theorem, it is necessary to allow complementation of the span of block basic sequences with respect to a permutation of $(e_n)_{n=1}^\infty$. One may show that the canonical basis of $\ell_p(\ell_r^n)$ where $r \neq p$ has the property that every block basic sequence spans a complemented subspace, but obviously it is not equivalent to the canonical basis of $\ell_p$ or $c_0$ (see the problems).

In [200], Lindenstrauss and Tzafriri solved the *complemented subspace problem* discussed in Chapter 2. We cannot quite prove this yet in full generality, since it requires more machinery, but in this section we will see the proof in the case of spaces with unconditional basis.

**Theorem 9.4.4.** *Let X be a Banach space with unconditional basis. If every closed subspace of X is complemented in X, then X is isomorphic to* $\ell_2$.

*Proof.* Let $(x_n)_{n=1}^\infty$ be an unconditional basis of such an $X$. By Theorem 9.4.2, $(x_n)_{n=1}^\infty$ is equivalent either to the canonical basis of $c_0$ or to the canonical basis of $\ell_p$ for some $1 \le p < \infty$.

Suppose that $(x_n)_{n=1}^\infty$ is equivalent to the canonical basis of $\ell_p$ for some $1 < p < \infty$, $p \neq 2$. We know that in this case, $\ell_p$ is isomorphic to $\ell_p(\ell_2^n)$ and that the canonical basis of $\ell_p(\ell_2^n)$ is not equivalent to the standard basis of $\ell_p$. Therefore $X$ contains an unconditional basis $(u_n)_{n=1}^\infty$ equivalent to the canonical basis of $\ell_p(\ell_2^n)$. Repeating the argument at the beginning of the proof with $(u_n)_{n=1}^\infty$ would lead to a contradiction.

Thus the possibilities for $X$ are reduced to three spaces: $X$ is either $c_0$, $\ell_1$, or $\ell_2$. To complete the proof we need only show that $c_0$ and $\ell_1$ have uncomplemented subspaces. In fact, in the case of $\ell_1$ we have already seen examples (Corollary 2.3.3).

Let us consider first the case of $c_0$. For each $n$, $\ell_1^n$ embeds isometrically in $\ell_\infty^{2^n}$. This follows from the fact that the norm of each element $(a_i)_{i=1}^n$ in $\ell_1^n$ can be written, using duality, as

$$\|(a_i)_{i=1}^n\| = \max \Big| \sum_{k=1}^n \varepsilon_k a_k \Big|,$$

the maximum being taken over the $2^n$ possible choices for the sequence of signs $(\varepsilon_k)_{k=1}^n$. Thus the embedding of $\ell_1^n$ into $\ell_\infty^{2^n}$ is given by the map

$$(a_i)_{i=1}^n \mapsto \Big( \sum_{i=1}^n \varepsilon_i a_i \Big)_{(\varepsilon_i)_{i=1}^n \in \{-1,1\}^n} \in \ell_\infty^{2^n}.$$

Hence, $c_0(\ell_1^n)$ embeds in $c_0(\ell_\infty^{2^n})$, which is isometrically isomorphic to $c_0$. As before, the subspace $c_0(\ell_1^n)$ cannot be complemented in $c_0$ because the canonical basis of $c_0(\ell_1)$ is not equivalent to the standard $c_0$-basis. □

*Remark 9.4.5.* In this proof we could have also shown that $\ell_1$ has an uncomplemented subspace using an argument similar to that for $c_0$: For each $n$, the space $L_1([0,1],\Sigma_n)$ is isometric to $\ell_1^{2^n}$, and by Khintchine's inequality, it contains an isomorphic copy of $\ell_2^n$ (namely, the space spanned by $\{r_1, r_2, \dots, r_n\}$) with isomorphism constants uniform on $n$. Then $\ell_1(\ell_2^n)$ embeds in $\ell_1(\ell_1^{2^n})$, which is isometrically isomorphic to $\ell_1$. If the subspace $\ell_1(\ell_2^n)$ were complemented in $\ell_1$, then it would be isomorphic to $\ell_1$ and so, as a consequence, $\ell_1$ would have an unconditional basis equivalent to the canonical basis of $\ell_1(\ell_2^n)$, which is not true.

## 9.5 The Existence of Conditional Bases

In this section we prove an earlier result of Pełczyński and Singer from 1964 [249] to the effect that every Banach space with a basis has a basis that is not unconditional. The original argument was more involved and does not use Zippin's theorem (Theorem 9.1.8), which it predates.

**Definition 9.5.1.** A normalized basis $(x_n)_{n=1}^\infty$ of a Banach space $X$ is called *conditional* if it is not unconditional.

In Chapter 3 we saw that $c_0$ has at least one conditional basis, the summing basis. On the other hand, the vectors $e_1, e_1 - e_2, e_2 - e_3, e_3 - e_4, \dots$, form a conditional basis of $\ell_1$, where as usual, $(e_n)_{n=1}^\infty$ denotes the standard $\ell_1$-basis. As for $\ell_2$, the existence of a conditional basis requires a bit of elaboration. This was originally proved by Babenko [17] using harmonic analysis methods. Our proof is based on a later argument by McCarthy and Schwartz [218]. However, the McCarthy–Schwartz argument is in a certain sense a very close relative of the Babenko approach.

**Theorem 9.5.2.** *$\ell_2$ has a conditional basis.*

*Proof.* Let $(e_n)_{n=1}^\infty$ be the canonical orthonormal basis of $\ell_2$. We pick a sequence of nonnegative real numbers $(a_n)_{n=1}^\infty$ such that

$$\sum_{n=1}^\infty a_n = \infty, \quad \sum_{n=1}^\infty n a_n^2 < \infty.$$

One may suppose that $a_n \sim 1/(n \log n)$ for $n$ large to get such a sequence.

We now define a sequence $(f_n)_{n=1}^\infty$ by

$$f_{2n-1} = e_{2n-1}$$

and

$$f_{2n} = e_{2n} + \sum_{j=1}^n a_j e_{2n+1-2j}.$$

We will investigate conditions under which $(f_n)_{n=1}^{\infty}$ is firstly a basis and secondly an unconditional basis.

Let us define an infinite matrix $B = (b_{ij})$ by

$$b_{ij} = \begin{cases} a_k, & j - i = 2k - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$B = \begin{pmatrix} 0 & a_1 & 0 & a_2 & 0 & a_3 & 0 \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots \\ 0 & 0 & 0 & a_1 & 0 & a_2 & 0 \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \dots \end{pmatrix}.$$

Now $B$ as a matrix acts on $c_{00}$ (when we regard each entry as an infinite column vector). Furthermore, $f_j = (I + B)e_j$.

Notice that $B^2$ can be computed (since every column has at most finitely many nonzero entries), and in fact, $B^2 = 0$. Consider the partial sum operators with respect to the basis $P_n$, say. In matrix terms we have

$$P_n = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$$

as a partitioned matrix. We also have $BP_nB = 0$.

The matrix $I + B$ is invertible (as a linear endomorphism of $c_{00}$) with inverse $I - B$. It follows that $(f_j)_{j=1}^{\infty}$ is always a Hamel basis of the countable-dimensional space $c_{00}$. The partial sum operators with respect to this Hamel basis are given by $(I + B)P_n(I - B) = I + BP_n - P_nB$. For $(f_n)_{n=1}^{\infty}$ to be a basis of $\ell_2$ simply requires that the operators $BP_n - P_nB$ extend to a uniformly bounded sequence of operators on $\ell_2$. Now $BP_n - P_nB$ is just the restriction of the matrix $B$ to the set of $(i, j)$ such that $i \le n < j$ (i.e., to the top right-hand corner). We claim that this operator is actually the restriction of a Hilbert–Schmidt operator, since

$$\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} |b_{ij}|^2 \le \sum_{k=1}^{\infty} k a_k^2.$$

It follows that we have a uniform bound

$$\|BP_n - P_nB\| \le (\sum_{k=1}^{\infty} k a_k^2)^{1/2}.$$

The uniform bound establishes that $(f_n)_{n=1}^{\infty}$ is a basis of $\ell_2$.

Assume that $(f_n)_{n=1}^{\infty}$ is unconditional. Then, since $1 \leq \|f_n\| \leq M$ for some $M$, $(f_n)_{n=1}^{\infty}$ must be equivalent to the canonical $\ell_2$-basis, and the operator $I + B$ must define a bounded operator on $\ell_2$; thus so does $B$. On the other hand, summing over the top left-hand corner square, we obtain

$$\Big\langle B(\sum_{j=1}^{2n} e_j), \sum_{j=1}^{2n} e_j\Big\rangle = \sum_{i=1}^{2n}\sum_{j=1}^{2n} b_{ij} = \sum_{k=1}^{n}(n-k+1)a_k.$$

Thus, if $B$ defines a bounded operator, then

$$\sum_{k=1}^{n}(n-k+1)a_k \leq 2n\|B\|,$$

i.e.,

$$\sum_{k=1}^{n}(1-\frac{k-1}{n})a_k \leq 2\|B\|.$$

Letting $n \to \infty$, we would conclude that $\sum_{k=1}^{\infty} a_k < \infty$, which would contradict our initial choice. □

Babenko's argument is based on considering weighted $L_2$-spaces. We consider complex Hilbert spaces. Let $w$ be a density function on $\mathbb{T}$ and consider the space $L_2(w(\theta)d\theta)$. Then it may be shown that the sequence $\{1, e^{i\theta}, e^{-i\theta}, e^{2i\theta}, \dots\}$ is a basis of $L_2(w\,d\theta)$ if and only if the Riesz projection $f \mapsto \sum_{n\geq 0}\hat{f}(n)e^{in\theta}$ (or the Hilbert transform) acts boundedly on $L_2(w\,d\theta)$. This happens if and only if $w$ is an $A_2$-weight (e.g., see [118]). On the other hand, unconditionality implies

$$\|f\|_{L_2(w\,d\theta)} \approx \left(\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2\right)^{1/2} \approx \|f\|_{L_2(d\theta)},$$

so that $w, w^{-1} \in L_\infty$. So, to give an example one needs an $A_2$-weight $w$ with $w$ or $w^{-1}$ unbounded. Babenko used the weight $|\theta|^\alpha$, where $0 < \alpha < 1$. However, the argument given in Theorem 9.5.2 can also be rephrased as a proof of the existence of unbounded $A_2$-weights.

We are headed to show the result of Pełczyński and Singer [249] that every Banach space with a basis has a conditional basis. To this end, first we need a few lemmas. Our next lemma gives us a criterion for the construction of a new basis of a Banach space with a given basis.

**Lemma 9.5.3.** *Suppose that $(e_n)_{n=1}^{\infty}$ is a basis of a Banach space $X$ and that $(r_n)_{n=0}^{\infty}$ is an increasing sequence of integers with $r_0 = 0$. For each $n$ let $E_n$ be the closed subspace spanned by the basis elements $\{e_{r_{n-1}+1}, \dots, e_{r_n}\}$. Further assume that $(f_n)_{n=1}^{\infty}$ is a sequence in $X$ such that:*

*(i)* $(f_{r_{n-1}+1},\dots,f_{r_n})$ *is a basis of* $E_n$ *for all* $n$*;*
*(ii)* $\sup_n K_n = M < \infty$, *where* $K_n$ *is the basis constant of* $(f_{r_{n-1}+1},\dots,f_{r_n})$.

*Then* $(f_n)_{n=1}^{\infty}$ *is a basis of* $X$.

*Proof.* Let $\mathsf{K_b}$ be the basis constant of $(e_n)_{n=1}^{\infty}$ and let $(S_N)_{N=1}^{\infty}$ be the sequence of natural projections associated with this basis. Since $[f_n] = [e_n] = X$, it suffices to show that there is a constant $C > 0$ such that given $m$ and $p$ in $\mathbb{N}$ with $m \le p$, the inequality

$$\Big\|\sum_{k=1}^{m}\alpha_k f_k\Big\| \le C\Big\|\sum_{k=1}^{p}\alpha_k f_k\Big\|$$

holds for any scalars $(\alpha_k)_{k=1}^{p}$.

Given any two integers $m, p$ with $m \le p$, there are integers $n, q$ such that $r_{n-1} < m \le r_n$ and $r_{q-1} < p \le r_q$. We have two possibilities: either $n < q$ or $n = q$. Assume first that $n < q$. Then,

$$\begin{aligned}\Big\|\sum_{k=1}^{m}\alpha_k f_k\Big\| &\le \Big\|\sum_{k=1}^{r_{n-1}}\alpha_k f_k\Big\| + \Big\|\sum_{k=r_{n-1}+1}^{m}\alpha_k f_k\Big\| \\ &\le \Big\|S_{r_{n-1}}\Big(\sum_{k=1}^{p}\alpha_k f_k\Big)\Big\| + M\Big\|\sum_{k=r_{n-1}+1}^{r_n}\alpha_k f_k\Big\| \\ &\le \mathsf{K_b}\Big\|\sum_{k=1}^{p}\alpha_k f_k\Big\| + M\Big\|S_{r_n}\Big(\sum_{k=1}^{p}\alpha_k f_k\Big) - S_{r_{n-1}}\Big(\sum_{k=1}^{p}\alpha_k f_k\Big)\Big\| \\ &\le (\mathsf{K_b} + 2\mathsf{K_b}M)\Big\|\sum_{k=1}^{p}\alpha_k f_k\Big\|.\end{aligned}$$

If $n = q$, analogously we have

$$\begin{aligned}\Big\|\sum_{k=1}^{m}\alpha_k f_k\Big\| &\le \Big\|\sum_{k=1}^{r_{n-1}}\alpha_k f_k\Big\| + \Big\|\sum_{k=r_{n-1}+1}^{m}\alpha_k f_k\Big\| \\ &\le \Big\|S_{r_{n-1}}\Big(\sum_{k=1}^{p}\alpha_k f_k\Big)\Big\| + M\Big\|\sum_{k=r_{n-1}+1}^{p}\alpha_k f_k\Big\| \\ &\le \mathsf{K_b}\Big\|\sum_{k=1}^{p}\alpha_k f_k\Big\| + M\Big\|S_{r_n}\Big(\sum_{k=1}^{p}\alpha_k f_k\Big) - S_{r_{n-1}}\Big(\sum_{k=1}^{p}\alpha_k f_k\Big)\Big\| \\ &\le (\mathsf{K_b} + 2\mathsf{K_b}M)\Big\|\sum_{k=1}^{p}\alpha_k f_k\Big\|.\end{aligned}$$

□

The following two lemmas are due to Zippin [309].

**Lemma 9.5.4.** *Let $E$, $F$ be two closed subspaces of codimension* 1 *of a Banach space $X$. Then there exists an isomorphism $T : E \to F$ such that $\|T\|\|T^{-1}\| \le 25$.*

*Proof.* Unless $E = F$, $E \cap F$ is a subspace of $X$ of codimension 2. Let us pick $x_0 \in E \setminus (E \cap F)$ such that $1 = \|x_0\| d(x_0, E \cap F) \le 2$. Analogously, pick $x_1 \in F$ such that $1 = \|x_1\| d(x_1, E \cap F) \le 2$.

Each element of $E$ can be written in a unique way in the form $\lambda x_0 + y$ for some scalar $\lambda$ and some $y \in E \cap F$. Analogously, the elements of $F$ admit a unique representation in the fashion $\lambda x_1 + y$, where $\lambda \in \mathbb{R}$ and $y \in E \cap F$. Define $T : E \to F$ as $T(\lambda x_0 + y) = \lambda x_1 + y$. On the one hand, we have

$$\|\lambda x_1 + y\| \le |\lambda|\, \|x_1\| + \|y\| \le 2|\lambda| + \|y\| \le 2 \max\big\{|\lambda|, \|y\|\big\}. \tag{9.12}$$

On the other,

$$\|\lambda x_0 + y\| = |\lambda| \Big\| x_0 + \frac{y}{|\lambda|} \Big\| = |\lambda| \Big\| x_0 - \big(-\frac{y}{|\lambda|}\big) \Big\| \ge |\lambda| d(x_0, E \cap F) = |\lambda|$$

and

$$\|y + \lambda x_0\| \ge \|y\| - 2|\lambda|.$$

Hence,

$$\|y + \lambda x_0\| \ge \max\big\{|\lambda|, \|y\| - 2|\lambda|\big\} \ge \max\left\{|\lambda|, \frac{1}{3}\|y\|\right\}. \tag{9.13}$$

Combining (9.12) and (9.13), we obtain

$$\|T(\lambda x_0 + y\| \le 5\, \|\lambda x_0 + y\|,$$

so $\|T\| \le 5$. We would follow exactly the same steps to find a bound for $\|T^{-1}\|$, which would yield $\|T\|\|T^{-1}\| \le 25$. □

**Lemma 9.5.5.** *Suppose that $(e_n)_{n=1}^{\infty}$ is a basis of a Banach space $X$ and that $(u_n)_{n=1}^{\infty}$ is a block basic sequence of $(e_n)_{n=1}^{\infty}$. Then there exists a basis $(f_n)_{n=1}^{\infty}$ of $X$ such that $(u_n)_{n=1}^{\infty}$ is a subbasis of $(f_n)_{n=1}^{\infty}$.*

*Proof.* For each $n \in \mathbb{N}$ suppose that $u_n$ is normalized and supported on the basis elements $\{e_{r_{n-1}+1}, \dots, e_{r_n}\}$, where $(r_n)_{n=1}^{\infty}$ is an increasing sequence of positive integers with $r_1 = 1$. Let $E_n = [e_{r_{n-1}+1}, \dots, e_{r_n}]$. By the Hahn–Banach theorem there exists $u_n^*$ in the dual space of the finite-dimensional normed space $E_n$ such that $u_n^*(u_n) = \|u_n\| = 1$. Let $F_n = \ker u_n^*$. Then $F_n$ is a subspace of $E_n$ of codimension 1. By Lemma 9.5.4 there is an isomorphism

$$T_n : [e_{r_{n-1}+1}, \dots, e_{r_n - 1}] \longrightarrow F_n$$

with $\|T_n\|\|T_n\|^{-1} \leq 25$. Pick $f_i = T_n(e_i)$ for $i = r_{n-1}+1, \dots, r_n - 1$. Then $\{f_{r_{n-1}+1}, \dots, f_{r_n-1}\}$ is a basis of $F_n$ with basis constant bounded by $25\mathsf{K}_\mathsf{b}$, $\mathsf{K}_\mathsf{b}$ being the basis constant of $(e_n)_{n=1}^\infty$. Thus, if we take $f_{r_n} = u_n$ for each $n$, by Lemma 9.5.3 the sequence $(f_n)_{n=1}^\infty$ is a basis of $X$ that satisfies the lemma. □

**Theorem 9.5.6 (Pełczyński–Singer).** *Let $X$ be any Banach space with a basis. Then $X$ has a conditional basis.*

*Proof.* Assume that every basis of $X$ is unconditional and let $(e_n)_{n=1}^\infty$ be one of them. Suppose $(u_k)_{k=1}^\infty$ is a block basic sequence of $(e_n)_{n=1}^\infty$. Then, using Lemma 9.5.5, $X$ has a basis $(f_n)_{n=1}^\infty$ of which $(u_k)_{k=1}^\infty$ is a subsequence. Moreover, $(f_n)_{n=1}^\infty$ is unconditional by our assumption; hence $[u_k]$ is a complemented subspace in $X$. This argument will also apply to every permutation of $(e_n)_{n=1}^\infty$. Hence every block basic sequence of every permutation of $(e_n)_{n=1}^\infty$ spans a complemented subspace. By Theorem 9.4.2, $(e_n)_{n=1}^\infty$ must be equivalent to the canonical basis of $c_0$ or $\ell_p$ for some $1 \leq p < \infty$. This is a contradiction, because on the one hand, $\ell_p$ has an unconditional basis that is not equivalent to the canonical basis of the space if $1 < p < \infty$, $p \neq 2$, as we saw in Proposition 8.3.7, and on the other hand, $c_0$, $\ell_1$, and $\ell_2$ have conditional bases. □

## Problems

**9.1.** Suppose $(x_n)_{n=1}^\infty$ is a basis for a Banach space $X$. Suppose there is a constant $C \geq 1$ such that whenever $p_0 = 0 < p_1 < \cdots$ and $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ are two normalized block basic sequences of $(x_n)_{n=1}^\infty$ of the form

$$u_n = \sum_{i=p_{n-1}+1}^{p_n} a_i x_i,$$

$$v_n = \sum_{i=p_{n-1}+1}^{p_n} b_i x_i,$$

then $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ are $C$-equivalent. Show that the closed linear span of a block basic sequence of $(x_n)_{n=1}^\infty$ is always complemented.

**9.2.** Show that every block basic sequence of $\ell_p(\ell_r^n)$, where $1 \leq r \neq p < \infty$, spans a complemented subspace.

**9.3.** Show that $\ell_p$ for $1 \leq p < \infty$ has a unique (up to equivalence) symmetric basis.

**9.4 (Lorentz Sequence Spaces).** For every $1 \leq p < \infty$ and every nonincreasing sequence of positive numbers $w = (w_n)_{n=1}^\infty$ we consider the *Lorentz sequence space* $d(w, p)$ of all sequences of scalars $x = (a_n)_{n=1}^\infty$ for which

$$\|x\| = \sup \left( \sum_{n=1}^{\infty} |a_{\pi(n)}|^p w_n \right)^{1/p} < \infty, \tag{9.14}$$

where $\pi$ ranges over all permutations of $\mathbb{N}$. One easily checks that $d(w,p)$ equipped with the norm defined by (9.14) is a Banach space.

(a) Show that if $\inf_n w_n > 0$, then $d(w,p) \approx \ell_p$.
(b) Show that if $\sum_{n=1}^{\infty} w_n < \infty$, then $d(w,p) \approx \ell_\infty$.

Therefore, to avoid trivial cases we shall assume that $w_1 = 1$, $\lim_{n\to\infty} w_n = 0$, and $\sum_{n=1}^{\infty} w_n = \infty$.

(c) Show that no nontrivial Lorentz sequence space is isomorphic to an $\ell_p$-space.
(d) Show that the unit vectors $(e_n)_{n=1}^{\infty}$ form a normalized symmetric basis for $d(w,p)$.

The reader interested in knowing more about Lorentz sequence spaces will find these properties and other, deeper, ones in [203].

**9.5 (Lindenstrauss and Tzafriri [201]).** Let $F$ be an Orlicz function satisfying the additional condition that for some $q < \infty$ the function $F(x)/x^q$ is decreasing.

(a) Let $E_F$ be the subset of $\mathcal{C}[0,1]$ defined as the closure of the set of all functions of the form $F_t(x) = F(tx)/F(t)$ for $0 < t \le 1$. Show that $E_F$ is compact.
(b) Let $C_F$ be the closed convex hull of $E_F$. Show that every normalized block basic sequence has a subsequence equivalent to the canonical basis of $\ell_G$ for some $G \in C_F$. Conversely, show that for every $G \in C_F$ there is a normalized block basic sequence equivalent to the canonical $\ell_G$-basis.
(c) Show that every symmetric basic sequence in $\ell_F$ is equivalent to the canonical basis of some $\ell_G$, where $G \in C_F$.
(d) Show that if $G \in E_F$, then $\ell_G$ is isomorphic to a complemented subspace of $\ell_F$.

**9.6 (Lindenstrauss and Tzafriri [201]).** (Continuation of 9.5) For $0 < s < 1$ define $T_s(F) \in \mathcal{C}[0,1]$ by $T_sF(x) = F(sx)/F(s)$.

(a) Show that $T_s : C_F \to C_F$ is continuous.
(b) Show that there is a common fixed point for $\{T_s : 0 < s < 1\}$ and hence that $x^p \in C_F$ for some $1 \le p < \infty$. (This uses the Schauder fixed point theorem, Theorem F.4.) Deduce that every $\ell_F$ has a closed subspace isomorphic to some $\ell_p$.

For a more precise result see [202].

**9.7 (Zippin [309] (Compare with Problem 3.9)).**

(a) Let $X$ be a Banach space with a basis that is not boundedly complete. Show that $X$ has a normalized basis $(x_n)_{n=1}^{\infty}$ such that for some subsequence $(x_{p_n})_{n=1}^{\infty}$ we have $\sup_n \|\sum_{j=1}^{n} x_{p_j}\| < \infty$. Deduce that $X$ has a basis that is not shrinking.
(b) Show that $X$ is reflexive whenever (i) every basis is shrinking, or (ii) every basis is boundedly complete.

**9.8 ([151]).** Let $X$ be a Banach space with a basis and suppose $X$ has the following property: whenever $(x_n)_{n=1}^{\infty}$ is a basis of $X$ and $(\sum_{j=1}^{n} a_j x_j)_{n=1}^{\infty}$ is a weakly Cauchy sequence, then $\sum_{j=1}^{\infty} a_j x_j$ converges.

(a) Show that every weakly Cauchy block basic sequence of a basis $(x_n)_{n=1}^{\infty}$ is weakly null. [*Hint*: Use Zippin's lemma (Lemma 9.5.5).]

(b) Show that if $(y_n)_{n=1}^{\infty}$ is a weakly Cauchy sequence, then there exist a subsequence $(y_{n_k})_{k=1}^{\infty}$ and a sequence $(z_k)_{k=1}^{\infty}$ of the form

$$z_k = \sum_{j=1}^{p_k} a_j x_j + \sum_{j=p_k+1}^{p_{k+1}-1} b_j x_j$$

such that $\lim_{k\to\infty} \|y_{n_k} - z_k\| = 0$.

(c) Show that $X$ is weakly sequentially complete.

**9.9.** Show that every unconditional basis of $L_p$ $(1 < p < \infty)$ has a subsequence equivalent to the canonical basis of $\ell_p$. Deduce that:

(a) If $p \neq 2$, then $L_p$ has no symmetric basis.

(b) If $(f_n)_{n=1}^{\infty}$ is a greedy basis of $L_p$, then there exist $0 < c < C < \infty$ such that

$$cn^{1/p} \le \Big\| \sum_{k=1}^{n} f_n \Big\|_p \le Cn^{1/p}.$$

**9.10 (Edelstein and Wojtaszczyk [83]).** Let $(x_n)_{n=1}^{\infty}$ be a normalized unconditional basis of $\ell_1 \oplus \ell_2$. Show that one can partition $\mathbb{N}$ into two infinite sets $\mathbb{A}$ and $\mathbb{B}$ such that $(x_n)_{n\in\mathbb{A}}$ is equivalent to the canonical basis of $\ell_1$ and $(x_n)_{n\in\mathbb{B}}$ is equivalent to the canonical basis of $\ell_2$. [*Hint:* Suppose $x_n = (y_n, z_n)$ with $y_n \in \ell_1$ and $z_n \in \ell_2$. Let $x_n^* = (y_n^*, z_n^*) \in \ell_\infty \oplus \ell_2$. Let $\mathbb{A} = \{n : \ y_n^*(y_n) \ge \frac{1}{2}\}$.]

# Chapter 10
# Greedy-Type Bases

Suppose that $X$ is a Banach space and that $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is a basis of $X$. An *m-term approximation* with respect to $\mathcal{B}$ is a map $T_m : X \to X$ such that for each $x \in X$, $T_m(x)$ is a linear combination of at most $m$ elements of $\mathcal{B}$. An *approximation algorithm* is a sequence $(T_m)_{m=1}^{\infty}$ of such maps.

The most natural approximation algorithm is the *linear algorithm* $(S_m)_{m=1}^{\infty}$ given by the partial sum projections $S_m(x) = \sum_{j=1}^{m} e_j^*(x)e_j$. For each $m$, $S_m$ provides a near-best $m$-term approximation for every $x \in X$ from the linear subspace $[e_1, \dots, e_m]$. That is, if for each $x \in X$ we define its *best m-term linear approximation error* as

$$E_m(x) = \inf\left\{ \left\| x - \sum_{j=1}^{m} \alpha_j e_j \right\| : (\alpha_j)_{j=1}^{m} \text{ scalars} \right\},$$

we have

$$\|x - S_m(x)\| \le C\,E_m(x), \quad m = 1, 2, \dots,$$

for some constant $C \ge 1$ independent of $x \in X$ and $m \in \mathbb{N}$. Thus the linear approximation theory with respect to bases turns out to be simple and convenient if one is happy with this level of accuracy.

Motivated by the problem of finding more efficient algorithms (i.e., that improve the error in the approximation), researchers brought nonlinearity into play by allowing the elements used in the approximation to depend on the vector $x$ being approximated instead of picking them from a fixed linear space. The corresponding approximation algorithms constructed in this way define maps that need not be linear or continuous. Konyagin and Temlyakov introduced in [176] the *greedy algorithm* $(G_m)_{m=1}^{\infty}$, where $G_m(x)$ is obtained by taking the first $m$ terms in decreasing order of magnitude from the series expansion of $x$ with respect to $\mathcal{B}$; when two terms are of equal size, we take them in the basis order (precise definitions are given below).



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This chapter deals with nonlinear approximation. Roughly speaking, our aim is to investigate how well $G_m(x)$ approximates $x \in X$ for various bases $\mathcal{B}$. The minimal requirement we might have to make this a reasonable method of approximation is that $G_m(x)$ converge to $x$ for every $x \in X$. This induces the notion of *quasi-greedy basis*. At the opposite extreme, the maximal requirement is that $G_m(x)$ essentially provide the best possible approximation to $x$ by a sum of $m$ elements of the basis, which leads to the notion of *greedy basis*. The formal definitions of quasi-greedy basis and greedy basis were given by Konyagin and Temlyakov in [176]; studying these types of bases will be the subject of Sections 10.2 and 10.4, respectively. In between these two extreme cases we find the intermediate concept of *almost greedy* basis, whose theory was developed by Dilworth, Kalton, Kutzarova, and Temlyakov in [68] and which we cover in Section 10.5. If $(e_n)_{n=1}^{\infty}$ is a basis of a Banach space $X$, then $(e_n^*)_{n=1}^{\infty}$ is a basic sequence in $X^*$, and if $X$ is reflexive, then $(e_n^*)_{n=1}^{\infty}$ is a basis for $X^*$. In Section 10.6 we discuss how the above three properties for bases dualize. The chapter closes with Section 10.7, where we enunciate without proof some important theoretical results on existence and uniqueness of greedy-type bases in certain Banach spaces that are outside the scope of this book.

## 10.1 General Framework

Let $X$ be a Banach space. Throughout this chapter $\mathcal{B} = (e_n)_{n=1}^{\infty}$ will be a semi-normalized (Schauder) basis in $X$ (i.e., $1/\mathsf{c} \le \|e_n\| \le \mathsf{c}$ for all $n$, for some $\mathsf{c}$) with biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$.

For each $m = 1, 2, \dots,$ we let $\Sigma_m[\mathcal{B}, X]$ denote the collection of all $x$ in $X$ that can be expressed as a linear combination of $m$ elements of $\mathcal{B}$,

$$\Sigma_m[\mathcal{B}, X] = \left\{ \sum_{n\in A} a_n e_n \colon A \subset \mathbb{N},\ |A| = m,\ a_n \in \mathbb{R} \right\}.$$

When the basis $\mathcal{B}$, the space $X$, or both are clear from the context we will suppress them from the above notation and use just the symbol $\Sigma_m$. Clearly, $\Sigma_m \subset \Sigma_k$ whenever $m \le k$. Let us notice that the space $\Sigma_m$ is not linear: the sum of two elements from $\Sigma_m$ is generally not in $\Sigma_m$, it is in $\Sigma_{2m}$.

The fundamental question here is how to construct for each $x \in X$ and each $m = 1, 2, \dots$ an element $y_m \in \Sigma_m$ such that the error of the approximation of $x$ by $y_m$, given by the quantity $\|x - y_m\|$, is small.

The answer to this question in some particular cases is simple. For instance, if $X = H$ is a Hilbert space and $\mathcal{B}$ is an orthonormal basis, every element $x \in H$ has an expansion in the form

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n,$$

and by Parseval's identity,

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2.$$

One easily realizes that a best approximation to $x$ from $\Sigma_m$ exists, and it is obtained as follows. We order the Fourier coefficients $(\langle x, e_n\rangle)_{n=1}^{\infty}$ of $x$ according to the absolute value of their size, and we choose $A_m(x)$ as a set of indices $n$ of cardinality $m$ for which $|\langle x, e_n\rangle|$ is largest. Then

$$y_m = \sum_{n\in A_m(x)} \langle x, e_n\rangle e_n$$

is a best approximation to $x$ from $\Sigma_m$, and

$$\|x - y_m\|^2 = \sum_{n\notin A_m(x)} |\langle x, e_n\rangle|^2.$$

This is an example of what is known as a *greedy algorithm*.

There are two essentially equivalent ways to formalize this idea and simultaneously generalize it to seminormalized bases in an arbitrary Banach space $X$.

The first one is to consider for each $\epsilon > 0$ the *thresholding operator*,

$$T_\epsilon[\mathcal{B}, X](x) := T_\epsilon(x) = \sum_{\{n:|e_n^*(x)|>\epsilon\}} e_n^*(x)e_n.$$

Since $\lim_n e_n^*(x) = 0$, we are summing only a finite number of terms, and so the maps $(T_\epsilon)_{\epsilon>0}$ are well defined.

In the second approach we fix $m \in \mathbb{N}$, and for $x \in X$ define a *greedy sum* of $x$ of order $m$ by

$$G_m[\mathcal{B}, X](x) := G_m(x) = \sum_{n\in A_m(x)} e_n^*(x)e_n,$$

where $A_m(x)$ is an $m$-element set of indices such that

$$\min\{|e_n^*(x)| : n \in A_m(x)\} \geq \max\{|e_n^*(x)| : n \notin A_m(x)\}.$$

Of course, the sets $A_m(x)$ may not be uniquely determined by the previous conditions; hence a given $x \in X$ can have more than one greedy sum of any order. However, notice that if $\min\{|e_n^*(x)| : n \in A_m(x)\} > \max\{|e_n^*(x)| : n \notin A_m(x)\}$, then $\sum_{n\in A_m(x)} e_n^*(x)e_n$ is the only greedy sum of $x$ of order $m$. In this case we will use the term *strictly greedy sum* of $x$ of order $m$.

Strictly greedy sums and thresholding operators are closely related. To every $x \in X$ there corresponds a subset of integers

$$\mathbb{N}_x = \{m \in \mathbb{N} : \text{ there is a strictly greedy sum of } x \text{ of order } m\}.$$

Then for $\epsilon > 0$ and $m \in \mathbb{N}_x$, we have

$$T_\epsilon(x) = G_{m_\epsilon}(x) \text{ and } G_m(x) = T_{\epsilon_m}(x), \tag{10.1}$$

where

$$m_\epsilon = |\{n \in \mathbb{N} : |e_n^*(x)| > \epsilon\}| \quad \text{and} \quad \max_{n \notin A_m(x)} |e_n^*(x)| \le \epsilon_m < \min_{n \in A_m(x)} |e_n^*(x)|.$$

The most natural way to construct a greedy sum of a vector $x$ is to start with an injective map $\pi : \mathbb{N} \to \mathbb{N}$ such that $(|e_{\pi(n)}^*(x)|)_{n=1}^\infty$ is nonincreasing and then consider the partial sums

$$G_m(x) = \sum_{n=1}^m e_{\pi(n)}^*(x) e_{\pi(n)}$$

of the formal series $\sum_{n=1}^\infty e_{\pi(n)}^*(x) e_{\pi(n)}$. Every sequence $(G_m(x))_{m=1}^\infty$ thus obtained will be called a *greedy approximation* of $x$, and $\pi$ will be said to be a *greedy ordering* for $x$. If the support of $x$ is infinite, then so is $\mathbb{N}_x$; hence it can be enumerated in the form $\mathbb{N}_x = (m_j)_{j=1}^\infty$ with the indices $m_j$ increasing. The sequence $(G_{m_j}(x))_{j=1}^\infty$ is a subsequence of every greedy approximation of $x$, and we will refer to it as the *strictly greedy approximation* of $x$.

The greedy approximations of a vector $x$ need not be unique. However, we can use the natural ordering existing in $\mathbb{N}$ to construct for each $x$ a uniquely determined greedy ordering as follows. Define $\rho : \mathbb{N} \to \mathbb{N}$ such that $\{n : e_n^*(x) \neq 0\} \subset \rho(\mathbb{N})$ and such that if $j < k$, then either $|e_{\rho(j)}^*(x)| > |e_{\rho(k)}^*(x)|$ or $|e_{\rho(j)}^*(x)| = |e_{\rho(k)}^*(x)|$ and $\rho(j) < \rho(k)$. With this convention, the $m$th greedy sum of $x$, given by

$$\mathcal{G}_m[\mathcal{B}, X](x) := \mathcal{G}_m(x) = \sum_{n=1}^m e_{\rho(n)}^*(x) e_{\rho(n)},$$

is now uniquely determined, and $(\mathcal{G}_m(x))_{m=1}^\infty$ forms a greedy approximation of $x$. The sequence of maps $(\mathcal{G}_m)_{m=1}^\infty$ is known as the *greedy algorithm* associated to $\mathcal{B}$ in $X$. On a few occasions our reasoning will require considering greedy sums of null order, so we agree to put $\mathcal{G}_0(x) = 0$ for all $x$.

Notice that $(\mathcal{G}_m)_{m=1}^\infty$ are neither linear nor continuous. Nevertheless, they are homogeneous, i.e., $\mathcal{G}_m(\lambda x) = \lambda \mathcal{G}_m(x)$ for $\lambda \in \mathbb{R}$, thanks to which we can define their norm,

$$\|\mathcal{G}_m\| = \sup_{\|x\| \le 1} \|\mathcal{G}_m(x)\|.$$

Note that if $T\colon X \to Y$ is an isomorphism between Banach spaces, then

$$T(\mathcal{G}_m[\mathcal{B}, X](x)) = \mathcal{G}_m[T(\mathcal{B}), Y](T(x));$$

thus all the concepts related to greediness that we will introduce later on will be invariant under isomorphism.

## 10.2 Quasi-Greedy Bases

Let $\mathcal{B} = (e_n)_{n=1}^\infty$ be a basis in a Banach space $X$. Given $x = \sum_{n=1}^\infty e_n^*(x)e_n$, the $m$th greedy sum $\mathcal{G}_m(x) = \sum_{n=1}^m e_{\rho(n)}^*(x)e_{\rho(n)}$ is a partial sum of the rearranged (formal) series

$$\sum_{n=1}^\infty e_{\rho(n)}^*(x)e_{\rho(n)}. \tag{10.2}$$

In order to understand how well the greedy sums $(\mathcal{G}_m(x))_{m=1}^\infty$ approximate $x$, the first natural question that comes to mind is, when does the series in (10.2) converge? From the definition of unconditional basis (see Definitions 2.4.1 and 3.1.1) it follows that $\sum_{n=1}^\infty e_{\pi(n)}^*(x)e_{\pi(n)}$ converges to $x$ for every permutation $\pi$ of $\mathbb{N}$; in particular, so does the rearrangement of (10.2). As it happens, unconditionality is not a necessary condition for the convergence of this specific series.

**Definition 10.2.1.** A basis $\mathcal{B}$ for a Banach space $X$ is said to be *quasi-greedy* if the sequence $(\mathcal{G}_m(x))_{m=1}^\infty$ converges to $x$ (in the norm-topology of $X$) for all $x \in X$.

Bases need not be quasi-greedy, as the next example shows.

*Example 10.2.2.* Consider the sequence space $X = \{(a_n)_{n=1}^\infty \colon \sum_{n=1}^\infty a_n \text{ converges}\}$ endowed with the norm $\|(a_n)_{n=1}^\infty\| = \max_N |\sum_{n=N}^\infty a_n|$. The unit vector basis $(e_n)_{n=1}^\infty$ of $X$ is isometrically equivalent to the summing basis in $c_0$ (see Example 3.1.2). Pick $x = (a_n)_{n=1}^\infty \in X$ given by

$$x = \left( \underbrace{1, -1}, \underbrace{\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}}, \dots, \underbrace{\frac{1}{k}, \frac{-1}{k^2}, \dots, \frac{-1}{k^2}}, \dots \right).$$

This vector consists of blocks of length $k+1$, and the sum of the coordinates in each block is zero. For $j \in \mathbb{N}$, pick $\epsilon \in (1/(j^2+1), 1/j^2)$. We have

$$\|T_\epsilon(x)\| \geq \sum_{\{n : |a_n| > \epsilon\}} a_n = \sum_{n=j+1}^{j^2} \frac{1}{n} \geq \int_{j+1}^{j^2+1} \frac{1}{x}\, dx = \ln \frac{j^2+1}{j+1},$$

which combined with (10.1) gives $\sup_n \|\mathcal{G}_n(x)\| = \infty$. Hence $(\mathcal{G}_n(x))_{n=1}^\infty$ does not converge.

The next result of Wojtaszczyk [305, Theorem 1] characterizes quasi-greedy bases in terms of the uniform boundedness of $(\mathcal{G}_m)_{m=1}^\infty$. Nothing to be surprised about, if it were not for the fact that these maps are neither linear nor continuous!

**Theorem 10.2.3.** *A basis $\mathcal{B} = (e_n)_{n=1}^\infty$ in a Banach space $X$ is quasi-greedy if and only if there is a constant $\mathsf{C} \geq 1$ such that $\|\mathcal{G}_m(x)\| \leq \mathsf{C}\|x\|$ for all $x \in X$ and $m \in \mathbb{N}$.*

The proof of Theorem 10.2.3 relies on Lemmas 10.2.5 and 10.2.6. We will also use several times a simple argument that we record in Lemma 10.2.4. Given a finite subset $A \subset \mathbb{N}$, we denote by $P_A : X \to X$ the (bounded and linear) projection onto the vector space $[e_n : n \in A]$,

$$P_A(x) = \sum_{n \in A} e_n^*(x) e_n,$$

and let $P_{A^c} = I_X - P_A$. In particular, we have $P_{\{1,\dots,m\}} = S_m$, the partial sum projection associated to $(e_n)_{n=1}^\infty$. For every greedy sum $G_m(x)$ there is a unique $A \subset \mathbb{N}$ with $|A| = m$ such that $G_m(x) = P_A(x)$. In the sequel we will use repeatedly the constants

$$\mathsf{k} := \sup_{|A|=1} \|P_A\| = \sup_n \|e_n\| \|e_n^*\| \quad \text{and } \mathsf{K} := \sup_{n \in \mathbb{N}} \|e_n^*\|.$$

**Lemma 10.2.4.** *Let $\mathcal{B}$ be a basis in a Banach space $X$. Given $x \in X$, suppose that $y = P_B(x)$ is a greedy sum (respectively, strictly greedy sum) of $x$. Then for every finite subset $A$ of $\mathbb{N}$, the element $P_{A^c}(y) = P_{B \setminus A}(x)$ is a greedy sum (respectively, strictly greedy sum) of $P_{A^c}(x) = x - P_A(x)$.*

**Lemma 10.2.5.** *Let $\mathcal{B} = (e_n)_{n=1}^\infty$ be a basis in a Banach space $X$. The following are equivalent:*

*(a) $G_m(x) \to x$ for every $x \in X$ and every greedy approximation $(G_m(x))_{m=1}^\infty$.*
*(b) $\mathcal{B}$ is quasi-greedy.*
*(c) For every $x \in X$ there is a greedy approximation $(G_m(x))_{m=1}^\infty$ such that $G_m(x) \to x$.*
*(d) For every $x \in X$ with infinite support its strictly greedy approximation converges to $x$.*
*(e) $\lim_{\epsilon \to 0} T_\epsilon(x) = x$ for every $x \in X$.*

*Proof.* The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ are obvious, and $(d) \Leftrightarrow (e)$ is an easy consequence of (10.1). To complete the chain of implications, let us prove (a) with the assumption of (d).

Fix $x \in X$, a greedy ordering $\pi$ of $x$, and considerer the greedy approximation $(G_m(x))_{m=1}^\infty$ of $x$ given by

$$G_m(x) = \sum_{j=1}^m e_{\pi(j)}^*(x) e_{\pi(j)}, \quad m = 1, 2, \dots.$$

Pick a sequence of real numbers $(\delta_n)_{n=1}^{\infty}$ in such a way that $\sum_{j=1}^{\infty} |\delta_j| < \infty$ and $(|e_{\pi(j)}^*(x) - \delta_j|)_{j=1}^{\infty}$ is strictly decreasing. Let $y = \sum_{j=1}^{\infty} \delta_j e_{\pi(j)}$ and consider the perturbation of $x$ defined by $z = x - y$. Notice that $e_{\pi(j)}^*(z) = e_{\pi(j)}^*(x) - \delta_j$ for all $j$ and that $e_n^*(z) = 0$ if $n \notin \pi(\mathbb{N})$. Hence $\mathbb{N}_z = \mathbb{N}$, $\pi$ is the only greedy ordering for $z$, and

$$G_m(z) = G_m(x) - \sum_{j=1}^{m} \delta_j e_{\pi(j)}$$

for all $m$. By hypothesis we know that $G_m(z) \to z$, and the absolute convergence of the series defining $y$ yields that $\sum_{j=1}^{m} \delta_j e_{\pi(j)} \to y$. Combining, we obtain that $G_m(x) \to x$. □

**Lemma 10.2.6.** *Let $\mathcal{B} = (e_n)_{n=1}^{\infty}$ be a basis in a Banach space $X$. The following are equivalent:*

*(a) There exists a constant $\mathsf{C}$ such that for all $x \in X$, all $m \in \mathbb{N}$, and all greedy sums $G_m(x)$,*

$$\|G_m(x)\| \leq \mathsf{C}\|x\|. \tag{10.3}$$

*(b) There exists a constant $\mathsf{C}$ such that $\|\mathcal{G}_m(x)\| \leq \mathsf{C}\|x\|$ for all $x \in X$ and all $m \in \mathbb{N}$.*
*(c) For every $m \in \mathbb{N}$ and every $x \in X$ there exists a greedy sum $G_m(x)$ such that $\|G_m(x)\| \leq \mathsf{C}\|x\|$, where $\mathsf{C}$ is an absolute constant.*
*(d) There exists a constant $\mathsf{C}$ such that $\|G_m(x)\| \leq \mathsf{C}\|x\|$, for all $x \in X$ and all strictly greedy sums $G_m(x)$.*
*(e) There exists a constant $\mathsf{C}$ such that $\|G_m(x)\| \leq \mathsf{C}\|x\|$ for all $x \in X$ of finite support and all strictly greedy sums $G_m(x)$.*

*Moreover, the least constant $\mathsf{C}$ in any of the above estimates is also the least constant in all the others.*

*Proof.* The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$ (maintaining the constant $\mathsf{C}$) are obvious. Let us show that $(e) \Rightarrow (a)$ with the same constant. To that end, let $x \in X$ and let $G_m(x) = P_A(x)$ with $|A| = m$ be a greedy sum of $x$. Fix $\epsilon > 0$. Since $(e_n)_{n=1}^{\infty}$ is a basis for $X$, there is a set of integers $B$ containing $A$ such that $y = P_B(x)$ satisfies $\|x - y\| < \epsilon$. We can easily construct a small perturbation $z$ of $y$ (in the sense that $\|z - y\| < \epsilon$) such that supp $(z) =$ supp $(y)$ and $G_m(x)$ is a strictly greedy sum of $z$. Then, by our hypothesis,

$$\|G_m(x)\| \leq \mathsf{C}\|z\| \leq \mathsf{C}(\|x\| + 2\epsilon).$$

Making $\epsilon \to 0$ yields the desired conclusion. □

The spirit of the equivalence between (a), (b), (c), and (d) in Lemma 10.2.6 is the same as in Lemma 10.2.5, namely in every statement involving greedy sums we can replace the sentence "for every greedy sum $G_m(x)$" with either of the three following sentences: either "for the greedy sum $\mathcal{G}_m(x)$," or "for some greedy sum $G_m(x)$," or "for every strictly greedy sum $G_m(x)$, when it exists." The equivalence between (e) and the other four statements allows us to connect Lemma 10.2.6 with the next lemma, which is crucial in the proof of Theorem 10.2.3.

**Lemma 10.2.7.** *Suppose that (e) in Lemma 10.2.6 does not hold. Then for every positive constant* $\mathsf{C}$ *and for every finite set* $A \subset \mathbb{N}$*, there exists* $x \in X$ *with* $|\mathrm{supp}(x)| < \infty$ *and* $\mathrm{supp}(x) \cap A = \emptyset$ *such that for some strictly greedy sum* $G_m(x)$ *of* $x$ *we have* $\|G_m(x)\| > \mathsf{C}\|x\|$.

*Proof.* Fix a constant $\mathsf{C} > 0$. Given any finite $A \subset \mathbb{N}$, put $M = \max_{E \subset A} \|P_E\|$. By our assumption there exists a finitely supported $y$ in $X$ and a strictly greedy sum of $y$, $G_r(y) = P_F(y)$, such that $\|G_r(y)\| > (\mathsf{C}(1+M)+M)\,\|y\|$. By Lemma 10.2.4, $G_m(x) = P_{A^c}(G_r(y))$ is a strictly greedy sum of $x = P_{A^c}(y)$ of order $m \le r$. Notice that $G_m(x) = G_r(y) - P_{A\cap F}(y)$ and that $x = y - P_A(y)$ has finite support disjoint from $A$. Moreover, $\|x\| \le (1+M)\|y\|$ and

$$\|G_m(x)\| \ge \|G_r(y)\| - \|P_{A\cap F}(y)\| > (\mathsf{C}(1+M)+M)\|y\| - M\|y\|,$$

all of which together gives $\|G_m(x)\| > \mathsf{C}\|x\|$. □

*Proof of Theorem 10.2.3.* To show the forward implication, assume that the claim fails. Then, statement (e) in Lemma 10.2.6 also fails, whence Lemma 10.2.7 kicks in. We will use it to construct recursively a pair of sequences $(x_k)_{k=1}^{\infty}$, $(y_k)_{k=1}^{\infty}$ of elements of $X$ fulfilling the following properties for each $k$:

(1) supp $(x_k)$ is finite and disjoint from supp $(x_i)$ ( $i = 1, \dots, k-1$);
(2) $y_k$ is a strictly greedy sum of $x_k$;
(3) $\|x_k\| \le 2^{-k}$;
(4) $\|y_k\| > 1$; and
(5) $\max\{|e_n^*(x_k)| : n \in \mathbb{N}\} < \min\{|e_n^*(x_{k-1})| : n \in \mathrm{supp}\,(x_{k-1})\}$.

Suppose we have manufactured pairs of vectors $\{x_1, y_1\}, \dots, \{x_{k-1}, y_{k-1}\}$ (note that the construction also works for the initial step $k = 1$). Let $\mu = \min\{|e_n^*(x_{k-1})| : n \in \mathrm{supp}\,(x_{k-1})\}$ and put $\mathsf{C}_k = \max\{2^k, 2\mathsf{K}\mu^{-1}\}$, where $\mathsf{K} = \sup_n \|e_n^*\|$. Using Lemma 10.2.7, there exist $x_k$ of support disjoint from $\cup_{i=1}^{k-1}\mathrm{supp}\,(x_i)$ and a strictly greedy sum $y_k$ of $x_k$ such that $\|y_k\| > \mathsf{C}_k\|x_k\|$. By homogeneity we can choose $x_k$ having norm $\|x_k\| = \mathsf{C}_k^{-1} \le 2^{-k}$, so that $\|y_k\| > 1$. Whence, for every $n \in \mathbb{N}$ we have

$$|e_n^*(x_k)\| \le \|e_n^*\|\|x_k\| \le \mathsf{K}\frac{\mu}{2\mathsf{K}} < \mu,$$

and so (5) holds.

Now, the series $\sum_{k=1}^{\infty} x_k$ converges to some $x$ in $X$. Note that both $\sum_{k=1}^{j-1} x_k := G_{m_j}(x)$ and $y_j + \sum_{k=1}^{j-1} x_k := G_{r_j}(x)$ are strictly greedy sums of $x$. Since $\|G_{r_j}(x) -$

$G_{m_j}(x)\| = \|y_j\| > 1$, and $m_j < r_j < m_{j+1}$ we infer that the greedy algorithm of $x$ does not converge to $x$.

For the converse, we will see that (10.3) implies the convergence of the greedy algorithm. Fix $x \in X$. For $\epsilon > 0$ there is $A \subset \mathbb{N}$ such that $y = P_A(x)$ satisfies $\|x - y\| < \epsilon/(1 + \mathsf{C})$. There is no loss of generality in assuming that $e_j^*(x) \neq 0$ for all $j \in A$. Let $\delta = \min\{|e_j^*(x)| : j \in A\} > 0$ and let $B = \{j \in \mathbb{N} : |e_j^*(x)| \geq \delta\}$. Clearly, $A \subset B$.

Let $G_m(x) = P_E(x)$ be a greedy sum of $x$ order $m \geq |B|$. Note that $A \subset B \subset E$. By Lemma 10.2.4, $P_{A^c}(G_m(x)) = G_m(x) - y$ is a greedy sum of $P_{A^c}(x) = x - y$. Therefore,

$$\|x - G_m(x)\| \leq \|x - y\| + \|G_m(x) - y\| \leq \|x - y\| + \mathsf{C}\|x - y\| = (1 + \mathsf{C})\|x - y\| < \epsilon.$$

□

**Definition 10.2.8.** The least constant in (10.3) is called the *quasi-greedy constant* of $\mathcal{B}$ and will be denoted by $\mathsf{C}_{\mathsf{qg}}[\mathcal{B}, X]$ or, simply $\mathsf{C}_{\mathsf{qg}}$.

Unconditional bases are a special kind of quasi-greedy bases. However, there exist quasi-greedy bases that are not unconditional, as the next example shows.

*Example 10.2.9.* A conditional quasi-greedy basis.

In $c_{00}$ we define the norm of a sequence $(a_n)_{n=1}^\infty$ by the formula

$$\|(a_n)_{n=1}^\infty\| = \max\left\{\left(\sum_{n=1}^\infty |a_n|^2\right)^{1/2}, \sup_N \left|\sum_{n=1}^N \frac{a_n}{\sqrt{n}}\right|\right\}.$$

Let $X$ be the completion of $c_{00}$ in $c_0$ under this norm and let $\mathcal{B} = (e_n)_{n=1}^\infty$ be the unit vector basis in $X$.

On the one hand, if we let $H_m = \sum_{n=1}^m 1/n$, we have

$$\left\|\sum_{n=1}^m \frac{1}{\sqrt{n}} e_n\right\| = \max\left\{\left(\sum_{n=1}^m \frac{1}{n}\right)^{1/2}, \sum_{n=1}^m \frac{1}{n}\right\} = H_m.$$

On the other hand,

$$\begin{aligned}\left\|\sum_{n=1}^m \frac{(-1)^n}{\sqrt{n}} e_n\right\| &= \max\left\{\left(\sum_{n=1}^m \frac{1}{n}\right)^{1/2}, \sup_{N \leq m} \left|\sum_{n=1}^N \frac{(-1)^n}{n}\right|\right\} \\ &= \left(\sum_{n=1}^m \frac{1}{n}\right)^{1/2} \\ &= \sqrt{H_m}.\end{aligned}$$

Since $\lim_m H_m = \infty$, we conclude that $\mathcal{B}$ is a (normalized) conditional basis in $X$.

The see that $\mathcal{B}$ is quasi-greedy, by Lemma 10.2.6(e) and the correspondence between thresholding sums and strictly greedy sums made explicit in (10.1), it suffices to show that the thresholding operators $T_\epsilon$ are uniformly bounded. To that end, it is in turn easy to realize that we need only obtain a constant $C$ such that

$$\left|\sum_{n\in A}\frac{a_n}{\sqrt{n}}\right| \le C, \tag{10.4}$$

for every $(a_n)_{n=1}^\infty$ of finite support with $\|(a_n)_{n=1}^\infty\| \le 1$, every $\epsilon > 0$, and every $N \in \mathbb{N}$, where $A = \{n \le N : |a_n| > \epsilon\}$. Note also that since for each $n$, $|a_n| \le (\sum_{n=1}^\infty |a_n|^2)^{1/2} \le \|(a_n)\| \le 1$, we can assume $0 < \epsilon < 1$.

Set an index $L = \lfloor \epsilon^{-2} \rfloor$ such that $1/2 \le \epsilon^2 L \le 1$. Then if $N \le L$,

$$\begin{aligned}
\left|\sum_{n\in A}\frac{a_n}{\sqrt{n}}\right| &\le \left|\sum_{n\le N}\frac{a_n}{\sqrt{n}}\right| + \left|\sum_{\substack{n\le N\\ |a_n|<\epsilon}}\frac{a_n}{\sqrt{n}}\right| \\
&\le 1 + \epsilon\sum_{n=1}^{N}\frac{1}{\sqrt{n}} \\
&\le 1 + 2\epsilon\sqrt{N} \\
&\le 3.
\end{aligned}$$

If $L < N$, by the triangle inequality we have

$$\left|\sum_{n\in A}\frac{a_n}{\sqrt{n}}\right| \le \left|\sum_{\substack{n\le L\\ |a_n|>\epsilon}}\frac{a_n}{\sqrt{n}}\right| + \left|\sum_{\substack{L<n\le N\\ |a_n|>\epsilon}}\frac{a_n}{\sqrt{n}}\right|.$$

The first summand is bounded above by 3. As for the second, by Hölder's inequality,

$$\begin{aligned}
\left|\sum_{\substack{L<n\le N\\ |a_n|>\epsilon}}\frac{a_n}{\sqrt{n}}\right| &\le \sum_{\substack{L<n\le N\\ |a_n|>\epsilon}}\frac{|a_n|}{\sqrt{n}} \\
&\le \left(\sum_{L<n\le N} n^{-3/2}\right)^{1/3}\left(\sum_{\substack{L<n\le N\\ |a_n|>\epsilon}} |a_n|^{3/2}\right)^{2/3}
\end{aligned}$$

$$\le 2^{1/3} L^{-1/6} \left( \sum_{\substack{L<n\le N \\ |a_n|>\epsilon}} |a_n|^{3/2} \sqrt{\frac{|a_n|}{\epsilon}} \right)^{2/3}$$

$$\le 2^{1/3} L^{-1/6} \epsilon^{-1/3}$$

$$= 2^{1/3} (L\epsilon^2)^{-1/6}$$

$$\le 2^{1/3} 2^{1/6} = 2^{1/2},$$

and (10.4) holds with $C = 3 + \sqrt{2}$.

Despite the fact that quasi-greedy bases need not be unconditional, they preserve some vestiges of unconditionality, as we see next.

**Proposition 10.2.10.** *Suppose that $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is a quasi-greedy basis. Then:*

*(a) Whenever A and B are finite subsets of integers with $B \subset A$,*

$$\left\| \sum_{n\in B} e_n \right\| \le \mathsf{C}_{\mathsf{qg}} \left\| \sum_{n\in A} e_n \right\|. \tag{10.5}$$

*(b) $\mathcal{B}$ is unconditional for constant coefficients, i.e., for every finite subset A of $\mathbb{N}$ and every choice of signs $\varepsilon_n = \pm 1$, we have*

$$(2\mathsf{C}_{\mathsf{qg}})^{-1} \left\| \sum_{n\in A} e_n \right\| \le \left\| \sum_{n\in A} \varepsilon_n e_n \right\| \le 2\mathsf{C}_{\mathsf{qg}} \left\| \sum_{n\in A} e_n \right\|. \tag{10.6}$$

*Proof.* Let $B \subset A \subset \mathbb{N}$, with $A$ finite. Note that both $g = \sum_{n\in B} e_n$ and $h = \sum_{n\in A\setminus B} e_n$ are greedy sums of $x = \sum_{n\in A} e_n$. In the same way, both $g$ and $-h$ are greedy sums of $y = \sum_{n\in B} e_n - \sum_{n\in A\setminus B} e_n$, so we just need to apply Lemma 10.2.6 (a) to get

$$\|g\| \le \mathsf{C}_{\mathsf{qg}}\|x\|, \quad \|h\| \le \mathsf{C}_{\mathsf{qg}}\|x\|, \quad \|g\| \le \mathsf{C}_{\mathsf{qg}}\|y\|, \quad \text{and} \quad \|-h\| \le \mathsf{C}_{\mathsf{qg}}\|y\|.$$

The first inequality gives (a). Combining and applying the triangle inequality, we obtain

$$\|y\| = \|g-h\| \le 2\mathsf{C}_{\mathsf{qg}}\|x\| \quad \text{and} \quad \|x\| = \|g+h\| \le 2\mathsf{C}_{\mathsf{qg}}\|y\|.$$

Taking into account that $\sum_{n\in A} \varepsilon_n e_n = \sum_{n\in B} e_n - \sum_{n\in A\setminus B} e_n$ for given signs $\varepsilon_n = \pm 1$ and for some $B$ yields (b). □

**Corollary 10.2.11.** *Suppose $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is quasi-greedy. For every $A \subset \mathbb{N}$ finite and real numbers $(a_n)_{n\in A}$,*

$$\left\| \sum_{n\in A} a_n e_n \right\| \le 2\mathsf{C}_{\mathsf{qg}} \max_{n\in A} |a_n| \left\| \sum_{n\in A} e_n \right\|. \tag{10.7}$$

*Proof.* The result follows from (10.6) by convexity. □

**Theorem 10.2.12.** *Let $\mathcal{B} = (e_n)_{n=1}^{\infty}$ be a quasi-greedy basis in a Banach space X.*

*(a) For every $x \in X$, every greedy ordering $\pi$ of x, and every $m \in \mathbb{N}$,*

$$|e_{\pi(m)}^*(x)| \left\| \sum_{j=1}^{m} e_{\pi(j)} \right\| \leq 4\mathsf{C}_{\mathsf{qg}}^2 \, \|x\| \,. \tag{10.8}$$

*(b) For every $A \subset \mathbb{N}$ finite and any real numbers $(a_n)_{n\in A}$,*

$$\min_{n\in A} |a_n| \left\| \sum_{n\in A} e_n \right\| \leq 4\mathsf{C}_{\mathsf{qg}}^2 \left\| \sum_{n\in A} a_n e_n \right\| . \tag{10.9}$$

*Proof.* (a) For $1 \leq j \leq m$, let $G_j(x) = \sum_{k=1}^{j} e_{\pi(k)}^*(x) e_{\pi(k)}$ be a greedy sum of order $j$ of $x$. Put $b_j = 1/|e_{\pi(j)}^*(x)|$ and pick signs $(\epsilon_j)_{j=1}^{m}$ such that $b_j(G_j(x) - G_{j-1}(x)) = \epsilon_j e_{\pi(j)}$. Let $b_0 = b_{m+1} = 0$. Then, by (10.6), Abel's summation formula, and (10.3),

$$\begin{aligned}
\left\| \sum_{j=1}^{m} e_{\pi(j)} \right\| &\leq 2\mathsf{C}_{\mathsf{qg}} \left\| \sum_{j=1}^{m} \epsilon_j e_{\pi(j)} \right\| \\
&= 2\mathsf{C}_{\mathsf{qg}} \left\| \sum_{j=1}^{m} b_j (G_j(x) - G_{j-1}(x)) \right\| \\
&= 2\mathsf{C}_{\mathsf{qg}} \left\| \sum_{j=0}^{m} (b_j - b_{j+1}) G_j(x) \right\| \\
&\leq 2\mathsf{C}_{\mathsf{qg}} \sum_{j=0}^{m} |b_j - b_{j+1}| \|G_j(x)\| \\
&\leq 2\mathsf{C}_{\mathsf{qg}}^2 \sum_{j=0}^{m} |b_j - b_{j+1}| \|x\| = 4\mathsf{C}_{\mathsf{qg}}^2 b_m \|x\|.
\end{aligned}$$

(b) is a consequence of (a). □

**Corollary 10.2.13.** *Suppose $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is a quasi-greedy basis of X with quasi-greedy constant $\mathsf{C}_{\mathsf{qg}}$. For every $x \in X$ and every finite subset $A \subset \operatorname{supp}(x)$ we have*

$$\|P_A(x)\| \leq 16\mathsf{C}_{\mathsf{qg}}^4 \frac{\max\{|e_n^*(x)| : n \in A\}}{\min\{|e_n^*(x)| : n \in A\}} \|x\|. \tag{10.10}$$

*Proof.* Let $\mu = \min\{|e_n^*(x)| : n \in A\}$ and $\nu = \max\{|e_n^*(x)| : n \in A\}$ and consider $B = \{n \in \mathbb{N} : \mu \le |e_n^*(x)| \le \nu\}$. Notice that for $\epsilon > 0$ small enough, $\sum_{n\in B} e_n^*(x)e_n = T_{\mu-\epsilon}(x) - T_\nu(x)$. Combining (10.7), (10.9), and (10.3) yields

$$\|P_A(x)\| \le 2\mathsf{C}_{\mathrm{qg}}\nu \left\| \sum_{n\in B} e_n \right\| \le 8\mathsf{C}_{\mathrm{qg}}^3 \frac{\nu}{\mu} \left\| T_{\mu-\epsilon}(x) - T_\nu(x) \right\| \le 16\mathsf{C}_{\mathrm{qg}}^4 \frac{\nu}{\mu} \|x\|.$$

□

An immediate application of this corollary is that if a basis $(e_n)_{n=1}^\infty$ is quasi-greedy, then there is a constant $C$ such that

$$\Big\| \sum_{j=1}^n \epsilon_j a_j e_j \Big\| \le C \Big\| \sum_{j=1}^n a_j e_j \Big\|$$

whenever $\epsilon_j = \pm 1$, *provided* we make the requirement that all the nonzero coefficients $(a_j)$ be of approximately the same size, e.g., $1 \le |a_j| \le 2$. This is just one more qualitative indicator of the unconditionality traces found in quasi-greedy bases, to the extent that as has been recently proved (see [4]), quasi-greedy bases with $C_{\mathrm{qg}} = 1$ are unconditional with suppression-unconditional constant $\mathsf{K}_{\mathrm{su}} = 1$!

The conditionality of a basis $\mathcal{B}$ can be quantified in terms of the growth of the sequence

$$\mathsf{k}_m[\mathcal{B}, X] := \mathsf{k}_m = \sup_{|A|\le m} \|P_A\|, \qquad m = 1, 2, \ldots. \tag{10.11}$$

In fact, $\mathcal{B}$ is unconditional if and only if $\mathsf{k}_m = \mathcal{O}(1)$. For every basis $(e_n)_{n=1}^\infty$ in a Banach space one always has the estimate $\mathsf{k}_m \le \mathsf{k}\, m$, where $\mathsf{k} = \sup_{|A|=1} \|P_A\| = \sup_n \|e_n\| \|e_n^*\|$, and this is the best one can hope for in general. Indeed, the summing basis $(f_n)_{n=1}^\infty$ of $c_0$ satisfies $\mathsf{k}_m[\mathcal{B}, X] \ge m$ for each $m$. However, when the basis is quasi-greedy, the size of the members of the sequence $(\mathsf{k}_m)_{m=1}^\infty$ is controlled by a slowly growing function, as the following theorem shows [67].

**Theorem 10.2.14.** *If* $\mathcal{B} = (e_n)_{n=1}^\infty$ *is quasi-greedy, then*

$$\mathsf{k}_m = \mathcal{O}(\log_2(m)). \tag{10.12}$$

*Proof.* Consider an integer $m \ge 2$ and let $p = \lfloor \log_2(m) \rfloor$, be such that $2^p \le m < 2^{p+1}$. Let $x \in X$ with $\|x\| = 1$ so that $|e_n^*(x)| \le \mathsf{K}$ for all $n \in \mathbb{N}$.

Put $B_0 = \{n \in \mathbb{N} \colon |e_n^*(x)| \le \mathsf{K}2^{-p}\}$, and for $1 \le j \le p$ let

$$B_j = \{n \in \mathbb{N} \colon \mathsf{K}2^{-j} < |e_n^*(x)| \le \mathsf{K}2^{-j+1}\}.$$

The sets $(B_j)_{j=0}^p$ form a partition of $\mathbb{N}$. Let $A \subset \mathbb{N}$ with $|A| = m$. Using (10.10),

$$\|P_{A\cap B_j}(x)\| \le 32\mathsf{C}_{\mathrm{qg}}^4, \quad j = 1, \ldots, p.$$

By the triangle inequality,

$$\|P_{A\cap B_0}(x)\| \le |A\cap B_0|\mathsf{cK}2^{-p} \le m\mathsf{cK}\frac{2}{m} = 2\mathsf{cK}.$$

From all of this we obtain

$$\|P_A(x)\| \le 32\mathsf{C}_{\mathsf{qg}}^4 p + 2\mathsf{cK} \le 32\mathsf{C}_{\mathsf{qg}}^4 \log_2(m) + 2\mathsf{cK},$$

which yields the claim. □

*Remark 10.2.15.* It turns out that the estimate (10.12) is sharp for general Banach spaces [102], yet it can be improved in special cases such as Hilbert spaces and the spaces $L_p$ and $\ell_p$ when $1 < p < \infty$ ([101]; see [9] for further developments).

Given a quasi-greedy basis $\mathcal{B}$ in a Banach space $X$, the approximants $G_m[\mathcal{B}, X](x)$ change for each $x$ as we rescale the basis vectors. To have an idea of the robustness of the greedy algorithm associated to $\mathcal{B}$, it is useful to know how re-scaling affects its convergence. The following proposition from [305] grants us certain flexibility in making adjustments in a given basis $\mathcal{B}$ with an eye to greedy approximation.

**Proposition 10.2.16.** *Let $(e_n)_{n=1}^{\infty}$ be a quasi-greedy basis in a Banach space $X$. Suppose $(\lambda_n)_{n=1}^{\infty}$ is a sequence of real numbers such that $0 < \inf_n |\lambda_n| \le \sup_n |\lambda_n| < \infty$. Then the perturbed basis $\tilde{\mathcal{B}} = (\tilde{e}_n)_{n=1}^{\infty} = (\lambda_n e_n)_{n=1}^{\infty}$ is also quasi-greedy.*

*Proof.* Let $a = \sup_n |\lambda_n|$ and assume (by homogeneity) that $\inf_n |\lambda_n| = 1$. Let $G_m[\tilde{\mathcal{B}}, X](x) = P_A(x)$ be a greedy sum of $x$ with respect to the basis $\tilde{\mathcal{B}}$. Put $\mu = \min_{n\in A} |e_n^*(x)|/|\lambda_n|$, so that

$$\{n : |e_n^*(x)|/|\lambda_n| > \mu\} \subset A \subset \{n : |e_n^*(x)|/|\lambda_n| \ge \mu\}.$$

Define sets $E = \{n\colon |e_n^*(x)| > a\mu\}$ and $F = \{n\colon |e_n^*(x)| \ge \mu\}$. Clearly $E \subset A \subset F$, and $T_{a\mu}(x) = P_E(x)$ is a strictly greedy sum of $x$ with respect to the basis $\mathcal{B}$. By (10.3) and (10.10),

$$\|G_m[\tilde{\mathcal{B}}, X](x)\| \le \|T_{a\mu}(x)\| + \|P_{A\setminus E}(x)\| \le \left(\mathsf{C}_{\mathsf{qg}} + 16a\mathsf{C}_{\mathsf{qg}}^4\right)\|x\|,$$

and the proof is over by appealing to Theorem 10.2.3. □

Notice that if $(e_n)_{n=1}^{\infty}$ is an unconditional basis and $(\lambda_n)_{n=1}^{\infty}$ is as in Proposition 10.2.16, then $(\lambda_n e_n)_{n=1}^{\infty}$ is a basis equivalent to the original one; hence it is unconditional. Proposition 10.2.16 establishes an analogous result for quasi-greedy bases.

## 10.3 Democratic Bases

Let us now introduce a property enjoyed by some bases that is fundamental for our considerations [176].

**Definition 10.3.1.** A basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$ of a Banach space $X$ is said to be *democratic* if blocks of the same size of $\mathcal{B}$ have uniformly comparable norms, i.e., there is a constant $\mathsf{C} \geq 1$ such that for every two finite subsets $A$, $B$ of $\mathbb{N}$ with $|A| = |B|$,

$$\left\| \sum_{n \in A} e_n \right\| \leq \mathsf{C} \left\| \sum_{n \in B} e_n \right\| .$$

The least such $\mathsf{C}$ is called the *democracy constant* of $\mathcal{B}$ and will be denoted by $\mathsf{C}_{\mathsf{d}}$.

The lack of democracy of a basis $\mathcal{B}$ exhibits some sort of asymmetry. To measure how much a basis $\mathcal{B}$ deviates from being democratic, we consider its *upper democracy function*, also known as the *fundamental function of* $\mathcal{B}$,

$$\varphi_u[\mathcal{B}, X](m) := \varphi_u(m) = \sup_{|A|=m} \left\| \sum_{n \in A} e_n \right\| , \qquad m = 1, 2, \ldots,$$

and its *lower democracy function*,

$$\varphi_l[\mathcal{B}, X](m) := \varphi_l(m) = \inf_{|A|=m} \left\| \sum_{n \in A} e_n \right\| , \qquad m = 1, 2, \ldots .$$

A basis $\mathcal{B}$ is democratic if and only if the sequences $(\varphi_u[\mathcal{B}, X](m))_{m=1}^{\infty}$ and $(\varphi_l[\mathcal{B}, X](m))_{m=1}^{\infty}$ are uniformly comparable term by term, in which case

$$\mathsf{C}_{\mathsf{d}} = \sup_{m \in \mathbb{N}} \frac{\varphi_u(m)}{\varphi_l(m)} < \infty.$$

Of course, the democracy functions $\varphi_l[\mathcal{B}, X]$ and $\varphi_u[\mathcal{B}, X]$ may vary as we consider different bases $\mathcal{B}$ within the same Banach space $X$.

The attentive reader will have noticed that the notion of democratic basis was already hinted at in the previous chapter when we showed (inside the proof of Theorem 9.1.8) that if $\mathcal{B}$ is a perfectly homogeneous basis in a Banach space, its fundamental function determines completely the basis (and the space). The following lemmas gather elementary properties of the democracy functions of bases that will become handy later on. From now on, to avoid using unnecessary constants that may divert attention from the essential, given sequences of positive real numbers $(\alpha_N)_{N=1}^{\infty}$ and $(\beta_N)_{N=1}^{\infty}$, the notation $\alpha_N \lesssim \beta_N$ means that $\sup_N \alpha_N / \beta_N < \infty$. Likewise, we write $\alpha_N \approx \beta_N$ to mean $\alpha_N \lesssim \beta_N$ and $\beta_N \lesssim \alpha_N$.

**Lemma 10.3.2.** *Let $\mathcal{B}$ be a basis in a Banach space $X$ and let $m \in \mathbb{N}$. Then* $0 < \varphi_l(m) \le \varphi_u(m) < \infty$.

**Lemma 10.3.3.** *Let $r \in \mathbb{N}$. A basis is democratic if and only if*

$$\sup_{m \ge r} \frac{\varphi_u(m)}{\varphi_l(m)} < \infty.$$

**Lemma 10.3.4.** *Let $\mathcal{B}$ be a basis with basis constant $\mathsf{K}_\mathsf{b} = \sup_{m\in\mathbb{N}} \|S_m\|$. Then:*

*(a) The functions $\varphi_u$ and $\varphi_l$ are essentially nondecreasing, i.e., for $m \le r$,*

$$\varphi_u(m) \le \mathsf{K}_\mathsf{b}\varphi_u(r) \;\; \textit{and} \;\; \varphi_l(m) \le \mathsf{K}_\mathsf{b}\varphi_l(r).$$

*(b) The sequence $(\varphi_u(m)/m)_{m=1}^\infty$ is nonincreasing.*

*Proof.* Let us see (b) and leave the proof of (a) as an exercise. For every finite set $A$ with $|A| = m \ge 2$ let us write

$$\sum_{n\in A} e_n = \frac{1}{m-1} \sum_{k\in A} \sum_{n\in A\setminus\{k\}} e_n,$$

from which we obtain

$$\left\| \sum_{n\in A} e_j \right\| \le \frac{m}{m-1} \varphi_u(m-1).$$

Maximizing over $A$ completes the proof. □

**Lemma 10.3.5.** *Let $\mathcal{B}$ be a democratic basis in a Banach space $X$ and let $\tilde{\mathcal{B}}$ be a basis of a finite-dimensional Banach space $Y$. The direct sum of $\mathcal{B}$ and $\tilde{\mathcal{B}}$ is a democratic basis in $X \oplus Y$.*

The last results of this section combine the unconditionality-like properties of quasi-greedy bases with the democracy functions.

**Lemma 10.3.6.** *Let $\mathcal{B} = (e_n)_{n=1}^\infty$ and $\tilde{\mathcal{B}} = (\tilde{e}_n)_{n=1}^\infty$ be quasi-greedy bases for the Banach spaces $X$ and $Y$ respectively. Let $A$ and $B$ be finite sets of integers. Suppose that $x \in X$ and $y \in Y$ are such that* $\max\{|e_n^*(x)| : n \in A\} \le \min\{|\tilde{e}_n^*(y)| : n \in B\}$. *Then*

$$\|P_A(x)\| \le 16\mathsf{C}_{\mathsf{qg}}[\mathcal{B}, X]\mathsf{C}_{\mathsf{qg}}^3[\tilde{\mathcal{B}}, Y] \frac{\varphi_u[\mathcal{B}, X](|A|)}{\varphi_l[\tilde{\mathcal{B}}, Y](|B|)} \|P_B(y)\|. \tag{10.13}$$

*Proof.* The result follows readily from (10.7) and (10.9). □

*Remark 10.3.7.* Most of the time, Lemma 10.3.6 will be applied with $X = Y$, $\mathcal{B} = \tilde{\mathcal{B}}$, and $x = y$. In the case that $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are unconditional, with respective unconditional constants $\mathsf{K}_\mathsf{u}[\mathcal{B}, X]$ and $\mathsf{K}_\mathsf{u}[\tilde{\mathcal{B}}, Y]$, inequality (10.13) holds with the term $16\mathsf{C}_{\mathsf{qg}}[\mathcal{B}, X]\mathsf{C}_{\mathsf{qg}}^3[\tilde{B}, Y]$ replaced by $\mathsf{K}_\mathsf{u}[\mathcal{B}, X]\mathsf{K}_\mathsf{u}[\tilde{\mathcal{B}}, Y]$.

**Proposition 10.3.8.** *Suppose that $\mathcal{B}$ is a quasi-greedy basis in a Banach space $X$. If $(\varphi_u(m))_{m=1}^{\infty}$ is bounded then $\mathcal{B}$ is equivalent to the canonical $c_0$ basis.*

*Proof.* By Corollary 10.2.11, the proof is analogous to the proof of Lemma 9.1.4. □

## 10.4 Greedy Bases

In approximation theory it is convenient to know from a theoretical point of view whether the greedy algorithm is efficient, in the sense that the error we make for each $m$ in approximating any $x$ in $X$ by $\mathcal{G}_m(x)$ is uniformly comparable with the smallest theoretical error in the $m$-term approximation of $x$ with respect to the basis $\mathcal{B}$, given by

$$\sigma_m[\mathcal{B}, X](x) := \sigma_m(x) = \inf_{y \in \Sigma_m} \|x - y\|.$$

To formalize this idea, Konyagin and Temlyakov introduced the concept of greedy basis in [176].

**Definition 10.4.1.** A basis $\mathcal{B}$ of a Banach space $X$ is *greedy* if there is an absolute constant $\mathsf{C} \geq 1$ such that

$$\|x - \mathcal{G}_m(x)\| \leq \mathsf{C}\sigma_m(x), \quad \forall\, m \in \mathbb{N}, \ \forall x \in X. \tag{10.14}$$

The least $\mathsf{C}$ in (10.14) is called the *greedy constant* of $\mathcal{B}$, and is denoted by $\mathsf{C}_\mathsf{g}$.

Notice that since $0 \in \Sigma_m$ for all $m$, we have

$$\sigma_m(x) \leq \|x\|, \qquad \forall x \in X.$$

Also, since $\cup_{n=1}^{\infty} \Sigma_n$ is dense in $X$, then

$$\lim_{m \to \infty} \sigma_m(x) = 0.$$

Consequently, every greedy basis is quasi-greedy.

*Remark 10.4.2.* As the attentive reader will have guessed, the definition of greedy basis is equivalent to the fulfillment of the condition

$$\|x - G_m(x)\| \leq \mathsf{C}\sigma_m(x), \quad \forall\, m \in \mathbb{N}, \ \forall x \in X,$$

for some constant $\mathsf{C}$, for all greedy sums $G_m(x)$ (see Problem 10.2).

*Example 10.4.3.* If $\mathcal{B}$ is the unit vector basis in $\ell_p$ $(1 \leq p < \infty)$ or $c_0$, then for every $x$ in the space and every integer $m$ we have $\sigma_m(x) = \|x - G_m(x)\|$ for every

$m$-term greedy sum $G_m(x)$, which is the best we can hope for. In other words, $\mathcal{B}$ is a greedy basis and $C_g = 1$. This property can be extended to symmetric bases (see Problem 10.6).

*Example 10.4.4.* Examples of bases that fail to be greedy.

(a) Consider $X = \ell_p \oplus \ell_q$, $1 \le p < q < \infty$, and let $\mathcal{B} = (e_n)_{n=1}^{\infty}$ be the direct sum of the natural unit vector bases of the two spaces. That is, in our basis we have

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\| = \left( \sum_{k=0}^{\infty} |a_{2k+1}|^p \right)^{1/p} + \left( \sum_{k=1}^{\infty} |a_{2k}|^q \right)^{1/q}.$$

The basis $\mathcal{B}$ is unconditional and therefore quasi-greedy. Let us show that the approximations provided by $(\mathcal{G}_m)_{m=1}^{\infty}$ can be very far from optimal. Fix a small $\delta > 0$ and for each $m$ define a vector $z_m = \sum_{k=1}^{2m} (1 + (-1)^k \delta) e_k$. We have $\mathcal{G}_m(z_m) = \sum_{k=1}^{m} (1+\delta) e_{2k}$, whence $\|z_m - \mathcal{G}_m(z_m)\| = (1-\delta) m^{1/p}$. But

$$\sigma_m(z_m) \le \|\mathcal{G}_m(z_m)\| = (1+\delta) m^{1/q},$$

so that

$$m^{\frac{1}{p} - \frac{1}{q}} \frac{1-\delta}{1+\delta} \sigma_m(z_m) \le \|z_m - \mathcal{G}_m(z_m)\|.$$

The point here is to diagnose the reason why $\mathcal{B}$ fails to be greedy. Since for $N \in \mathbb{N}$,

$$\left\| \sum_{n=1}^{N} e_{2n} \right\| = N^{1/q},$$

while

$$\left\| \sum_{n=1}^{N} e_{2n+1} \right\| = N^{1/p},$$

we note that $\mathcal{B}$ fails to be democratic.

(b) Consider $\mathcal{B} = (f_n)_{n=1}^{\infty}$, the summing basis of $c_0$ (see Example 3.1.2). We infer from Example 10.2.2 and the fact that greedy bases are quasi-greedy that $\mathcal{B}$ is not greedy. It can be illustrative to see a direct proof that condition (10.14) is violated, which we do next.

Given $\sum_{n=1}^{\infty} a_n f_n \in c_0$, its norm is computed using the formula

$$\left\| \sum_{n=1}^{\infty} a_n f_n \right\| = \sup_{k \in \mathbb{N}} \left| \sum_{n=k}^{\infty} a_n \right|.$$

For each $N \in \mathbb{N}$, let

$$x = x(N) = \sum_{n=1}^{N} f_{2n-1} - \sum_{n=1}^{N} \left(1 - \frac{1}{N}\right) f_{2n}.$$

Since $\mathcal{G}_N(x) = \sum_{j=1}^{N} f_{2n-1}$, we have

$$\|x - \mathcal{G}_N(x)\| = \left\| - \sum_{n=1}^{N} \left(1 - \frac{1}{N}\right) f_{2n} \right\| = N - 1.$$

But $\sigma_N(x) \leq \|x\| = 1$, so that

$$\frac{\|x - \mathcal{G}_N(x)\|}{\sigma_N(x)} \geq N - 1 \to \infty.$$

The main result of this section is a very satisfactory intrinsic characterization by Konyagin and Temlyakov of greedy bases [176].

**Theorem 10.4.5.** *A basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$ of a Banach space $X$ is greedy if and only if it is unconditional and democratic.*

*Proof.* Assume that $\mathcal{B}$ is greedy with greedy constant $\mathsf{C_g}$. Fix $x \in X$ and let $A \subset \mathbb{N}$ be of cardinality $m$. Consider the vector

$$y = P_{A^c}(x) + \alpha \sum_{n \in A} e_n = x + \sum_{n \in A} (\alpha - e_n^*(x)) e_n,$$

where $\alpha > \sup_{n \notin A} |e_n^*(x)|$. Clearly, $\sigma_m(y) \leq \|x\|$ and $\mathcal{G}_m(y) = \alpha \sum_{n \in A} e_n$. Thus, the assumption of greediness yields

$$\|P_{A^c}(x)\| = \|y - \mathcal{G}_m(y)\| \leq \mathsf{C_g} \sigma_m(y) \leq \mathsf{C_g} \|x\|,$$

which implies that $\mathcal{B}$ is unconditional with suppression-unconditional constant bounded by $\mathsf{C_g}$ (see Problem 10.15).

To show that $\mathcal{B}$ is democratic, let us pick any two subsets of integers $A, B$ of the same cardinality. Consider $x = \sum_{n \in A \cup B} e_n$. Since $\sum_{n \in B \setminus A} e_n$ is a greedy sum of $x$ of order $m = |A \setminus B| = |B \setminus A|$, and $\sum_{n \in A \setminus B} e_n \in \Sigma_m$, we have

$$\left\| \sum_{n \in A} e_n \right\| = \left\| x - \sum_{n \in B \setminus A} e_n \right\| \leq \mathsf{C_g} \left\| x - \sum_{n \in A \setminus B} e_n \right\| = \mathsf{C_g} \left\| \sum_{n \in B} e_n \right\|,$$

so that $\mathcal{B}$ is democratic with democratic constant bounded by $\mathsf{C_g}$.

For the converse, assume that $\mathcal{B}$ is unconditional with unconditional constant $\mathsf{K_u}$ and democratic with democracy constant $\mathsf{C_d}$. Fix $x \in X$ and $m \in \mathbb{N}$. Pick $y = \sum_{n\in B} a_n e_n \in \Sigma_m$. Consider a greedy sum $G_m(x) = P_A(x)$. Note that

$$\|x - G_m(x)\| = \|P_{(AUB)^c}(x) + P_{B\setminus A}(x)\| \le \|P_{(AUB)^c}(x)\| + \|P_{B\setminus A}(x)\|.$$

By definition of greedy sum,

$$\max_{n\in B\setminus A} |e_n^*(x)| \le \min_{n\in A\setminus B} |e_n^*(x)|.$$

Let $r = |A \setminus B| = |B \setminus A|$. Using Remark 10.3.7, we have

$$\|P_{B\setminus A}(x)\| \le \mathsf{K}_\mathsf{u}^2 \frac{\varphi_u(r)}{\varphi_l(r)} \|P_{A\setminus B}(x)\| \le \mathsf{K}_\mathsf{u}^2 \mathsf{C_d} \|P_{A\setminus B}(x)\|.$$

The unconditionality of $\mathcal{B}$ implies

$$\|P_{(B\cup A)^c}(x)\| = \|P_{(B\cup A)^c}(x-y)\| \le \mathsf{K_{su}} \|x-y\|.$$

Analogously,

$$\|P_{A\setminus B}(x)\| = \|P_{A\setminus B}(x-y)\| \le \mathsf{K_{su}} \|x-y\|.$$

Combining, we obtain

$$\|x - G_m(x)\| \le (\mathsf{K_{su}} + \mathsf{K_{su}} \mathsf{K}_\mathsf{u}^2 \mathsf{C_d}) \|x-y\|,$$

and, with Remark 10.3.7 in mind, the proof is over by taking the infimum on $y$. □

**Corollary 10.4.6.** *Every subsymmetric basis in a Banach space is greedy.*

*Proof.* Suppose that $\mathcal{B} = (e_n)_{n=1}^\infty$ is subsymmetric and nongreedy. Since $\mathcal{B}$ is unconditional, it must fail to be democratic. We recursively construct sequences of mutually disjoint subsets of integers $(A_k)_{k=1}^\infty$ and $(B_k)_{k=1}^\infty$ such that $|A_k| = |B_k|$, $\max(A_{k-1} \cup B_{k-1}) < \min(A_k \cup B_k)$, and $\|\sum_{n\in A_k} e_n\| > k\|\sum_{n\in B_k} e_n\|$. Suppose we constructed $(A_j)_{j=1}^{k-1}$ and $(B_j)_{j=1}^{k-1}$. Set $N = 1 + \max(A_{k-1} \cup B_{k-1})$. Using Lemma 10.3.5, we infer that the basic sequence $(e_n)_{n=N}^\infty$ is not democratic. Therefore, there exist $A_k$, $B_k$ subsets of $\{n \in \mathbb{N} : n \ge N\}$ such that $|A_k| = |B_k|$ and $\|\sum_{n\in A_k} e_n\| > k\|\sum_{n\in B_k} e_n\|$.

Now consider strictly increasing sequences of integers $(n_j^a)_{j=1}^\infty$ and $(n_j^b)_{j=1}^\infty$ such that $\{n_j^a : j \in \mathbb{N}\} = \cup_{k=1}^\infty A_k$ and $\{n_j^b : j \in \mathbb{N}\} = \cup_{k=1}^\infty B_k$. Since the basic sequences $(e_{n_j^a})_{j=1}^\infty$ and $(e_{n_j^b})_{j=1}^\infty$ are not equivalent, we reach a contradiction to $\mathcal{B}$ being subsymmetric. □

The next result was obtained by Temlyakov [292].

**Proposition 10.4.7.** *The normalized Haar system $\mathcal{H}_p = (h_n^p)_{n=1}^\infty$ is a greedy basis in $L_p[0,1]$ for $1 < p < \infty$.*

*Proof.* Since $\mathcal{H}_p$ is unconditional (Theorem 6.1.7), by Theorem 10.4.5 we need only show that $\mathcal{H}_p$ is democratic. To this end, by Lemma 10.3.5 it will be enough to estimate $\| \sum_{n\in A} h_n^p\|$ for finite subsets of integers $A$ such that $1 \notin A$. We will use the following property of geometric series, of easy verification.

**Lemma 10.4.8.** *Let $1 < r < \infty$ and $0 < p < \infty$. There are positive constants $\mathsf{c}_{r,p}$ and $\mathsf{C}_{r,p}$ such that for every finite set of integers $A$,*

$$\mathsf{c}_{r,p}\left(\sum_{k\in A} r^{pk}\right)^{1/p} \le \left(\sum_{k\in A} r^{2k}\right)^{1/2} \le \mathsf{C}_{r,p}\left(\sum_{k\in A} r^{pk}\right)^{1/p}.$$

Note that the functions $(h_n^p)_{n=2}^\infty$ attain only the values 0 and $\pm 2^{k/p}$, and that for each $t \in [0,1]$ only one of those functions attains $2^{k/p}$ in absolute value. Lemma 10.4.8 with $r = 2^{1/p}$ gives constants $\mathsf{c}_p$ and $\mathsf{C}_p$ such that for every finite subset $A \subset \mathbb{N} \setminus \{1\}$ and for all $t \in [0,1]$,

$$\mathsf{c}_p\left(\sum_{n\in A} |h_n^p(t)|^p\right)^{1/p} \le \left(\sum_{n\in A} |h_n^p(t)|^2\right)^{1/2} \le \mathsf{C}_p\left(\sum_{n\in A} |h_n^p(t)|^p)\right)^{1/p}.$$

Taking $L_p$-norms in the above inequalities and using that $\|h_n^p\|_p = 1$ yields

$$\mathsf{c}_p|A|^{1/p} \le \left\|\left(\sum_{n\in A} |h_n^p|^2\right)^{1/2}\right\|_p \le \mathsf{C}_p|A|^{1/p}. \tag{10.15}$$

From Theorem 6.2.13 and the unconditionality $\mathcal{H}_p = (h_n^p)_{n=1}^\infty$, for each $1 < p < \infty$ there exist constants $A_p'$ and $B_p'$ such that for every $(a_n) \in c_{00}$,

$$A_p'\left\|\left(\sum_{n=1}^\infty |a_n|^2|h_n^p|^2\right)^{1/2}\right\|_p \le \left\|\sum_{n=1}^\infty a_n h_n^p\right\|_p \le B_p'\left\|\left(\sum_{n=1}^\infty |a_n|^2|h_n^p|^2\right)^{1/2}\right\|_p. \tag{10.16}$$

Thus combining with (10.15), one gets

$$A_p'\mathsf{c}_p|A|^{1/p} \le \left\|\sum_{n\in A} h_n^p\right\|_p \le B_p'\mathsf{C}_p|A|^{1/p},$$

for every finite $A \subset \mathbb{N}\setminus\{1\}$, which implies the democracy of $(h_n^p)_{n=2}^\infty$. □

The natural examples of greedy bases in the classical spaces that we have seen in this section can become useful for producing greedy bases when combined with simple theoretical methods. We will see in Section 10.6 that duality may serve this purpose. For now, we will take advantage of the fact that being greedy is an isomorphic property, i.e., if $(e_n)_{n=1}^{\infty}$ is a greedy basis in $X$ and $T : X \to Y$ is a linear isomorphism, then $(T(e_n))_{n=1}^{\infty}$ is a greedy basis in $Y$, to give two practical applications that will reinforce the importance of the Haar system.

*Example 10.4.9.* (a) Consider $L_p[0,1]$, $1 < p < \infty$. If $\mathcal{W}_p$ is a good wavelet basis (cf. [304, Theorem 8.13]) normalized in $L_p$, then $\mathcal{W}_p$ is equivalent to $\mathcal{H}_p$; thus all such systems are greedy.

(b) It is known (cf. [304, Chapter 9]) that normalized good wavelet bases in Besov spaces $B^p_{\alpha,p}$ are equivalent to the canonical basis in $\ell_p$, $1 \le p < \infty$, thus greedy.

## 10.5 Almost Greedy Bases

One may argue that comparing the quantity $\|x - \mathcal{G}_m(x)\|$ with $\sigma_m(x)$ in order to measure up to the standards of greedy basis is a bit unfair. Indeed, $\mathcal{G}_m(x)$ gives an approximation of $x$ of the form $P_A(x)$, while to compute $\sigma_m(x)$ we allow a much bigger class of approximants, namely, all sums of the form $\sum_{n\in A} a_n e_n$ with $|A| = m$. For unconditional bases this distinction is not substantial. If $\mathcal{B}$ is unconditional, then for every $A$ with $|A| = m$ and every vector $y = \sum_{n\in A} a_n e_n$, we have

$$\|x - P_A(x)\| = \|P_{A^c}(x)\| = \|P_{A^c}(x-y)\| \le \mathsf{K}_{\mathsf{su}}\|x-y\|,$$

so that

$$\sigma_m(x) \le \inf_{|A|=m} \|x - P_A(x)\| \le \mathsf{K}_{\mathsf{su}}\sigma_m(x).$$

Thus, in order to relax the requirement that a basis be greedy, we will start by comparing the error in approximating $x$ by $\mathcal{G}_m(x)$ for each $m$ with the best $m$-term approximation error that can be attained exclusively with $m$-term projections, i.e.,

$$\tilde{\sigma}_m[\mathcal{B},X](x) := \tilde{\sigma}_m(x) = \inf\{\|x - P_A(x)\| : |A| = m\}. \tag{10.17}$$

In this spirit, the article [68] introduces a property for bases that is intermediate between quasi-greedy and greedy.

**Definition 10.5.1.** A basis $\mathcal{B}$ is said to be *almost greedy* if there exists a constant $\mathsf{C} \ge 1$ such that

$$\|x - \mathcal{G}_m(x)\| \le \mathsf{C}\tilde{\sigma}_m(x), \quad \forall\, m \in \mathbb{N}, \ \forall x \in X. \tag{10.18}$$

The smallest $\mathsf{C}$ for which this inequality holds is denoted by $\mathsf{C}_{\mathsf{ag}}$ and will be called the *almost greedy constant* of $\mathcal{B}$.

*Remark 10.5.2.* Note that almost greedy bases in fact satisfy

$$\|x - G_m(x)\| \leq \mathsf{C}_{\mathsf{ag}} \tilde{\sigma}_m(x), \quad \forall\, m \in \mathbb{N}, \ \forall x \in X,$$

for all greedy sums $G_m(x)$ (see Problem 10.3).

Since $\sigma_m(x) \leq \tilde{\sigma}_m(x)$, greedy bases are almost greedy. In turn, note that

$$\sigma_m(x) \leq \tilde{\sigma}_m(x) \leq \|x - S_m(x)\| \to 0 \qquad \text{as} \ \ m \to \infty;$$

hence almost greedy bases are in particular quasi-greedy.

It turns out, rather remarkably, that the definition of almost greedy bases has some nice equivalent formulations. Indeed, Dilworth, Kalton, Kutzarova, and Temlyakov gave in [68] the following important characterization of almost greedy bases in the spirit of Konyagin and Temlyakov's characterization of greedy bases (Theorem 10.4.5).

**Theorem 10.5.3.** *A basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$ in a Banach space $X$ is almost greedy if and only if it is quasi-greedy and democratic.*

*Proof.* We have already seen that almost greedy bases are quasi-greedy. The proof that greedy bases are democratic (see Theorem 10.4.5) carries over to show that almost greedy bases are democratic by replacing $\mathsf{C}_{\mathsf{g}}$ with $\mathsf{C}_{\mathsf{ag}}$.

For the converse, let $x \in X$ and $G_m(x) = P_A(x)$ a greedy sum of order $m$ of $x$, and let $B$ be an arbitrary set of indices with $|B| = m$. Let us write

$$x - P_A(x) = P_{B^c}(x) + P_{B\setminus A}(x) - P_{A\setminus B}(x),$$

so that $\|x - P_A(x)\| \leq \|P_{B^c}(x)\| + \|P_{B\setminus A}(x)\| + \|P_{A\setminus B}(x)\|$. Using (10.13) and taking into account that $\max_{j\in B\setminus A} |e_j^*(x)| \leq \min_{n\in A\setminus B} |e_n^*(x)|$, we have

$$\|P_{B\setminus A}(x)\| \leq 16\mathsf{C}_{\mathsf{qg}}^4 \frac{\varphi_u(r)}{\varphi_l(r)} \left\|P_{A\setminus B}(x)\right\| \leq 16\mathsf{C}_{\mathsf{qg}}^4 \mathsf{C}_{\mathsf{d}} \left\|P_{A\setminus B}(x)\right\|,$$

where $r = |A \setminus B| = |A \setminus B|$. By Lemma 10.2.4, $P_{A\setminus B}(x)$ is a greedy sum of $P_{B^c}(x)$, so that

$$\|P_{A\setminus B}(x)\| \leq \mathsf{C}_{\mathsf{qg}} \|P_{B^c}(x)\|.$$

Putting it all together, we obtain

$$\|x - G_m(x)\| \leq (16\mathsf{C}_{\mathsf{qg}}^5 \mathsf{C}_{\mathsf{d}} + \mathsf{C}_{\mathsf{qg}} + 1)\|P_{B^c}(x)\|.$$

Taking the infimum over all subsets $B$ with $|B| = m$ yields the claim. □

*Example 10.5.4.* An almost greedy basis that is not greedy.

Aside from being quasi-greedy, the basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$ in Example 10.2.9 is democratic. Indeed, if $|A| = m$, then

$$\left(\sum_{n\in A} 1\right)^{1/2} = m^{1/2},$$

while

$$\sum_{n\in A} \frac{1}{\sqrt{n}} \le \sum_{n=1}^{m} \frac{1}{\sqrt{n}} \le \int_0^m \frac{dx}{\sqrt{x}} = 2m^{1/2}.$$

Hence,

$$m^{1/2} \le \left\| \sum_{n\in A} e_n \right\| \le 2m^{1/2}.$$

Thus $\mathcal{B}$ is almost greedy. It cannot be greedy, because it is not unconditional.

The next characterization of almost greedy bases by Dilworth et al. [68] says that the greedy algorithm for almost greedy bases is essentially the best if one allows a small percentage increase in $m$, so the terminology is justified.

**Theorem 10.5.5.** *Suppose $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is a basis in a Banach space $X$. The following conditions are equivalent:*

*(a) $\mathcal{B}$ is almost greedy.*
*(b) For every $\lambda > 1$ there exists a constant $\mathsf{C}_\lambda$ such that*

$$\|x - \mathcal{G}_{\lceil \lambda m \rceil}(x)\| \le \mathsf{C}_\lambda \sigma_m(x), \quad \forall\, m \in \mathbb{N}, \; \forall x \in X.$$

*(c) For some $\lambda > 1$ there exists a constant $\mathsf{C}$ such that*

$$\|x - \mathcal{G}_{\lceil \lambda m \rceil}(x)\| \le \mathsf{C} \sigma_m(x), \quad \forall\, m \in \mathbb{N}, \; \forall x \in X. \tag{10.19}$$

*Proof.* To show $(a) \Rightarrow (b)$ we need a lemma that roughly speaking tells us that the gap between $\tilde{\sigma}$ and $\sigma$ depends on the proximity between the democracy functions of the basis.

**Lemma 10.5.6.** *Suppose $\mathcal{B}$ is quasi-greedy. Then, for all $m, r \in \mathbb{N}$,*

$$\tilde{\sigma}_{m+r}(x) \le \left(1 + \mathsf{C}_{\mathsf{qg}} + 16\mathsf{C}_{\mathsf{qg}}^5 \frac{\varphi_u(m)}{\varphi_l(r)}\right) \sigma_m(x), \qquad \forall x \in X. \tag{10.20}$$

*Proof.* Take $y \in \Sigma_m$ and let $A = \operatorname{supp}(y)$. Consider $z = x - y$. Pick $G_r(z) = P_B(z)$ a greedy sum of $z$ of order $r$. Since $|A| \le m$ and $|B| = r$, there is a set $E \subset \mathbb{N}$ with $A \cup B \subset E$ and $|E| = m + r$. Let us write

$$x - P_E(x) = P_{E^c}(x) = P_{E^c}(z) = z - P_B(z) - P_{E\setminus B}(z)$$

to obtain

$$\tilde{\sigma}_{m+r}(x) \le \|x - P_E(x)\| \le \|z\| + \|P_B(z)\| + \|P_{E\setminus B}(z)\|.$$

Since $\max\{|e_n^*(z)| : n \in E \setminus B\} \le \min\{|e_n^*(z)| : n \in B\}$, applying (10.13) gives

$$\|P_{E\setminus B}(z)\| \le 16\mathsf{C}_{\mathsf{qg}}^4 \frac{\varphi_u(m)}{\varphi_l(r)} \|P_B(z)\|.$$

By (10.3), $\|P_B(z)\| \le \mathsf{C}_{\mathsf{qg}}\|z\|$. Combining, and taking the infimum on $y$, we obtain the desired result. □

Now assume that $\mathcal{B}$ is almost greedy, hence quasi-greedy and democratic by Theorem 10.5.3. Given $x \in X$ and $m \in \mathbb{N}$, let $r = \lceil \lambda m \rceil - m$. By Lemma 10.3.4,

$$\frac{\varphi_u(m)}{\varphi_l(r)} \le \mathsf{C}_{\mathsf{d}} \frac{\varphi_u(m)}{\varphi_u(r)} \le \mathsf{C}_{\mathsf{d}} \max\left\{\mathsf{K}_{\mathsf{b}}, \frac{m}{r}\right\} \le \mathsf{C}_{\mathsf{d}} \max\left\{\mathsf{K}_{\mathsf{b}}, \frac{1}{\lambda - 1}\right\}.$$

Therefore, using Lemma 10.5.6 we have

$$\|x - \mathcal{G}_{\lceil \lambda m \rceil}(x)\| \le \mathsf{C}_{\mathsf{ag}}\mathsf{C}_{\mathsf{d}} \left(1 + \mathsf{C}_{\mathsf{qg}} + 16\mathsf{C}_{\mathsf{qg}}^5\right) \max\left\{\mathsf{C}_{\mathsf{qg}}, \frac{1}{\lambda - 1}\right\} \sigma_m(x),$$

i.e., (b) holds.

$(b) \Rightarrow (c)$ is trivial.

Assume $(c)$ holds. We infer (see Problem 10.4) that in fact,

$$\|x - G_{\lceil \lambda m \rceil}(x)\| \le \mathsf{C}\sigma_m(x), \quad \forall\, m \in \mathbb{N},\ \forall x \in X, \tag{10.21}$$

for every greedy sum $G_{\lceil \lambda m \rceil}(x)$ of $x$. Then, since $\sigma_m(x) \le \|x\|$, we have

$$\|G_{\lceil \lambda m \rceil}(x)\| \le (1 + \mathsf{C})\, \|x\|, \ \ \forall\, m \in \mathbb{N},\ \forall x \in X.$$

Let $G_r(x)$ be a greedy sum of $x$. Pick $m \in \mathbb{N} \cup \{0\}$ such that $\lceil \lambda m \rceil \le r < \lceil \lambda(m+1) \rceil$. Observe that $r - \lceil \lambda m \rceil \le \lceil \lambda(m+1) \rceil - \lceil \lambda m \rceil - 1 \le \lambda$. Therefore, with the notation introduced in (10.11),

$$\|G_r(x)\| \le \|G_{\lceil \lambda m \rceil}(x)\| + \|G_{\lceil \lambda m \rceil}(x) - G_r(x)\| \le (1 + \mathsf{C} + \mathsf{k}_{\lfloor \lambda \rfloor})\|x\|.$$

Hence $\mathcal{B}$ is quasi-greedy by Theorem 10.2.3.

To show that $\mathcal{B}$ is democratic, suppose $|A| = m$ and $|B| = \lceil \lambda m \rceil$. Choose a set $E$ with $|E| = m + \lceil \lambda m \rceil$ such that $A \cup B \subset E \subset \mathbb{N}$ and consider $x = \sum_{n\in E} e_n$. Then $\sum_{n\in E\setminus A} e_n$ is a greedy sum of $x$ of order $\lceil \lambda m \rceil$, while $\sum_{n\in E\setminus B} e_n \in \Sigma_m$. Therefore, using (10.21), we obtain

$$\left\| \sum_{n\in A} e_n \right\| = \left\| x - \sum_{n\in E\setminus A} e_n \right\| \leq \mathsf{C} \left\| x - \sum_{n\in E\setminus B} e_n \right\| = \mathsf{C} \left\| \sum_{n\in B} e_n \right\|. \tag{10.22}$$

Maximizing over $A$ and minimizing over $B$ yields $\varphi_u(m) \leq \mathsf{C}\varphi_l(\lceil \lambda m \rceil)$. Given $r \in \mathbb{N}$ such that $\lceil \lambda \rceil \leq r$, pick out $m \in \mathbb{N}$ such that $\lceil \lambda m \rceil \leq r < \lceil \lambda(m+1) \rceil$. Since $m < \lambda m$, we have $m < r$. Moreover, $r < \lambda(m+1) \leq 2\lambda m$. Then, using Lemma 10.3.4, we have

$$\varphi_u(r) = r\frac{\varphi_u(r)}{r} \leq r\frac{\varphi_u(m)}{m} \leq \mathsf{C}\frac{r}{m}\varphi_l(\lceil \lambda m \rceil) \leq \mathsf{C}\mathsf{K}_\mathsf{b}\frac{r}{m}\varphi_l(r) \leq 2\lambda \mathsf{C}\mathsf{K}_\mathsf{b}\varphi_l(r).$$

An appeal to Lemma 10.3.3 concludes the proof. □

## 10.6 Greedy Bases and Duality

In this section we approach the following question: given a basis $\mathcal{B} = (e_n)_{n=1}^\infty$ of a Banach space $X$ with some greedy-like property, what can be said about the sequence of its biorthogonal functionals $\mathcal{B}^* := (e_n^*)_{n=1}^\infty$ in the dual space $X^*$?

Let $Z$ be the subspace of $X^*$ generated by $\mathcal{B}^*$. By Proposition 3.2.1, $\mathcal{B}^*$ is a (seminormalized) basis of $Z$. Moreover, by Corollary 3.2.4 the basic sequence $\mathcal{B}^{**}$ in $Z^*$ is equivalent to the basis $\mathcal{B}$ via the natural identification

$$e_n^{**}(x^*) = x^*(e_n), \quad \forall x^* \in Z. \tag{10.23}$$

Let us start by analyzing the democracy of $\mathcal{B}^*$. Set $\varphi_u^*(m) = \varphi_u[\mathcal{B}^*, Z](m)$ and $\varphi_l^*(m) = \varphi_l[\mathcal{B}^*, Z](m)$, for $m \in \mathbb{N}$. The elementary computation

$$m = \left(\sum_{n\in A} e_n^*\right)\left(\sum_{n\in A} e_n\right) \leq \left\| \sum_{n\in A} e_n^* \right\| \left\| \sum_{n\in A} e_n \right\| \text{ if } |A| = m \tag{10.24}$$

sheds some interesting information. To begin, we have

$$m \leq \varphi_u(m)\varphi_u^*(m), \qquad \forall m \in \mathbb{N},$$

and wonder about the fulfillment of the reverse inequality.

**Definition 10.6.1.** A basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$ of a Banach space $X$ is said to be *bidemocratic* with constant $\mathsf{C_b}$ if

$$\varphi_u(m)\varphi_u^*(m) \leq \mathsf{C_b} m, \quad \forall m \in \mathbb{N}. \tag{10.25}$$

**Theorem 10.6.2.** *Let $\mathcal{B}$ be an unconditional basis of a Banach space $X$. The following are equivalent:*

*(a) $\mathcal{B}$ is bidemocratic;*
*(b) $\mathcal{B}$ and $\mathcal{B}^*$ are both democratic (thus greedy).*

*Proof.* Suppose $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is bidemocratic with constant $\mathsf{C_b}$. If $A$ is any $m$-element subset of $\mathbb{N}$, the estimates (10.24) and (10.25) give

$$\varphi_u(m) \leq \mathsf{C_b}\frac{m}{\varphi_u^*(m)} \leq \mathsf{C_b}\frac{\left\|\sum_{n\in A} e_n^*\right\| \left\|\sum_{n\in A} e_n\right\|}{\varphi_u^*(m)} \leq \mathsf{C_b}\left\|\sum_{n\in A} e_n\right\|.$$

Hence $\varphi_u(m) \leq \mathsf{C_b}\varphi_l(m)$, and so $\mathcal{B}$ is democratic. A dual argument shows that $\mathcal{B}^*$ is also democratic.

For the converse, let $\mathsf{C_d}$ and $\mathsf{C_d^*}$ be the democracy constants of $\mathcal{B}$ and $\mathcal{B}^*$ respectively, and let $\mathsf{K_u}$ be the unconditional constant of $\mathcal{B}$ (hence of $\mathcal{B}^*$). Fix $m \in \mathbb{N}$. Define, for $E \subset \mathbb{N}$,

$$\mathcal{K}_E = \left\{x^* \in [e_n^* : n \in E] : \|x^*\| \leq 1, \sum_{n\in E} x^*(e_n) \geq \frac{\varphi_l(m)}{\mathsf{K_u}}\right\}.$$

Observe that $\mathcal{K}_E$ is convex and closed in $X^*$. If $|E| = m$, there exists $x^* \in [e_n^* : n \in E]$ with $\|x^*\| \leq 1$ such that

$$\sum_{n\in E} x^*(e_n) = x^*\left(\sum_{n\in E} e_n\right) \geq \frac{1}{\mathsf{K_u}}\left\|\sum_{n\in E} e_n\right\|$$

(see Problem 10.16). Since $\|\sum_{n\in E} e_n\| \geq \varphi_l(m)$, we conclude that $\mathcal{K}_E \neq \emptyset$. Notice that $\mathcal{K}_E$ increases with $E$. Therefore $\mathcal{K}_E \neq \emptyset$ for every set $E$ of cardinality $|E| \geq m$.

Now fix $A \subset \mathbb{N}$ of cardinality $|A| = 2m$. Pick $v^* \in \mathcal{K}_A$ such that $\sum_{n\in A}(v^*(e_n))^2$ is minimized. Now let $G_m^*(v^*) = P_B^*(v^*)$ be a greedy sum of $v^*$ of order $m$. Note that $|B| = |A \setminus B| = m$. Pick $z^* \in \mathcal{K}_{A\setminus B} \subset \mathcal{K}_A$. By the geometric properties of minimizing vectors on convex subsets of Hilbert spaces we have

$$\sum_{n\in A}(v^*(e_n))^2 \leq \sum_{n\in A} v^*(e_n)z^*(e_n) = \sum_{n\in A\setminus B} v^*(e_n)z^*(e_n) = z^*(y), \tag{10.26}$$

where $y = \sum_{n\in A\setminus B} v^*(e_n)e_n$. Observe that by the definition of $G_m^*(v^*)$,

$$\max_{n\in A\setminus B} |v^*(e_n)| = \max_{n\in A\setminus B} |e_n^{**}(v^*)| \leq \min_{n\in B} |e_n^{**}(v^*)|.$$

Using Remark 10.3.7, we obtain

$$z^*(y) \leq \|z^*\|\|y\| \leq \|y\| \leq \mathsf{K}_\mathsf{u}^2 \frac{\varphi_u(m)}{\varphi_l^*(m)} \|P_B^*(Z)\| \leq \mathsf{K}_\mathsf{u}^3 \frac{\varphi_u(m)}{\varphi_l^*(m)}. \tag{10.27}$$

Finally, by the Cauchy–Schwarz inequality,

$$\frac{\varphi_l^2(m)}{\mathsf{K}_\mathsf{u}^2} \leq \left( \sum_{n\in A} v^*(e_n) \right)^2 \leq m \sum_{n\in A} (v^*(e_n))^2. \tag{10.28}$$

Combining (10.26), (10.27), and (10.28) yields

$$\varphi_u(m)\varphi_u^*(m) \leq \mathsf{C}_\mathsf{d}^2 \mathsf{C}_\mathsf{d}^* \frac{\varphi_l^2(m)\varphi_l^*(m)}{\varphi_u(m)} \leq \mathsf{C}_\mathsf{d}^2 \mathsf{C}_\mathsf{d}^* \mathsf{K}_\mathsf{u}^5 m.$$

□

**Theorem 10.6.3.** *Let $\mathcal{B} = (e_n)_{n=1}^\infty$ be a greedy basis in a Banach space $X$. Suppose $\varphi_u(m) \approx m^\alpha$ for some $0 < \alpha < 1$. Then $\mathcal{B}^*$ is also greedy.*

*Proof.* From Theorem 10.4.5 we know that $\mathcal{B}$ is unconditional, say with unconditional constant $\mathsf{K}_\mathsf{u}$; hence the basic sequence $\mathcal{B}^*$ is also unconditional. To show that $\mathcal{B}^*$ is democratic under our assumption on $\varphi_u$, by Theorem 10.6.2 it suffices to see that $\varphi_u^*(m) \leq Cm^{1-\alpha}$ for all $m$, for some constant $C$.

Let $B \subset \mathbb{N}$ be of cardinality $m$. Pick out $x \in X$ with $\|x\| = 1$ such that

$$\left\| \sum_{j\in B} e_j^* \right\| \leq 2 \sum_{j\in B} |e_j^*(x)|,$$

and take $\sigma : \{1, 2, \ldots, m\} \to B$ such that $|e_{\sigma(j)}^*(x)| \leq |e_{\sigma(i)}^*(x)|$ whenever $j \geq i$. The unconditionality of the basis gives

$$|e_{\sigma(j)}^*(x)|\varphi_l(j) \leq |e_{\sigma(j)}^*(x)| \left\| \sum_{i=1}^j e_{\sigma(i)} \right\| \leq \mathsf{K}_\mathsf{u}\|x\| = \mathsf{K}_\mathsf{u},$$

so that

$$\sum_{j\in B} |e_j^*(x)| = \sum_{j=1}^m |e_{\sigma(j)}^*(x)| \leq \mathsf{K}_\mathsf{u} \sum_{j=1}^m \varphi_l(j)^{-1}.$$

Then, if we let $\gamma = \inf_j \varphi_l(j) j^{-\alpha} > 0$,

$$\varphi_u^*(m) \leq 2K_\mathsf{u} \sum_{j=1}^m \varphi_l(j)^{-1} \leq 2\frac{K_\mathsf{u}}{\gamma} \sum_{j=1}^m j^{-\alpha} \leq \frac{2K_\mathsf{u}}{(1-\alpha)\gamma}\, m^{1-\alpha}.$$

□

This theorem of Wojtaszczyk [306] was generalized by Dilworth et al., who obtained (cf. [68, Theorem 5.1]) that if $(e_n)_{n=1}^{\infty}$ is a greedy basis of a Banach space $X$ with nontrivial type, then $(e_n^*)_{n=1}^{\infty}$ is a greedy *basis* of $X^*$ (note that every space with nontrivial type and an unconditional basis is reflexive by James's theorem).

**Corollary 10.6.4.** *If $\mathcal{B}$ is a greedy basis in $L_p[0,1]$, $1 < p < \infty$, then $\mathcal{B}^*$ is a greedy basis in $L_q[0,1]$, where $1/p + 1/q = 1$.*

*Proof.* Fix $1 < p < \infty$. Notice that in $L_p[0,1]$, all greedy bases $\mathcal{B}$ have essentially the same democracy functions, namely

$$\varphi_l[\mathcal{B}, L_p](m) \approx \varphi_u[\mathcal{B}, L_p](m) \approx m^{1/p}.$$

Indeed, suppose $\mathcal{B}$ is a greedy basis in $L_p$. Since $\mathcal{B}$ is unconditional, it has a subsequence $\tilde{\mathcal{B}}$ equivalent to the unit vector basis of $\ell_p$ (see [147]). Therefore,

$$\varphi_l[\tilde{\mathcal{B}}, L_p](m) \approx \varphi_u[\tilde{\mathcal{B}}, L_p](m) \approx m^{1/p}.$$

The proof is over once we have combined this with Theorem 10.6.3. □

*Example 10.6.5.* A greedy basis $\mathcal{B}$ such that $\mathcal{B}^*$ is not democratic (hence non-greedy).

Let $\mathcal{H}_1 = (h_n^1)_{n=1}^{\infty}$ be the Haar system normalized in $L_1[0,1]$. Since $\mathcal{H}_1$ is not unconditional, we cannot count on having an estimate like (10.16) for $p = 1$. However, if we consider the space $X$ of all sequences of scalars $(a_n)_{n=1}^{\infty}$ such that

$$\|(a_n)_{n=1}^{\infty}\| = \int_0^1 \Big( \sum_{n=1}^{\infty} (a_n h_n^1(t))^2 \Big)^{1/2} dt < \infty,$$

we end up with a Banach space $(X, \|\cdot\|)$ in which the unit vectors $\mathcal{B} = (e_n)_{n=1}^{\infty}$ are a normalized unconditional basis.

A similar argument as in the proof of Proposition 10.4.7 shows that $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is democratic with $\varphi_u[\mathcal{B}, X](m) \approx m$. Hence $\mathcal{B}$ is greedy in $X$.

Suppose that $\mathcal{B}^*$ is democratic. Then, by Theorem 10.6.2, $\mathcal{B}$ would be bidemocratic, and so $\varphi_u^*(m) \approx 1$. As we have seen (Proposition 10.3.8), this implies that $\mathcal{B}^*$ is equivalent to the canonical $c_0$-basis. By Corollary 3.2.4, $\mathcal{B}$ is equivalent to the unit vector basis of $\ell_1$. But this is an absurdity, because if we consider the normalized sequence of blocks of $(e_n)_{n=1}^{\infty}$ given by

$$x_k = \frac{1}{2^k} \sum_{n=2^k}^{2^{k+1}-1} e_k, \qquad k = 1, 2 \ldots,$$

then on the one hand, for every $(a_k)$ we would have

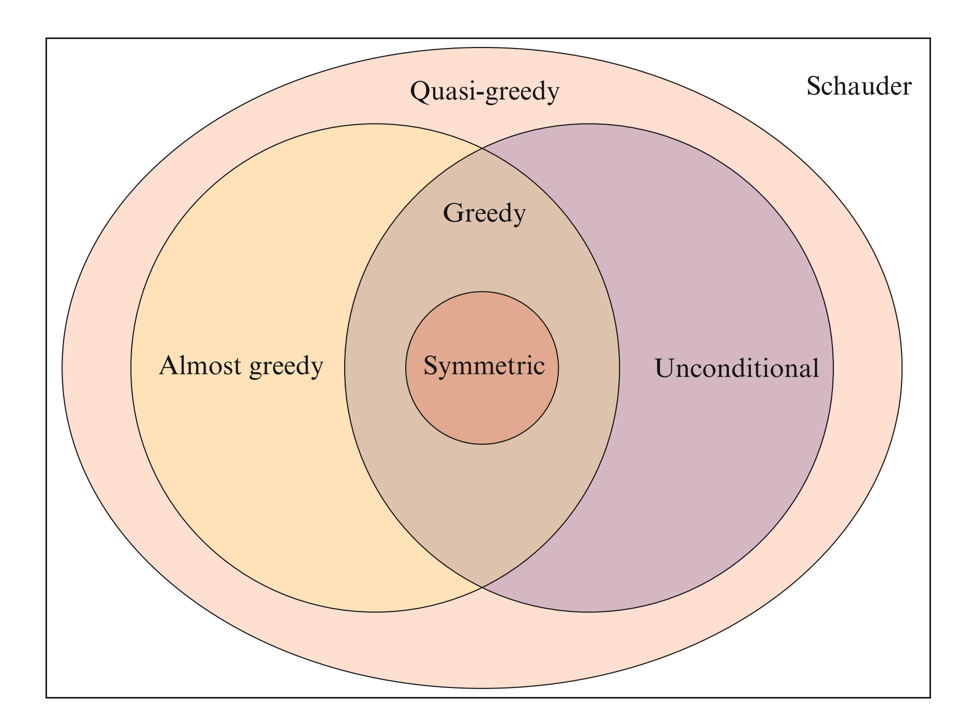
$$\Big\| \sum_{k=1}^{\infty} a_k x_k \Big\| = \Big( \sum_{k=1}^{\infty} a_k^2 \Big)^{1/2},$$

while on the other hand, every normalized block basic sequence of the canonical $\ell_1$-basis is known to be isometrically equivalent to itself (Lemma 2.1.1)!

A complete duality theory for greedy bases in Banach spaces was developed in [68] by Dilworth, Kalton, Kutzarowa, and Temlyakov. These authors also studied the duality properties of bases under weaker forms of greediness. As a sample, they showed that the dual basis of a quasi-greedy basis in a Hilbert space is also quasi-greedy and that in general, if a Banach space $X$ has nontrivial type and $\mathcal{B}$ is an almost greedy basis of $X$, then $\mathcal{B}^*$ is an almost greedy basic sequence (cf. [68, Corollary 4.5 and Theorem 5.4]). Let us remark that in contrast to greedy and almost greedy bases, the duality properties of quasi-greedy bases in Banach spaces have not been thoroughly investigated. Dilworth and Mitra constructed in [65] a conditional quasi-greedy basis $\mathcal{B}$ in $\ell_1$ whose dual basis $\mathcal{B}^*$ is not unconditional for constant coefficients, so $\mathcal{B}^*$ is not quasi-greedy.

## 10.7 The Zoo of Greedy-Like Bases in a Banach Space

With all the above, we are ready to play a game. Let us fix a Banach space $X$ and try to determine the population distribution of its bases according to the map in the figure.




Of course, one or more of these areas might not be populated, depending on the space. Other spaces might have essentially only one inhabitant in a certain zone and plenty in a different one, and so on. This simplistic board game serves to motivate very natural questions in Banach space theory:

- Given a Banach space $X$, pick any two zones. Is there a basis $\mathcal{B}$ in $X$ that belongs to one of them but not to the other? In this direction of work, Wojtaszczyk gave in [305] a general construction (improved in [101]) to produce quasi-greedy bases in some Banach spaces. His method yields the existence of conditional quasi-greedy bases in separable infinite-dimensional Hilbert spaces, in the spaces $\ell_p$ and $L_p[0, 1]$ for $1 < p < \infty$, and in the Hardy space $H_1$. Dilworth and Mitra showed in [65] that $\ell_1$ also has a conditional quasi-greedy basis.
- The question of existence. Once it has been determined that the natural basis of a space does not occupy a certain area in a Banach space, the problem is to establish whether that zone has any occupants at all. For instance, none of the Banach spaces $L_1$, $\mathcal{C}[0, 1]$, and the James space $\mathcal{J}$ have a greedy basis, since they do not have unconditional bases. One can also give other examples failing to have a greedy basis such as $\ell_1 \oplus \ell_2$, $\ell_1 \oplus c_0$, $c_0 \oplus \ell_2$, $\ell_1 \oplus \ell_2 \oplus c_0$, or any other mixed finite direct sum of the Banach spaces with a unique unconditional basis. This is a direct application of a classical result of Edelstein and Wojtaszczyk [83] that every normalized unconditional basis of each of those spaces is equivalent to the canonical basis, which is plainly not democratic. As it happens, no finite direct sum of two or more distinct spaces from the family $\{\ell_p : 1 \le p < \infty\} \cup c_0$ has a greedy basis.

  However, having an almost greedy basis is much easier. Indeed, a very general construction of almost greedy bases was given by Dilworth, Kalton, and Kutzarova in [67], where the following theorem is proved.

  **Theorem 10.7.1.** *Let X be a Banach space with a basis. Suppose that X has a complemented subspace Y with a symmetric basis. Then:*

  *(i) If Y is not isomorphic to $c_0$, then X has a quasi-greedy basis.*
  *(ii) If Y has finite cotype, then X has an almost greedy basis.*

  From this theorem one gets immediately that $L_1[0, 1]$ has an almost greedy basis (note that the Haar basis is not even a quasi-greedy basis for $L_1[0, 1]$, since it is not unconditional for constant coefficients). This is a pretty good result if one takes into account that $L_1[0, 1]$ does not have an unconditional basis; however, the proof is not constructive, and no specific example is given. This gap was filled in [109], where Gogyan constructs an explicit almost greedy basis in $L_1[0, 1]$. The paper [67] also contains a characterization of the $\mathcal{L}_\infty$ spaces that admit a quasi-greedy basis. In fact, the authors prove that $c_0$ is the only $\mathcal{L}_\infty$ space (up to isomorphism) to have a quasi-greedy basis. Thus, in particular, $\mathcal{C}[0, 1]$ does not have a quasi-greedy basis.
- Uniqueness. If a Banach space $X$ has a unique unconditional basis (up to equivalence), then it clearly has a unique greedy basis. Hence the canonical unit vector basis is (essentially) the only greedy basis in $c_0$, $\ell_1$, and $\ell_2$ (beware: there exist other Banach spaces apart from those three with a unique greedy basis [8]). Wojtaszczyk [305] gave an example of a basis in $L_p$ for $p > 2$ that is greedy and not equivalent to any subsequence of the Haar system. Using duality properties of greedy bases, such an example exists also for $1 < p < 2$. Subsequently it has

been proved that each of the spaces $L_p[0,1]$ and $\ell_p$ for $1 < p < \infty$ has a family of uncountably many greedy bases that are not even permutatively equivalent (cf. [284] and [69] respectively). The problem of uniqueness makes sense as well for weaker forms of greediness. For instance, Dilworth, Kalton, and Kutzarowa proved that $c_0$ has a unique quasi-greedy basis, and that in fact, it is the only infinite-dimensional Banach space to have a unique quasi-greedy basis [67].

- Now choose your favorite isomorphic property $(\mathsf{P})$ (for example being of type 2) and assume $X$ has $(\mathsf{P})$. Does this assumption affect the borders between zones and make some of them vanish?

Some readers might object that the figure should contain an enclosure for democratic bases. We will not argue that in doing, so the game could gain in complexity. However, we would rather regard democracy as a key for opening connecting doors between the already existing rings in some spaces.

## Problems

**10.1.** Let $\mathcal{B}$ be a seminormalized quasi-greedy basis in a Hilbert space $H$ with quasi-greedy constant $\mathsf{C}_{\mathsf{qg}} = 1$. Show that $\mathcal{B}$ is an orthogonal basis of $H$.

**10.2.** Let $\mathcal{B}$ be a basis in a Banach space $X$. Show that the following are equivalent:

(a) $\mathcal{B}$ is greedy.
(b) There is a constant $\mathsf{C}$ such that for every $x \in X$, for every $m \in \mathbb{N}$, and every $G_m(x)$ we have $\|x - G_m(x)\| \le \mathsf{C}\sigma_m(x)$.
(c) There is a constant $\mathsf{C}$ such that for every $x \in X$ and every $m \in \mathbb{N}$ there exists a greedy sum of $x$ of order $m$, $G_m(x)$, such that $\|x - G_m(x)\| \le \mathsf{C}\sigma_m(x)$.

Moreover, the least constant in (b) (respectively, (c)) is the greedy constant of $\mathcal{B}$.

**10.3.** Show that for a basis $\mathcal{B}$ in a Banach space $X$ the following conditions are equivalent:

(a) $\mathcal{B}$ is almost greedy.
(b) There exists a constant $\mathsf{C}$ such that for each $x \in X$ and every greedy sum $G_m(x)$,

$$\|x - G_m(x)\| \le \mathsf{C}\tilde{\sigma}_m(x). \tag{10.29}$$

(c) For every $m \in \mathbb{N}$ and $x \in X$ there exists a greedy sum $G_m(x)$ such that $\|x - G_m(x)\| \le \mathsf{C}\tilde{\sigma}_m(x)$, where $\mathsf{C}$ is a universal constant.

Moreover, the least constant $\mathsf{C}$ in (b) [respectively, (c)] is the almost greedy constant $\mathsf{C}_{\mathsf{ag}}$ of $\mathcal{B}$.

**10.4.** Suppose $\mathcal{B}$ is a basis in a Banach space $X$ and let $\lambda > 1$. Use a perturbation argument to show that the following conditions are equivalent:

(a) There exists a constant $\mathsf{C}$ such that for all $m \in \mathbb{N}$, all $x \in X$, and all $G_{\lceil \lambda m \rceil}(x)$ the greedy sum of $x$ of order $\lceil \lambda m \rceil$, $\|x - G_{\lceil \lambda m \rceil}(x)\| \leq \mathsf{C}\sigma_m(x)$.
(b) There exists a constant $\mathsf{C}$ such that for all $m \in \mathbb{N}$ and all $x \in X$, $\|x - \mathcal{G}_{\lceil \lambda m \rceil}(x)\| \leq \mathsf{C}\sigma_m(x)$.
(c) There exists a constant $\mathsf{C}$ such that for all $m \in \mathbb{N}$ and all $x \in X$ there is a greedy sum of $x$ of order $\lceil \lambda m \rceil$, $G_{\lceil \lambda m \rceil}(x)$, such that $\|x - G_{\lceil \lambda m \rceil}(x)\| \leq \mathsf{C}\sigma_m(x)$.

Moreover, if a constant $\mathsf{C}$ works in one of the estimates, it also works in the other two.

**10.5.** Let $(e_n)_{n=1}^{\infty}$ be an unconditional basis for a Banach space $X$ with suppression unconditional constant $\mathsf{K}_{\mathsf{su}} = 1$. Show that for each $x \in X$ and each $m = 1, 2, \ldots$ there exists $B \subset \mathbb{N}$ of cardinality $m$ such that $\sigma_m(x) = \|x - P_B(x)\|$. That is, if $\mathsf{K}_{\mathsf{su}} = 1$, then $\sigma_m(x) = \tilde{\sigma}_m(x)$, and the infimum in equation (10.17) is attained. Therefore we obtain the following immediate consequences:

(i) If $(e_n)_{n=1}^{\infty}$ is greedy with greedy constant $\mathsf{C}_{\mathsf{g}} = 1$, then

$$\|x - \mathcal{G}_m(x)\| = \sigma_m(x) = \min\{\|x - P_A(x)\| \colon |A| = m\}.$$

(ii) If $(e_n)_{n=1}^{\infty}$ is unconditional with $\mathsf{K}_{\mathsf{su}} = 1$ and

$$\|x - \mathcal{G}_m(x)\| = \min\{\|x - P_A(x)\| \colon |A| = m\}$$

for each $x \in X$ and every $m$, then $(e_n)$ is greedy with greedy constant $\mathsf{C}_{\mathsf{g}} = 1$ (cf. [305, Proposition 7]).

**10.6.** Prove that 1-symmetric bases are 1-greedy.

**10.7.** A basis $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is *partially greedy* if for some $\mathsf{C}$,

$$\|x - \mathcal{G}_m(x)\| \leq \mathsf{C}\, \|x - S_m(x)\|, \quad \forall\, m \in \mathbb{N}, \ \forall x \in X,$$

where $S_m$ denotes the $m$th partial sum projection associated to $\mathcal{B}$. Being partially greedy is an intermediate condition between almost greedy and quasi-greedy. Show that $\mathcal{B}$ is partially greedy if and only if it is quasi-greedy and *conservative*, i.e., for some constant $\mathsf{K}$,

$$\left\| \sum_{n \in A} e_n \right\| \leq \mathsf{K} \left\| \sum_{n \in B} e_n \right\|$$

whenever $|A| \leq |B|$ and $A < B$ (in the sense that $m \in A$ and $n \in B$ implies $m < n$) [68].

**10.8.** Show that a basis $\mathcal{B}$ is almost greedy if and only if there is $\mathsf{C} > 0$ such that

$$\|x - \mathcal{G}_m(x)\| \leq \mathsf{C} \inf_{k \leq m} \tilde{\sigma}_k(x), \quad \forall\, m \in \mathbb{N}, \ \forall x \in X.$$

**10.9.** The efficiency of the greedy algorithm $(\mathcal{G}_m)_{m=1}^\infty$ in a Banach space with respect to a certain basis $\mathcal{B}$ can be quantified by finding upper bounds for the growth of the sequence

$$E_m[\mathcal{B}, X] := E_m = \sup_{\|x\|\le 1} \frac{\|x - \mathcal{G}_m(x)\|}{\sigma_m(x)}, \qquad m = 1, 2, \dots$$

(with the convention that $\frac{0}{0} = 1$). Follow the steps in the proof of Theorem 10.4.5 to show that for every basis $\mathcal{B}$ in a Banach space $X$ we have

$$E_m \le \mathsf{k}_{2m} + 1 + \mathsf{k}_m^3 \mu_m, \qquad m = 1, 2, \dots,$$

where $\mu_m = \sup_{k\le m} \varphi_u(m)/\varphi_l(m)$. This estimate is an example of the so-called *Lebesgue-type inequalities* for greedy approximation. For a detailed study of such inequalities the reader may consult [70, 293].

**10.10.** Use the type and the cotype of a Hilbert space $H$ to show that every quasi-greedy basis in $H$ is almost greedy [305].

**10.11 ([177]).** Let $\mathcal{B} = (e_n)_{n=1}^\infty$ be a normalized basis in a Banach space $X$. Set $y_n = 2^n e_n$ and let $\tilde{\mathcal{B}} = (y_n)_{n=1}^\infty$. Clearly $\tilde{\mathcal{B}}$ is a basis in $X$. Prove that for each $x = \sum_{n=1}^\infty a_n y_n \in X$ we have

$$\lim_{\epsilon\to 0} \sum_{\{n : |a_n| > \epsilon\}} a_n y_n = x.$$

**10.12.** Show that if $\mathcal{B}$ is bidemocratic, then $\mathcal{B}$ and $\mathcal{B}^*$ are both unconditional for constant coefficients.

**10.13 ([68]).** In $\ell_2 \oplus \ell_p$ with $1 < p < 2$, let $(e_n^2)$ (respectively, $(e_n^p)$) denote the unit vector basis in $\ell_2$ (respectively, $\ell_p$). Define

$$f_{2n-1} = \frac{1}{\sqrt{2}}(e_n^2 + e_n^p) \text{ and } f_{2n} = \frac{1}{2} e_n^2 + \frac{\sqrt{3}}{2} e_n^p.$$

Show that:

(a) $(f_n)_{n=1}^\infty$ is a conditional basis in $\ell_2 \oplus \ell_p$.
(b) $(f_n)_{n=1}^\infty$ is a democratic basis with $\varphi_u(m) \approx m^{1/p}$.
(c) $(f_n^*)_{n=1}^\infty$ is a democratic basis in $(\ell_2 \oplus \ell_p)^*$ with $\varphi_u(m) \approx \sqrt{m}$.
(d) Deduce that $(f_n)$ is not bidemocratic.

**10.14.** Let $\mathcal{B}$ be a greedy basis in a Banach space $X$. Follow the steps of the proof of Theorem 10.6.3 to show that

$$\frac{\varphi_u[\mathcal{B}^*, Z](m)}{\varphi_l[\mathcal{B}^*, Z](m)} = \mathcal{O}(\log m).$$

**10.15.** Prove that a basis is unconditional if and only if there is a constant $\mathsf{C}$ such that $\|P_{A^c}\| \le \mathsf{C}$ for every finite set $A \subset \mathbb{N}$. Moreover, $\mathsf{K}_{\mathsf{su}} = \sup_{|A|<\infty} \|P_{A^c}\|$.

**10.16.** Suppose $(e_n)_{n=1}^{\infty}$ is unconditional with suppression unconditional basis $\mathsf{K}_{\mathsf{s}u}$. Given a finite subset $A \subset \mathbb{N}$ and $x \in [e_n : n \in A]$, there exists $x^* \in [e_n^* : n \in A]$ with $\|x^*\| \le 1$ such that

$$x^*(x) \ge \frac{1}{\mathsf{K}_{\mathsf{su}}}\|x\|.$$

**10.17.** Show that the direct sum of quasi-greedy bases is quasi-greedy.

**10.18.** Show that the direct sum of two democratic bases $\mathcal{B}_1$ in $X$ and $\mathcal{B}_2$ in $Y$ is democratic in $X \oplus Y$ if and only if $\varphi_u[\mathcal{B}_1, X](m) \approx \varphi_u[\mathcal{B}_2, Y](m)$.

**10.19.** A basis $(e_n)_{n=1}^{\infty}$ is said to be $\Gamma$*-superdemocratic* ($\Gamma \ge 1$) if the inequality

$$\left\|\sum_{k\in P} \theta_k e_k\right\| \le \Gamma \left\|\sum_{k\in Q} \epsilon_k e_k\right\|$$

holds for every two finite sets of integers $P$ and $Q$ of the same cardinality and every choices of signs $(\theta_k)_{k\in P}$ and $(\epsilon_k)_{k\in Q}$. Prove that a basis is superdemocratic if and only if it is democratic and unconditional for constant coefficients.

**10.20.** Show that $\varphi_u$ is subadditive, i.e.,

$$\varphi_u(m+n) \le \varphi_u(m) + \varphi_u(n), \quad \forall m, n \in \mathbb{N}.$$

**10.21.** Let $(e_n)_{n=1}^{\infty}$ be an almost greedy basis in a Banach space $X$. Suppose $(\lambda_n)_{n=1}^{\infty}$ is a sequence of real numbers such that $0 < \inf_n |\lambda_n| \le \sup_n |\lambda_n| < \infty$. Then the perturbed basis $\tilde{\mathcal{B}} = (\tilde{e}_n)_{n=1}^{\infty} = (\lambda_n e_n)_{n=1}^{\infty}$ is also almost greedy.

**10.22.** Give an example of a quasi-greedy basis that is neither unconditional nor almost greedy.

**10.23.** Look inside Corollary 10.2.13 to obtain the following improvement. Suppose $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is a quasi-greedy basis of $X$ with quasi-greedy constant $\mathsf{C}_{\mathsf{qg}}$. For every $x \in X$ and finite subset $A \subset \mathbb{N}$ we have

$$\|P_A(x)\| \le 8\mathsf{C}_{\mathsf{qg}}^2 \frac{\max\{|e_n^*(x)| : n \in A\}}{\min\{|e_n^*(x)| : n \in A \cap \operatorname{supp}(x)\}}\|x\|.$$

**10.24.** It is well known (see Problem 1.11) that the complex trigonometric system $(e^{in\theta})_{n\in\mathbb{Z}}$ is a basis for $L_p(\mathbb{T})$ $(1 < p < \infty)$. Prove that it is not quasi-greedy unless $p = 2$. (*Hint*: Use Khintchine's inequalities and the Dirichlet kernel to prove that it is not unconditional for constant coefficients.)

# Chapter 11
# $\ell_p$-Subspaces of Banach Spaces

In the previous chapters the spaces $\ell_p$ $(1 \leq p < \infty)$ and $c_0$ played a pivotal role in the development of the theory. This suggests that we should ask when we can embed one of these spaces in an arbitrary Banach space. For $c_0$ we have a complete answer: $c_0$ embeds into $X$ if and only if $X$ contains a WUC series that is not unconditionally convergent (Theorem 2.4.11).

In this chapter we present a remarkable theorem of Rosenthal from 1974 [273] that gives a precise necessary and sufficient condition for $\ell_1$ to be isomorphic to a subspace of a Banach space $X$; this is analogous to, but much more difficult than, the characterization of Banach spaces containing $c_0$. It requires us to develop so-called Ramsey theory, which has proved a very productive contributor to infinite-dimensional Banach space theory. Rosenthal's theorem asserts that either a Banach space contains $\ell_1$ or every bounded sequence has a weakly Cauchy subsequence.

The rest of the chapter is devoted to the construction of an important example, *Tsirelson space*. During the 1960s a likely picture of the structure of Banach spaces emerged in which the $\ell_p$-spaces and $c_0$ were considered potential building blocks. A question then arose as to whether every Banach space must contain a copy of one of these spaces. This was solved by Tsirelson [297], who constructed an elegant counterexample. Tsirelson's space has had a very profound influence on the further development of the subject.

## 11.1 Ramsey Theory

Let $\mathcal{P}\mathbb{N}$ denote the power set $2^{\mathbb{N}}$ of the natural numbers, i.e., the collection of all subsets of $\mathbb{N}$. The set $\mathcal{P}\mathbb{N}$ can be identified with the Cantor set $\Delta = \{0, 1\}^{\mathbb{N}}$ via the mapping $A \mapsto \chi_A$, where $\chi_A(n) = 1$ if $n \in A$ and 0 otherwise. Let $\mathcal{P}_\infty\mathbb{N}$ be the subset of $\mathcal{P}\mathbb{N}$ of all infinite subsets of $\mathbb{N}$. The complementary set of $\mathcal{P}_\infty\mathbb{N}$ in $\mathcal{P}\mathbb{N}$ of all finite subsets of $\mathbb{N}$ is denoted by $\mathcal{F}\mathbb{N}$.



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Given any $M \in \mathcal{P}\mathbb{N}$, $\mathcal{F}_r(M)$ will be the collection of all finite subsets of $M$ of cardinality $r$.

If $M \in \mathcal{P}_\infty M$ and $f : \mathcal{F}_r(\mathbb{N}) \to \mathbb{R}$ is any function, we will write

$$\lim_{A \in \mathcal{F}_r(M)} f(A) = \alpha$$

to mean that given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $A \in \mathcal{F}_r(\mathbb{N})$ and $A \subset [N, \infty)$, then $|f(A) - \alpha| < \epsilon$.

We shall start by proving a generalization of the original *Ramsey theorem* [265]. This is far too simple for our purposes, and we will need to go much further. The original Ramsey theorem corresponds to the case $r = 2$ of $(ii)$ of the following theorem. We will use Theorem 11.1.1 $(i)$ in the next chapter.

**Theorem 11.1.1 (Ramsey's Theorem [265]).**

*(i) Suppose $r \in \mathbb{N}$ and $f : \mathcal{F}_r(\mathbb{N}) \to \mathbb{R}$ is a bounded function. Then there exists $M \in \mathcal{P}_\infty(\mathbb{N})$ such that $\lim_{A \in \mathcal{F}_r(M)} f(A)$ exists.*

*(ii) If $\mathcal{A} \subset \mathcal{F}_r(\mathbb{N})$, then there exists $M \in \mathcal{P}_\infty(\mathbb{N})$ such that either $\mathcal{F}_r(M) \subset \mathcal{A}$ or $\mathcal{F}_r(M) \cap \mathcal{A} = \emptyset$.*

*Proof.* $(ii)$ follows directly from $(i)$ if we define $f(A) = \chi_{\mathcal{A}}(A)$.

The proof of $(i)$ is done by induction on $r$. For $r = 1$ it is trivially true. Assume that $r \geq 2$ and that $(i)$ holds for $r - 1$; we must deduce that $(i)$ is also true for $r$.

For distinct integers $m_1, \ldots, m_r$, put

$$f(m_1, m_2, \ldots, m_r) = f(\{m_1, \ldots, m_r\}).$$

We first use a diagonal procedure to obtain a subsequence (or subset) $M_1$ of $\mathbb{N}$ such that for every distinct $m_1, \ldots, m_{r-1}$,

$$\lim_{m_r \in M_1} f(m_1, m_2, \ldots, m_{r-1}, m_r) = g(m_1, m_2, \ldots, m_{r-1})$$

exists. Since $g$ is independent of the order of $m_1, \ldots, m_{r-1}$, we may write it as a bounded map $g : \mathcal{F}_{r-1}(\mathbb{N}) \to \mathbb{R}$. It follows from the inductive hypothesis that $M_1$ has an infinite subset $M_2$ such that

$$\lim_{A \in \mathcal{F}_{r-1}(M_2)} g(A) = \alpha$$

for some real $\alpha$.

If $A \in \mathcal{F}_{r-1}(M_2)$ and $\epsilon > 0$, we can find an integer $N = N(A, \epsilon)$ such that if $n \geq N(A, \epsilon)$ and $n \in M_2$, then $n \notin A$, and

$$|f(A \cup \{n\} - g(A))| < \epsilon.$$

We next choose an infinite subset of $M_2$. Pick $r-1$ initial points. Then if $m_1 < m_2 < \cdots < m_n$ have been chosen with $n \geq r-1$, pick $m_{n+1} > m_n$ such that

$$m_{n+1} > \max_{A \in \mathcal{F}_{r-1}\{m_1,\ldots,m_n\}} N(A, 2^{-n}).$$

Finally, let $M = \{m_j\}_{j=1}^{\infty}$.

Given $\epsilon > 0$ we may take $n \in \mathbb{N}$ such that on the one hand, if $A \subset [m_n, \infty)$ with $A \in \mathcal{F}_{r-1}(M)$, then $|g(A) - \alpha| < \frac{1}{2}\epsilon$, and on the other hand, $n$ is large enough that $2^{-n} < \frac{1}{2}\epsilon$. Suppose $A \in \mathcal{F}_r(M)$ with $A \subset [m_n, \infty)$. Let $m_k$ be its largest member and let $B = A \setminus \{m_k\}$. Then

$$|f(A) - g(B)| < 2^{-(k-1)} \leq 2^{-n} \leq \epsilon/2$$

and

$$|g(B) - \alpha| < \epsilon/2,$$

which shows that

$$|f(A) - \alpha| < \epsilon.$$

Hence

$$\lim_{A \in \mathcal{F}_r(M)} f(A) = \alpha.$$

□

We will need an infinite version of Theorem 11.1.1 (*ii*) when $\mathcal{A}$ becomes a subset of $\mathcal{P}_\infty\mathbb{N}$. This requires some topological restrictions.

The set $\mathcal{P}_\infty\mathbb{N}$ inherits a metric topology from the Cantor set, which we call the *Cantor topology*. Since $\mathcal{P}_\infty\mathbb{N}$ is a $G_\delta$-set in $\mathcal{P}\mathbb{N}$ and the Cantor set is compact, this topology can be given by a complete metric.

We shall also be interested in a second stronger topology that is known as the *Ellentuck topology*. If $A \in \mathcal{F}\mathbb{N}$ and $E \in \mathcal{P}_\infty\mathbb{N}$, we define $\mathcal{P}_\infty(A, E)$ to be the collection of all infinite subsets of $A \cup E$ that contain $A$. In the special case $A = \emptyset$, we write $\mathcal{P}_\infty(\emptyset, E) = \mathcal{P}_\infty(E)$.

Let us say that a set $\mathcal{U} \subset \mathcal{P}_\infty\mathbb{N}$ is *open for the Ellentuck topology* or *Ellentuck-open* if whenever $E \in \mathcal{U}$, there exists a finite set $A \subset E$ such that $\mathcal{P}_\infty(A, E) \subset \mathcal{U}$. This is easily seen to define a topology (the *Ellentuck topology*) on $\mathcal{P}_\infty\mathbb{N}$.

Our aim is to study a dichotomy result. We want to put conditions on a subset $\mathcal{V}$ of $\mathcal{P}_\infty\mathbb{N}$ such that either there is an $M \in \mathcal{P}_\infty\mathbb{N}$ with $\mathcal{P}_\infty(M) \subset \mathcal{V}$ or there is an $M \in \mathcal{P}_\infty\mathbb{N}$ with $\mathcal{P}_\infty(M) \cap \mathcal{V} = \emptyset$. If such a dichotomy holds, we say that $\mathcal{V}$ has the *Ramsey property* (or that $\mathcal{V}$ is a *Ramsey set*). However, it turns out to be easier to study a stronger property.

We say that $\mathcal{V}$ is *completely Ramsey* if for finite $A$ and infinite $E$ either there exists an $M \in \mathcal{P}_\infty(E)$ with $\mathcal{P}_\infty(A,M) \subset \mathcal{V}$ or there exists $M \in \mathcal{P}_\infty(E)$ with $\mathcal{P}_\infty(A,M) \cap \mathcal{V} = \emptyset$.

The main result in this section is a theorem of Galvin and Prikry [97] that says that a set that is Borel for the Ellentuck topology is completely Ramsey. In particular, this implies that a set that is Borel for the Cantor topology is completely Ramsey. Loosely speaking, this means that if we have a subset of $\mathcal{P}_\infty\mathbb{N}$ that may be defined by countably many conditions, then we expect it to be completely Ramsey. This is very useful, as we shall see, because most sets that arise in analysis are of this type. In fact, we will use only the special case of open sets for the Cantor topology, and this follows from the next result.

**Theorem 11.1.2.** *Suppose $\mathcal{U}$ is an Ellentuck-open set in $\mathcal{P}_\infty\mathbb{N}$. Then $\mathcal{U}$ is completely Ramsey.*

*Proof.* Let us introduce some notation. If $A$ is finite and $E$ is infinite, we shall say that $(A,E)$ is a *pair*. The pair $(A,E)$ is *good* (for $\mathcal{U}$) if there is an infinite subset $M$ of $E$ with $P_\infty(A,M) \subset \mathcal{U}$. Otherwise, we shall say that $(A,E)$ is *bad*. Of course, if $(A,E)$ is bad and $F \in \mathcal{P}_\infty(E)$, then $(A,F)$ is also bad. Notice also that if the symmetric difference $E\Delta F$ is finite, then $(A,E)$ and $(A,F)$ are either both good or both bad. We will show that if $(A,E)$ is bad, then there exists $M \in \mathcal{P}_\infty(E)$ with the property that $\mathcal{P}_\infty(A,M) \cap \mathcal{U} = \emptyset$. To achieve this we do not use the fact that $\mathcal{U}$ is Ellentuck open until the very last step.

*Step 1.* Suppose $(A_j)_{j=1}^m$ are finite sets and $E$ is an infinite set such that the pair $(A_j,E)$ is bad for $1 \le j \le m$. Then we claim that we can find $n \in E \setminus \cup_{j=1}^m A_j$ and $F \in \mathcal{P}_\infty(E)$ such that the pair $(A_j \cup \{n\}, F)$ is also bad for $1 \le j \le m$.

Suppose this is false. Then we may inductively pick an increasing sequence $(n_k)_{k=1}^\infty$, a decreasing sequence of infinite sets $(E_k)_{k=0}^\infty$ with $E_0 = E$, and a sequence $(p(k))_{k=1}^\infty$ of integers with $1 \le p(k) \le m$ such that $n_k \in E_{k-1} \setminus \cup_{j=1}^m A_j$ and $\mathcal{P}_\infty(A_{p(k)} \cup \{n_k\}, E_k) \subset \mathcal{U}$.

Now, there exists $1 \le p \le n$ such that the set $\{k \in \mathbb{N} : p(k) = p\}$ is infinite. Let $M = \{n_k : p(k) = p\}$. Suppose $G \in \mathcal{P}_\infty(A_p, M)$. Let $k$ be the least integer such that $n_k \in G$. Then $G \in \mathcal{P}_\infty(A_{p(k)} \cup \{n_k\}, E_k) \subset \mathcal{U}$. Hence $\mathcal{P}_\infty(A_p, M) \subset \mathcal{U}$, contradicting our hypothesis.

*Step 2.* We show that if a pair $(A,E)$ is bad, we can find $M \in \mathcal{P}_\infty(E)$ such that the pair $(B,M)$ is bad for *every* finite set $B$ with $A \subset B \subset A \cup M$.

This is achieved again by an inductive construction. To start the induction we use Step 1. Set $E_0 = E$; there exist $n_1 \in E_0$ and an infinite set $E_1 \in \mathcal{P}_\infty(E_0)$ for which the pair $(B,E_1)$ is bad if $A \subset B \subset A \cup \{n_1\}$. Suppose we have chosen sets $E_0, E_1, \dots, E_k$ with $E_j \subset E_{j-1}$ for $1 \le j \le n$, and integers $n_1, n_2, \dots, n_k$ with $n_j \in E_{j-1}$ for $1 \le j \le n$, such that $(B,E_j)$ is bad if $A \subset B \subset A \cup \{n_1, \dots, n_j\}$ for $1 \le j \le n$. Then, according to Step 1, we can find $n_{k+1} \in E_k$ with $n_{k+1} > n_k$ and $E_{k+1} \subset E_k$ such that $(B \cup \{n_{k+1}\}, E_{k+1})$ is bad for every $A \subset B \subset A \cup \{n_1, \dots, n_k\}$.

It remains to show that $M = \{n_1, n_2, \dots\}$ has the desired property. If $B$ is a finite subset of $A \cup M$, let $k$ be the largest natural number such that $n_k \in B$. Then $B \subset A \cup \{n_1, \dots, n_k\}$, so that $(B, E_k)$ is bad. However, $M \subset E_k \cup \{n_1, \dots, n_k\}$, so $(B, M)$ is also bad.

*Step 3.* Let us complete the proof, recalling finally that $\mathcal{U}$ is supposed Ellentuck open. If a pair $(A, E)$ is bad, we determine $M \subset E$ according to Step 2 such that $(B, M)$ is bad whenever $B$ is finite and $A \subset B \subset A \cup M$. Suppose $\mathcal{P}_\infty(A, M)$ meets $\mathcal{U}$, so there exists $G \in \mathcal{P}_\infty(A, M) \cap \mathcal{U}$. Since $\mathcal{U}$ is open, there exists a finite set $B$, which can be assumed to contain $A$, such that $\mathcal{P}_\infty(B, G) \subset \mathcal{U}$. This implies that $(B, M)$ is good, and we have reached a contradiction. Hence the only possible conclusion is that $\mathcal{P}_\infty(A, M) \cap \mathcal{U} = \emptyset$. □

Now we come to the theorem of Galvin and Prikry [97] mentioned before.

**Theorem 11.1.3.** *Let $\mathcal{V}$ be a subset of $\mathcal{P}_\infty\mathbb{N}$ that is Borel for the Ellentuck topology. Then $\mathcal{V}$ is completely Ramsey.*

*Proof.* We first remark that if $\mathcal{U}$ is dense and open for the Ellentuck topology, then Theorem 11.1.2 yields that for every pair $(A, E)$ there exists $M \in \mathcal{P}_\infty(E)$ with $\mathcal{P}_\infty(A, M) \subset \mathcal{U}$. This is because there is no pair $(A, M)$ with $\mathcal{P}_\infty(A, M) \cap \mathcal{U} = \emptyset$.

*Step 1.* We claim that for every pair $(A, E)$, if $B \subset E$ is finite, then there exists $M \in \mathcal{P}_\infty(B, E)$ such that $\mathcal{P}_\infty(A, M) \subset \mathcal{U}$.

Indeed, we list all subsets $(B_j)_{j=1}^N$ of $B$. Find $H_1 \in \mathcal{P}_\infty(E)$ such that $\mathcal{P}_\infty(A \cup B_1, H_1) \subset \mathcal{U}$ and then inductively $H_j \in \mathcal{P}_\infty(H_{j-1})$ such that $\mathcal{P}_\infty(A \cup B_j, H_j) \subset \mathcal{U}$. Finally, let $M = H_N$. If $G \in \mathcal{P}_\infty(A, M)$, let $G \cap B = B_j$. Then $G \in \mathcal{P}_\infty(A \cup B_j, M) \subset \mathcal{P}_\infty(A \cup B_j, H_j) \subset \mathcal{U}$.

*Step 2.* Suppose $\mathcal{G}$ is the intersection of a countable family of open dense sets for the Ellentuck topology. Then we can find a descending sequence of dense open sets $(\mathcal{U}_n)_{n=1}^\infty$ with $\mathcal{G} = \cap_{n=1}^\infty \mathcal{U}_n$. We will show that if $(A, E)$ is any pair, we can find $M \in \mathcal{P}_\infty(E)$ such that $\mathcal{P}_\infty(A, M) \subset \mathcal{G}$.

As usual, we inductively pick an increasing sequence of integers $(n_k)_{k=1}^\infty$ and a descending sequence of infinite sets $(E_k)_{k=0}^\infty$ with $E_0 = E$ such that $n_k \in E_j$ for all $j$ and $\mathcal{P}_\infty(A, E_k) \subset \mathcal{U}_k$. We pick $n_1 \in E_0$ arbitrarily and let $E_1 \subset E_0$ be such that $n_1 \in E_1$ and $\mathcal{P}_\infty(A, E_1) \subset \mathcal{U}_1$. If $n_1, \dots, n_{k-1}, E_1, \dots, E_{k-1}$ have been picked we choose $n_k \in E_{k-1}$ with $n_k > n_{k-1}$ and then use Step 1 to pick $E_k \subset E_{k-1}$ such that $\{n_1, \dots, n_k\} \subset E_k$ and $\mathcal{P}_\infty(A, E_k) \subset \mathcal{U}_k$.

Finally, let $M = \{n_1, n_2, \dots\}$. If $G \in \mathcal{P}_\infty(A, M)$, then for every $k$, we have $G \in \mathcal{P}_\infty(A, E_k)$, which implies $G \in \mathcal{U}_k$. Hence $G \in \mathcal{G}$.

*Step 3.* Let us complete the proof by supposing that $\mathcal{V}$ is a Borel set for the Ellentuck topology. Then there is a set $\mathcal{G}$ that is the intersection of a sequence of dense open sets $(\mathcal{U}_n)_{n=1}^\infty$, so that $\mathcal{G} \cap \mathcal{V} = \mathcal{G} \cap \mathcal{U}$ for some Ellentuck open set $\mathcal{U}$ (see the problems). If $(A, E)$ is any pair, we may first find $G \in \mathcal{P}_\infty(E)$ such that $\mathcal{P}_\infty(A, G) \subset \mathcal{G}$ by Step 2. Now there exists $M \in \mathcal{P}_\infty(G)$ such that either $\mathcal{P}_\infty(A, M) \subset \mathcal{U}$ or $\mathcal{P}_\infty(A, M) \cap \mathcal{U} = \emptyset$. But then either $\mathcal{P}_\infty(A, M) \subset \mathcal{V}$ or $\mathcal{P}_\infty(A, M) \cap \mathcal{V} = \emptyset$. □

## 11.2 Rosenthal's $\ell_1$ Theorem

The motivation for the main result in this section comes from the problem of finding a criterion for being able to extract a weakly Cauchy subsequence from any bounded sequence in a Banach space $X$. If $X$ is reflexive, this follows from the Eberlein–Šmulian theorem. What if $X$ is not reflexive? It was known to Banach that if $X^*$ is separable, then every bounded sequence in $X$ has a weakly Cauchy subsequence. But in other spaces this is not possible.

For instance, the canonical basis $(e_n)_{n=1}^\infty$ of $\ell_1$ has no weakly Cauchy subsequences. Rosenthal's $\ell_1$ theorem says that in some sense, this is the only possible example. Rosenthal proved this for real Banach spaces, and the necessary modifications for complex Banach spaces were given shortly afterward by Dor [72]. Our proof will work for both real and complex scalars.

**Theorem 11.2.1 (Rosenthal's $\ell_1$ Theorem [273]).** *Let $(x_n)_{n=1}^\infty$ be a bounded sequence in an infinite-dimensional Banach space X. Then either:*

*(a) $(x_n)_{n=1}^\infty$ has a subsequence that is weakly Cauchy, or*
*(b) $(x_n)_{n=1}^\infty$ has a subsequence that is basic and equivalent to the canonical basis of $\ell_1$.*

*Proof.* Let $(x_n)_{n=1}^\infty$ be a bounded sequence in a Banach space $X$ that has no weakly Cauchy subsequence. We will suppose that $\|x_n\| \le 1$ for all $n$. We begin by passing to a subsequence that is basic. This is achieved by Theorem 1.5.6, since obviously, the set $\{x_n\}_{n=1}^\infty$ does not have any weakly convergent subsequences. Thus we can assume that the sequence $(x_n)_{n=1}^\infty$ is already basic.

If $M$ is any infinite subset of $\mathbb{N}$, in order to measure how far the sequence of elements in $M$ is from being weakly Cauchy, we define

$$\operatorname{osc}(M) = \sup_{\|x^*\|\le 1} \lim_{k\to\infty} \sup_{\substack{m,n>k \\ m,n\in M}} |x^*(x_m) - x^*(x_n)|.$$

We claim that there exists $M \in \mathcal{P}_\infty \mathbb{N}$ such that if $M' \in \mathcal{P}_\infty(M)$, then $\operatorname{osc}(M') = \operatorname{osc}(M) > 0$.

Indeed, let us inductively define infinite sets $\mathbb{N} = M_0 \supset M_1 \supset M_2 \supset M_3 \supset \dots$ such that

$$\operatorname{osc}(M_k) < \inf_{M' \in \mathcal{P}_\infty(M_{k-1})} \operatorname{osc}(M') + k^{-1}, \qquad k = 1, 2, \dots.$$

Let $M$ be chosen by a diagonal procedure such that $M \subset M_k \cup F_k$, where each $F_k$ is finite. Then $M$ has the desired property that $\operatorname{osc}(M') = \operatorname{osc}(M)$ if $M' \in \mathcal{P}_\infty(M)$. Then, $\operatorname{osc}(M) > 0$ follows from the fact there is no weakly Cauchy subsequence.

We may make one further reduction by finding $u^* \in B_{X^*}$ and $M' \subset M$ such that $\lim_{n\in M'} u^*(x_n) = \theta$, where $|\theta| \ge \frac{1}{2}\operatorname{osc}(M)$.

Again for convenience of notation we may suppose that the original sequence has these properties, i.e., $\operatorname{osc}(M) = 4\delta > 0$ is constant for every infinite set $M$ and $\lim_{n\to\infty} u^*(x_n) = \theta$ for some $u^* \in B_{X^*}$ and $|\theta| > \delta$.

Since $(x_n)_{n=1}^\infty$ is basic and bounded away from zero, there exist biorthogonal functionals $(x_n^*)_{n=1}^\infty$ in $X^*$, and we have a bound $\|x_n^*\| \le B$ for some constant $B$.

Let $C = 1+\delta^{-1}+\delta^{-2}$. Let us consider the subset $\mathcal{V}$ of $\mathcal{P}_\infty\mathbb{N}$ of all $M = \{m_j\}_{j=1}^\infty$, where $(m_j)_{j=1}^\infty$ is strictly increasing such that there exists $x^* \in X^*$ with $\|x^*\| \le C$ and $x^*(x_{m_j}) = (-1)^j$ for all $j$.

It follows immediately from the weak$^*$ compactness of $\{x^* : \|x^*\| \le C\}$ that the set $\mathcal{V}$ is closed for the Cantor topology, and hence closed for the Ellentuck topology. Thus, $\mathcal{V}$ has the Ramsey property (note here that we use only Theorem 11.1.2).

Suppose $M$ is any infinite subset of $\mathbb{N}$. Since $\operatorname{osc}(M) = \delta$, we can find a subsequence $(m_j)_{j=1}^\infty$ of $M$ such that for some $y^* \in B_{X^*}$ we have $\lim_{j\to\infty} y^*(x_{m_{2j}}) = \alpha$ and $\lim_{j\to\infty} y^*(x_{m_{2j-1}}) = \beta$, where $|\alpha-\beta| \ge 2\delta$. Next let

$$v^* = \frac{2}{(\alpha-\beta)}y^* - \frac{\alpha+\beta}{\theta(\alpha-\beta)}u^*.$$

Then

$$\|v^*\| \le (1+\theta^{-1})\delta^{-1} \le \delta^{-1}+\delta^{-2}$$

and

$$\lim_{j\to\infty} v^*(x_{m_{2j}}) = 1, \quad \lim_{j\to\infty} v^*(x_{m_{2j-1}}) = -1.$$

By passing to a further subsequence, we can suppose that if $c_j = v^*(x_{m_j}) - (-1)^j$, then $|c_j| \le 2^{-j}B^{-1}$. Then consider

$$x^* = v^* + \sum_{j=1}^\infty c_j x_{m_j}^*.$$

We have

$$\|x^*\| \le 1+\delta^{-1}+\delta^{-2} = C.$$

Further, $x^*(x_{m_j}) = (-1)^j$.

It follows that $M' \in \mathcal{V}$, and thus there is no $M$ such that $\mathcal{P}_\infty(M)\cap\mathcal{V} = \emptyset$. Hence there is an infinite subset $M$ such that every $M' \in \mathcal{P}_\infty(M)$ is in $\mathcal{V}$.

Let $M = \{m_j\}_{j=1}^\infty$, where $(m_j)$ is increasing. Then the sequence $(m_{2j})_{j=1}^\infty$ has the property that for every sequence of signs $(\epsilon_j)$, we can find $x^*$ with $\|x^*\| \le C$ and $x^*(x_{m_{2j}}) = \epsilon_j$.

If $X$ is real, it is clear that for every sequence of scalars $(a_j)_{j=1}^n$, we can pick $\epsilon_j = \pm 1$ with $\epsilon_j a_j = |a_j|$ and then find $x^* \in X^*$ with $\|x^*\| \leq C$ such that $x^*(x_{m_{2j}}) = \epsilon_j$. Thus,

$$\Big\| \sum_{j=1}^n a_j x_{m_{2j}} \Big\| \geq \frac{1}{C} \sum_{j=1}^n |a_j|,$$

and so $(x_{m_{2j}})_{j=1}^\infty$ is equivalent to the canonical $\ell_1$-basis.

If $X$ is complex, the same reasoning shows that

$$\Big\| \sum_{j=1}^n a_j x_{m_{2j}} \Big\| \geq \frac{1}{C} \sum_{j=1}^n |\Re a_j|,$$

and similarly,

$$\Big\| \sum_{j=1}^n a_j x_{m_{2j}} \Big\| \geq \frac{1}{C} \sum_{j=1}^n |\Im a_j|.$$

Thus,

$$\Big\| \sum_{j=1}^n a_j x_{m_{2j}} \Big\| \geq \frac{1}{2C} \sum_{j=1}^n |a_j|.$$

□

**Corollary 11.2.2.** *A Banach space $X$ contains no copy of $\ell_1$ if and only if every bounded sequence in $X$ has a weakly Cauchy subsequence.*

*Remark 11.2.3.* If $X^*$ is separable, then $X$ cannot contain a copy of $\ell_1$. However, it is not easy to construct a separable Banach space for which $X^*$ is nonseparable but $X$ fails to contain a copy of $\ell_1$. This was done by James [133], who produced an example called the *James tree space*, $\mathcal{JT}$. We postpone the construction of this example to Chapter 15.

If $X$ is separable, there is a very fine distinction between the conditions that (a) $X^*$ is separable and (b) $X$ does not contain $\ell_1$. Let us illustrate this. If $X^*$ is separable, then the weak$^*$ topology on $B_{X^{**}}$ is a metrizable topology, and thus Goldstine's theorem guarantees that for every $x^{**} \in B_{X^{**}}$ there is a *sequence* $(x_n)_{n=1}^\infty$ in $B_X$ converging to $x^{**}$ weak$^*$ (this sequence is, of course, a weakly Cauchy sequence in $X$).

If $X$ does not contain $\ell_1$ but $X^*$ is not separable, then the weak$^*$ topology is no longer metrizable, yet remarkably, the same conclusion holds (this is due to Odell and Rosenthal [230]):

**Theorem 11.2.4.** *Let $X$ be a separable Banach space. Then $\ell_1$ does not embed into $X$ if and only if every $x^{**} \in X^{**}$ is the weak$^*$ limit of a sequence $(x_n)_{n=1}^\infty$ in $X$.*

## 11.3 Tsirelson Space

The question we want to address in this section is whether every Banach space contains a copy of one of the spaces $\ell_p$ for $1 \leq p < \infty$, or $c_0$. The motivation behind this question is that these spaces (which are prime!) appear to be in a certain sense the fundamental blocks from which all Banach spaces are constructed. Indeed, every space we have met so far contains one of these blocks. For example, every subspace of $\ell_p$ contains a copy of $\ell_p$. We also have seen that every subspace of $L_p$ for $p > 2$ contains a copy of one of the spaces $\ell_p$ or $\ell_2$ (Theorem 6.4.8). The case of subspaces of $L_p$ for $1 \leq p < 2$ is much more difficult and was not resolved until 1981, by Aldous. He showed [10] that every subspace of $L_p$ for $1 \leq p < 2$ also contains a copy of some $\ell_q$; Krivine and Maurey [183] subsequently gave an alternative argument based on the notion of stability. Nevertheless, the result is still not so easy and is beyond the scope of this book.

It was quite a surprise when in 1974, Tsirelson gave the first example of a Banach space not containing some $\ell_p$ $(1 \leq p < \infty)$ or $c_0$. Nowadays, the dual of the space constructed by Tsirelson has become known as *Tsirelson space*. Despite its apparently strange definition, it has turned out to be a remarkable springboard for further research.

Before getting to Tsirelson space we will need a result of James from 1964 [129]. He showed that if $\ell_1$ embeds in a Banach space, then it must embed very well (close to isometrically). This result, although quite simple, is also very significant, as we will discuss later.

**Theorem 11.3.1 (James's $\ell_1$ distortion theorem).** *Let $(x_n)_{n=1}^\infty$ be a normalized basic sequence in a Banach space $X$ that is equivalent to the canonical $\ell_1$-basis. Then given $\epsilon > 0$, there is a normalized block basic sequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that*

$$\Big\| \sum_{k=1}^N a_k y_k \Big\| \geq (1-\epsilon) \sum_{k=1}^N |a_k|$$

*for every sequence of scalars $(a_k)_{k=1}^N$.*

*Proof.* For each $n$ let $M_n$ be the least constant such that if $(a_k)_{k=1}^\infty \in c_{00}$ with $a_k = 0$ for $k \leq n$, then

$$\sum_{k=1}^\infty |a_k| \leq M_n \Big\| \sum_{k=1}^\infty a_k x_k \Big\|.$$

Then $(M_n)_{n=1}^\infty$ is a decreasing sequence with $\lim_{n\to\infty} M_n = M \geq 1$. Thus, for $n$ large enough, $M_n < (1-\epsilon)^{-\frac{1}{2}} M$.

Now we can pick a normalized block basic sequence $(y_n)_{n=1}^\infty$ of the form

$$y_n = \sum_{j=p_{n-1}+1}^{p_n} b_j x_j$$

such that

$$\sum_{j=p_{n-1}+1}^{p_n} |b_j| \geq (1-\epsilon)^{\frac{1}{2}} M, \qquad n = 1, 2, \ldots,$$

and such that $M_{p_0} < (1-\epsilon)^{-\frac{1}{2}} M$. Then,

$$\begin{aligned}\sum_{j=1}^{N} |a_j| &\leq (1-\epsilon)^{-\frac{1}{2}} M^{-1} \sum_{j=1}^{N} |a_j| \sum_{i=p_{j-1}+1}^{p_j} |b_i| \\ &\leq (1-\epsilon)^{-\frac{1}{2}} M^{-1} M_{p_0} \Big\| \sum_{j=1}^{N} a_j y_j \Big\| \\ &\leq (1-\epsilon)^{-1} \Big\| \sum_{j=1}^{N} a_j y_j \Big\|,\end{aligned}$$

and the result is proved. □

Next we construct Tsirelson's space. This is, as mentioned above, not the original space constructed by Tsirelson in 1974 [297] but its dual, as constructed by Figiel and Johnson [94].

**Theorem 11.3.2.** *There is a reflexive Banach space $\mathcal{T}$ that contains no copy of $\ell_p$ for $1 \leq p < \infty$, or $c_0$.*

*Proof.* Suppose $(I_1, \ldots, I_m)$ is a set of disjoint intervals of natural numbers. We say that $(I_1, \ldots, I_m)$ is *admissible* if $m < I_k$ for $k = 1, 2 \ldots, m$, i.e., each $I_k$ is contained in $[m+1, \infty)$.

We will adopt the convention that if $E$ is a subset of $\mathbb{N}$ (in particular, if $E$ is an interval of integers) and $\xi \in c_{00}$, we will write $E\xi$ for the sequence $(\chi_E(n)\xi(n))_{n=1}^{\infty}$, i.e., the sequence whose coordinates are $E\xi(n) = \xi(n)$ if $n \in E$ and $E\xi(n) = 0$ otherwise.

We define a norm, $\|\cdot\|_{\mathcal{T}}$, on $c_{00}$ by the formula

$$\|\xi\|_{\mathcal{T}} = \max\left\{ \|\xi\|_{c_0}, \sup \frac{1}{2} \sum_{j=1}^{m} \|I_j \xi\|_{\mathcal{T}} \right\}, \tag{11.1}$$

the supremum being taken over all admissible families of intervals. This definition is implicit, and we need to show that there is such a norm. But that follows by a relatively easy inductive argument. Let $\|\xi\|_0 = \|\xi\|_{c_0}$ and then define inductively for $n = 1, 2, \ldots,$

$$\|\xi\|_n = \max\left\{ \|\xi\|_{c_0}, \sup \frac{1}{2} \sum_{j=1}^{m} \|I_j \xi\|_{n-1} \right\},$$

where again, the supremum is taken over all admissible families of intervals. The sequence $(\|\xi\|_n)_{n=1}^{\infty}$ is increasing and bounded above by $\|\xi\|_{\ell_1}$. Hence it converges to some $\|\xi\|_{\mathcal{T}}$, and it follows readily that $\|\cdot\|_{\mathcal{T}}$ has all the required properties of a norm.

It is necessary also to show that the definition uniquely determines $\|\cdot\|_{\mathcal{T}}$. Indeed, suppose $\|\cdot\|_{\mathcal{T}}^*$ is another norm on $c_{00}$ satisfying (11.1). It is clear from the induction argument that $\|\xi\|_{\mathcal{T}}^* \geq \|\xi\|_{\mathcal{T}}$ for all $\xi \in c_{00}$. For $\alpha > 1$ let

$$S = \{\xi \in c_{00} : \|\xi\|_{\mathcal{T}}^* > \alpha\|\xi\|_{\mathcal{T}}\}.$$

If $S$ is nonempty, it has a member with minimal support. But an appeal to (11.1) gives a contradiction. Hence there is a unique norm on $c_{00}$ that is the solution of (11.1).

Let $\mathcal{T}$ be the completion of $(c_{00}, \|\cdot\|_{\mathcal{T}})$. The canonical unit vectors $(e_n)_{n=1}^{\infty}$ form a 1-unconditional basis of $\mathcal{T}$.

Suppose $\ell_p$ for some $1 < p < \infty$ or $c_0$ embeds in $\mathcal{T}$. Then, by the Bessaga–Pełczyński selection principle (Proposition 1.3.10), there is a normalized block basic sequence $(\xi_n)_{n=1}^{\infty}$ with respect to the canonical basis of $\mathcal{T}$ equivalent to the canonical basis. Suppose we fix $m$ and choose $n$ such that $\xi_n$ is supported in $[m+1, \infty)$. Then

$$\|\xi_n + \cdots + \xi_{n+m-1}\|_{\mathcal{T}} \geq \frac{1}{2}m$$

by the definition of $\|\cdot\|_{\mathcal{T}}$. This contradicts the equivalence with the $\ell_p$-basis (or the $c_0$-basis).

Let us show that $\ell_1$ cannot be embedded in $\mathcal{T}$. Assume it embeds. Then we can find a normalized block basic sequence equivalent to the $\ell_1$-basis. If $\epsilon < \frac{1}{4}$, by James's $\ell_1$ distortion theorem (Theorem 11.3.1) we pass to a sequence of blocks and assume we have a normalized block basic sequence $(\xi_n)_{n=0}^{\infty}$ such that

$$\Big\|\sum_{j=0}^{n} a_j\xi_j\Big\|_{\mathcal{T}} \geq (1-\epsilon)\sum_{j=0}^{n}|a_j|$$

for every scalars $(a_j)_{j=0}^{n}$.

Suppose $\xi_0$ is supported on $[1, r]$. For every $n$ we have

$$\Big\|\xi_0 + \frac{1}{n}\sum_{j=1}^{n}\xi_j\Big\|_{\mathcal{T}} \geq 2(1-\epsilon).$$

It is clear that

$$\Big\|\xi_0 + \frac{1}{n}\sum_{j=1}^{n}\xi_j\Big\|_{\mathcal{T}} > \Big\|\xi_0 + \frac{1}{n}\sum_{j=1}^{n}\xi_j\Big\|_{c_0},$$

so we must be able to find an admissible collection of intervals $(I_1, \ldots, I_k)$ such that

$$\Big\|\xi_0 + \frac{1}{n}\sum_{i=1}^{n}\xi_i\Big\|_{\mathcal{T}} = \frac{1}{2}\sum_{j=1}^{k}\Big\|I_j\big(\xi_0 + \frac{1}{n}\sum_{i=1}^{n}\xi_i\big)\Big\|_{\mathcal{T}}.$$

If $I_j\xi_0 = 0$ for every $j$, then

$$\frac{1}{2}\sum_{j=1}^{k}\Big\|I_j\big(\xi_0 + \frac{1}{n}\sum_{i=1}^{n}\xi_i\big)\Big\|_{\mathcal{T}} = \frac{1}{2}\sum_{i=1}^{k}\Big\|I_i\big(\frac{1}{n}\sum_{j=1}^{n}\xi_i\big)\Big\|_{\mathcal{T}} \le 1,$$

so we can assume that $I_j\xi_0 \neq 0$ for some $j$. But this means, by admissibility, that $k \le r$. Note that

$$\frac{1}{2}\sum_{j=1}^{k}\Big\|I_j\big(\xi_0 + \frac{1}{n}\sum_{i=1}^{n}\xi_i\big)\Big\|_{\mathcal{T}} \le \frac{1}{2}\sum_{j=1}^{k}\|I_j\xi_0\| + \frac{1}{2n}\sum_{j=1}^{k}\Big\|I_j\Big(\sum_{i=1}^{n}\xi_i\Big)\Big\|_{\mathcal{T}}.$$

The first term is estimated by $\|\xi_0\|_{\mathcal{T}} = 1$. For the second term we have

$$\frac{1}{2n}\sum_{j=1}^{k}\Big\|I_j\Big(\sum_{i=1}^{n}\xi_i\Big)\Big\|_{\mathcal{T}} \le \frac{1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{k}\|I_j\xi_i\|_{\mathcal{T}}.$$

There are at most $k \le r$ values of $i$ such that the support of $\xi_i$ meets at least two $I_j$. For such values of $i$ we have

$$\frac{1}{2n}\sum_{j=1}^{k}\|I_j\xi_i\|_{\mathcal{T}} \le \frac{1}{n}\|\xi_i\|_{\mathcal{T}} = \frac{1}{n}.$$

For all values of $i$ we have

$$\frac{1}{2n}\sum_{j=1}^{k}\|I_j\xi_i\|_{\mathcal{T}} \le \frac{1}{2n}.$$

Hence,

$$\Big\|\xi_0 + \frac{1}{n}\sum_{i=1}^{n}\xi_i\Big\|_{\mathcal{T}} \le 1 + \frac{k}{n} + \frac{n-k}{2n} = 1 + \frac{n+r}{2n}.$$

The right-hand side converges to $3/2$ as $n \to \infty$, and since $3/2 < 2(1-\epsilon)$, we have a contradiction.

By James's theorem (Theorem 3.3.4), since $\mathcal{T}$ contains no copy of $c_0$ or $\ell_1$, it must be reflexive. □

The construction of Tsirelson space was thus a disappointment to those who expected a nice structure theory for Banach spaces. It was, however, far from the end of the story. Tsirelson space (and its modifications) as an example has continued to play an important role in the area since 1974. See the book by Casazza and Shura from 1989 [45].

The major problem left open was the unconditional basic sequence problem, which was discussed at the end of Chapter 3. Tsirelson space played a significant role in the solution of this problem.

There is a curious and deep relationship between the unconditional basic sequence problem and James's $\ell_1$ distortion theorem (Theorem 11.3.1). James's result implies that if we put an equivalent norm $|||\cdot|||$ on $\ell_1$, then we will always be able to find an infinite-dimensional subspace on which this norm is a close multiple of the original norm. Thus, given $\epsilon > 0$, we can find an infinite-dimensional subspace $Y$ of $\ell_1$ and a constant $c > 0$ such that

$$c(1-\epsilon)\|\xi\|_1 \le |||\xi||| \le c(1+\epsilon)\|\xi\|_1, \qquad \xi \in Y.$$

Here $\|\cdot\|_1$ denotes the usual norm on $\ell_1$. James also showed the same property for $c_0$, and a problem arose as to whether a similar result might hold for arbitrary Banach spaces. The construction of Tsirelson space showed this to be false, using an earlier result of Milman [221]. However, it was left unresolved at that time whether one could specify a constant $M$ with the property that for every Banach space $X$ and every equivalent norm $|||\cdot|||$ there exist an infinite-dimensional subspace $Y$ and a constant $c > 0$ such that

$$cM^{-1}\|x\| \le |||x||| \le cM\|x\|, \qquad x \in Y.$$

This was solved negatively by Schlumprecht in 1991. He constructed an example (known nowadays as *Schlumprecht space*) that is a variant of Tsirelson's construction. Using this space, Odell and Schlumprecht [231] showed in 1994 that this property even fails in Hilbert spaces (and most other spaces). The Schlumprecht space was also a key ingredient in the Gowers–Maurey solution of the unconditional basic sequence problem [116].

## Problems

**11.1.** Show that if $X$ is a topological space and $\mathcal{V}$ is a Borel subset of $X$, then there exist a dense $G_\delta$-set $\mathcal{G}$ and an open set $\mathcal{U}$ such that $\mathcal{V}\cap\mathcal{G} = \mathcal{U}\cap\mathcal{G}$ (see Problem 4.7).

**11.2 (Johnson).** Let $(x_n)_{n=1}^\infty$ be a sequence in a Banach space $X$ with the property that every subsequence $(x_{n_k})_{k=1}^\infty$ contains a further subsequence $(x_{n_{k_j}})_{j=1}^\infty$ such that

$$\sup_{n\ge 1}\Big\|\sum_{j=1}^n(-1)^j x_{n_{k_j}}\Big\| < \infty.$$

Show that $(x_n)_{n=1}^\infty$ has a subsequence $(y_n)_{n=1}^\infty$ such that $\big(\sum_{j=1}^n y_j\big)_{n=1}^\infty$ is a WUC series. In particular, if $(x_n)_{n=1}^\infty$ is normalized, deduce that $(x_n)_{n=1}^\infty$ has a subsequence equivalent to the canonical basis of $c_0$.

**11.3 (James Distortion Theorem for $c_0$).** Let $(x_n)_{n=1}^\infty$ be a normalized basic sequence in a Banach space $X$ equivalent to the canonical $c_0$-basis. Show that given $\epsilon > 0$, there is a normalized block basic sequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that

$$\Big\| \sum_{k=1}^N a_k y_k \Big\| \geq (1-\epsilon) \max_k |a_k|$$

for every sequence of scalars $(a_k)_{k=1}^N$.

**11.4.** (a) Let $X$ be a nonreflexive Banach space and suppose $x^{**} \in X^{**} \setminus X$. Show that if $\epsilon > 0$, $V$ is a weak* neighborhood of $x^{**}$, and $x_1, \dots, x_n \in X$, there exists $x \in V \cap X^{**}$ such that

$$\Big| \|x_j + x^{**}\| - \|x_j + x\| \Big| < \epsilon, \qquad j = 1, 2, \dots, n.$$

(b) Show that if $X$ is a nonreflexive Banach space such that for some $x^{**} \in X^{**}$ we have $\|x^{**} + x\| = \|x^{**} - x\|$ for every $x \in X$, then $X$ contains a copy of $\ell_1$. [*Hint*: Use (a) and an inductive construction to find a basic sequence equivalent to the canonical $\ell_1$-basis.]

Part (b) is due to Maurey [213], who also proved the more difficult converse: if $X$ is *separable* and contains a copy of $\ell_1$, then there exists $x^{**} \in X^{**}$ with $\|x^{**} + x\| = \|x^{**} - x\|$ for all $x \in X$.

**11.5.** Show that Tsirelson space contains no symmetric basic sequence.

**11.6.** Let $|||\cdot|||$ be the norm on $c_{00}$ obtained by the implicit formula

$$|||\xi||| = \max\left( \|\xi\|_\infty, \sup \sum_{j=1}^{2n} |||I_j \xi||| \right),$$

where the supremum is over all $n$ and all collections of intervals $(I_j)_{j=1}^{2n}$ with $n < I_1 < I_2 < \cdots < I_{2n}$ (i.e., using $2n$ instead of $n$ in the definition of $\mathcal{T}$).

At the same time define two associated norms by

$$\|\xi\|_{T,1} = \sup \Big\{ \sum_{j=1}^{3} \|I_j \xi\|_{\mathcal{T}} \Big\},$$

where $(I_j)_{j=1}^3$ ranges over all triples of intervals $I_1 < I_2 < I_3$, and

$$\|\xi\|_{T,2} = \sup \Big\{ \sum_{j=1}^{8k} \|I_j \xi\|_{\mathcal{T}} \Big\},$$

the supremum being taken over all $k$ and all collections of intervals $(I_j)_{j=1}^{8k}$ such that $k < I_1 < I_2 < \cdots < I_{8k}$.

(a) Show that $\|\xi\|_{\mathcal{T},2} \le \|\xi\|_{\mathcal{T},1} \le 3\|\xi\|_{\mathcal{T}}$.
(b) Show by induction on the size of the support that

$$|||\xi||| \le \|\xi\|_{\mathcal{T},1}$$

and deduce that

$$\|\xi\|_{\mathcal{T}} \le |||\xi||| \le 3\|\xi\|_{\mathcal{T}}.$$

(c) Show that $\mathcal{T}$ is isomorphic to $\mathcal{T}^2$.

**11.7 (Casazza et al. [46]).** Let $J_1, \dots, J_m$ be disjoint intervals and suppose $\xi, \eta \in c_{00}$ are supported on $\cup_{j=1}^m J_k$ and satisfy $\|J_j\xi\|_{\mathcal{T}} = \|J_j\eta\|_{\mathcal{T}}$ for $1 \le j \le m$. The goal of this exercise is to show the following inequality:

$$\frac{1}{6}\|\xi\|_{\mathcal{T}} \le \|\eta\|_{\mathcal{T}} \le 6\|\xi\|_{\mathcal{T}}. \tag{11.2}$$

To this end, first we will show by induction on $m$ that $\|\xi\|_{\mathcal{T}} \le 2|||\eta|||$, where $|||\cdot|||$ is the norm we introduced in Problem 11.6. Suppose then that this has been proved for all collections of $m-1$ intervals, and $\xi$ and $\eta$ are given as above.

(a) Consider an admissible collection of intervals $n < I_1 < \cdots < I_n$. Let $\mathbb{A}$ be the set of all $j$ such that $J_j$ meets more than one $I_k$, together with the first $l$ such that $J_l$ meets at least one $I_k$.
Show that $|\mathbb{A}| \le n$, and that for each $j \in \mathbb{A}$,

$$\sum_{k=1}^{n} \|(I_k \cap J_j)\xi\|_{\mathcal{T}} \le 2\|J_j\eta\|_{\mathcal{T}}.$$

(b) Let $I_k' = I_k \setminus \cup_{j\in\mathbb{A}} J_j$ and

$$I_k'' = I_k' \cup \bigcup\{J_j : \; J_j \cap I_k' \neq \emptyset\}.$$

Show that $(I_k'')_{k=1}^n$ is admissible, and using the inductive hypothesis, show that

$$\|I_k''\xi\|_{\mathcal{T}} \le 2|||I_k''\eta|||, \qquad k = 1, 2, \dots, n.$$

(c) Complete the inductive proof that $\|\xi\|_{\mathcal{T}} \le 2|||\eta|||$.
(d) Prove the inequality (11.2).

**11.8 (Casazza et al. [46]).** Show that every block basic sequence in $\mathcal{T}$ is complemented. [*Hint*: Use the previous problem.]

# Chapter 12
# Finite Representability of $\ell_p$-Spaces

We are now going to switch gears and study local properties of infinite-dimensional Banach spaces. In Banach space theory the word *local* is used to denote finite-dimensional. We can distinguish between properties of a Banach space that are determined by its finite-dimensional subspaces and properties that require understanding of the whole space. For example, one cannot decide that a space is reflexive just by looking at its finite-dimensional subspaces, but properties like type and cotype that depend on inequalities with only finitely many vectors are local in character.

The key idea of the chapter is that while a Banach space need not contain any subspace isomorphic to a space $\ell_p$ $(1 \leq p < \infty)$ or $c_0$, as was shown by the existence of Tsirelson space, it will always contain such a space *locally*. The precise meaning of this statement will be made clear shortly.

There are two remarkable results of this nature due to Dvoretzky [80] from 1961 and Krivine [181] from 1976 that are the highlights of the chapter. The methods we use in this chapter are curiously infinite-dimensional in essence, although the results are local. In the following chapter we will consider a local and more quantitative approach to Dvoretzky's theorem.

## 12.1 Finite Representability

In this section we present the notions of finite representability and ultraproducts. Finite representability emerged as a concept in the Banach space scene in the late 1960s; it was originally introduced by James [131].

**Definition 12.1.1.** Let $X$ and $Y$ be infinite-dimensional Banach spaces. We say that $X$ is *finitely representable* in $Y$ if given any finite-dimensional subspace $E$ of $X$ and



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$\epsilon > 0$ there exist a finite-dimensional subspace $F$ of $Y$ with $\dim F = \dim E$ and a linear isomorphism $T : E \to F$ satisfying $\|T\|\|T^{-1}\| < 1 + \epsilon$; that is, in terms of the Banach–Mazur distance, $d(E, F) < 1 + \epsilon$.

*Example 12.1.2.* Every Banach space $X$ (not necessarily separable) is finitely representable in $c_0$. Indeed, given any finite-dimensional subspace $E$ of $X$ and $\epsilon > 0$, pick $\nu$ such that $\frac{1}{1-\nu} < 1+\epsilon$ and $\{e_1^*, \dots, e_N^*\}$ a $\nu$-net in $B_{E^*}$. Consider the mapping $T : E \to \ell_\infty^N$ defined by $T(e) = (e_j^*(e))_{j=1}^N$. Then, if we let $F = T(E)$, it is straightforward to check that $d(E, F) < 1 + \epsilon$.

*Remark 12.1.3.* (a) In Definition 12.1.1 we can assume that $\|T^{-1}\| = 1$ and $\|T\| < 1 + \epsilon$ (or vice versa) by replacing $T$ with a suitable multiple.

(b) If $X$ is finitely representable in $Y$, $X$ need not be isomorphic to a subspace of $Y$. For instance, Example 12.1.2 yields that $\ell_\infty$ is finitely representable in $c_0$, but it does not linearly embed in $c_0$. Another example is provided by $L_p$ $(1 \le p < \infty)$, which, despite the fact that it does not embed in $\ell_p$, is finitely representable in $\ell_p$, as we will see in Proposition 12.1.8.

**Proposition 12.1.4.** *If $X$ is finitely representable in $Y$, and $Y$ is finitely representable in $Z$, then $X$ is finitely representable in $Z$.*

*Proof.* Suppose $E$ is a finite-dimensional subspace of $X$ and $\epsilon > 0$. Then there exists a finite-dimensional subspace $F$ of $Y$ with $d(E,F) < (1 + \epsilon)^{1/2}$. Similarly, we can find a finite-dimensional subspace $G$ of $Z$ such that $d(F, G) < (1 + \epsilon)^{1/2}$. Then $d(E, G) \le d(E, F)d(F, G) < 1 + \epsilon$. □

**Definition 12.1.5.** An infinite-dimensional Banach space $X$ is said to be *crudely finitely representable (with constant $\lambda$)* in an infinite-dimensional Banach space $Y$ if there is a constant $\lambda > 1$ such that given any finite-dimensional subspace $E$ of $X$ there exist a finite-dimensional subspace $F$ of $Y$ with $\dim F = \dim E$ and a linear isomorphism $T\colon E \to F$ satisfying $\|T\|\|T^{-1}\| < \lambda$.

Thus $X$ is finitely representable in $Y$ if and only if $X$ is crudely finitely representable in $Y$ with constant $\lambda$ for every $\lambda > 1$.

We begin our study by considering finite representability in Hilbert spaces.

**Theorem 12.1.6.** *Given a (not necessarily separable) Banach space $X$, the following are equivalent:*

(*i*) *$X$ is crudely finitely representable in $\ell_2$.*
(*ii*) *There exists a constant $\lambda$ such that $d_E \le \lambda$ for every finite-dimensional subspace $E \subset X$.*
(*iii*) *$X$ is isomorphic to a Hilbert space.*

*Proof.* (*iii*) $\Rightarrow$ (*i*) is obvious. (*i*) $\Rightarrow$ (*ii*) follows from the (also obvious) fact that every $n$-dimensional subspace of $\ell_2$ is isometric to $\ell_2^n$. To obtain (*iii*) under the assumption of (*ii*), it suffices to notice that if (*ii*) holds, then $T_2(X)C_2(X) \le \lambda^2$, and then appeal to Theorem 7.4.7. □

**Lemma 12.1.7.** *Suppose $X$ is a separable Banach space and that $(E_n)_{n=1}^{\infty}$ is an increasing sequence of subspaces of $X$ such that $\cup_{n=1}^{\infty} E_n$ is dense in $X$.*

*(i) $X$ is finitely representable in a Banach space $Y$ if and only if given $n \in \mathbb{N}$ and $\epsilon > 0$, there exist a finite-dimensional subspace $F$ of $Y$ with $\dim F = \dim E_n$ and a linear isomorphism $T_n : E_n \to F$ satisfying $\|T_n\|\|T_n^{-1}\| < 1 + \epsilon$.*

*(ii) Let $\lambda > 1$ and suppose that $X$ has the property that given $n \in \mathbb{N}$, there exist a finite-dimensional subspace $F$ of $Y$ with $\dim F = \dim E_n$ and a linear isomorphism $T_n : E_n \to F$ satisfying $\|T_n\|\|T_n^{-1}\| \le \lambda$. Then, given any $\epsilon > 0$, $X$ is crudely finitely representable in $Y$ with constant $\lambda + \epsilon$.*

*Proof.* It is enough to prove $(ii)$. Suppose $X$ satisfies the property in the hypothesis that $E$ is a finite-dimensional subspace of $X$ and that $(e_j)_{j=1}^{N}$ is a basis of $E$ with basis constant $\mathsf{K_b}$. Then we can find $n$ such that there exist $x_j \in E_n$ for $1 \le j \le N$ with $2\mathsf{K_b} \sum_{j=1}^{n} \|x_j - e_j\| \|e_j\|^{-1} < \epsilon/(2\lambda + \epsilon)$. If we let $F = [x_j\colon 1 \le j \le n]$, the Principle of small perturbations (Theorem 1.3.9) yields $d(E, F) < (\lambda + \epsilon)/\lambda$. It is clear that $d(F, T_n(F)) \le \lambda$ and so $d(E, T_n(F)) < \lambda + \epsilon$. □

One of the reasons for developing the idea of finite representability is that we can express the obvious connection between the function spaces $L_p$ and the sequence spaces $\ell_p$ in this language:

**Proposition 12.1.8.** *The space $L_p$ is finitely representable in $\ell_p$ for $1 \le p < \infty$.*

*Proof.* For each $n$ let $E_n = [\chi_{((k-1)/2^n, k/2^n)}\colon 1 \le k \le 2^n]$ be the subspace spanned in $L_p$ by the characteristic functions on dyadic intervals. Since $E_n$ is isometrically isomorphic to a subspace of $\ell_p$, the proof follows by appealing to Lemma 12.1.7.

□

In fact, a converse statement is also true:

**Theorem 12.1.9.** *Let $X$ be a separable Banach space. If $X$ is finitely representable in $\ell_p$ ($1 \le p < \infty$), then $X$ is isometric to a subspace of $L_p$.*

*Proof.* Let $(x_n)_{n=1}^{\infty}$ be a dense sequence in $B_X$; by making a small perturbation where necessary, we can assume this sequence to be linearly independent in $X$. Let $q$ be the conjugate index of $p$. By hypothesis, for each $n \in \mathbb{N}$ there is a linear operator $T_n : E_n \to \ell_p$, where $E_n = [x_1, \dots, x_n]$, satisfying

$$\|x\| \le \|T_n x\| \le \left(1 + \frac{1}{n}\right)\|x\|, \quad x \in E_n.$$

Let $S : \ell_q \to X$ [respectively, $S : c_0 \to X$ if $q = \infty$] be the operator defined by

$$S(\xi) = \sum_{k=1}^{\infty} 2^{-k/p} \xi(k) x_k,$$

and for each $n$ let $V_n : \ell_q \to \ell_p$ [respectively, $V_n : c_0 \to \ell_p$ when $p = 1$] be given by

$$V_n(\xi) = \sum_{k=1}^{n} 2^{-k/p} \xi(k) T_n(x_k).$$

For $\xi_1, \dots, \xi_l, \eta_1, \dots, \eta_m \in c_{00}$ we would like to estimate the quantity

$$\sum_{i=1}^{l} \|V_n(\xi_i)\|^p - \sum_{i=1}^{m} \|V_n(\eta_i)\|^p.$$

Let $K = B_{\ell_q^*}$ [respectively, $K = B_{c_0^*}$ when $q = \infty$] with the weak* topology, and let $F$ be the continuous function on $K$ defined by

$$F(\xi^*) = \sum_{i=1}^{l} |\xi^*(\xi_i)|^p - \sum_{i=1}^{m} |\xi^*(\eta_i)|^p. \tag{12.1}$$

Note that $F(0) = 0$, so $\max_{s\in K} F(s) \geq 0$. Then, if we let $(e_n^*)_{n=1}^{\infty}$ denote the biorthogonal functionals associated to the canonical basis $(e_n)_{n=1}^{\infty}$ of $\ell_p$, we have

$$\begin{aligned}
\sum_{i=1}^{l} \|V_n(\xi_i)\|^p - \sum_{i=1}^{m} \|V_n(\eta_i)\|^p &= \sum_{j=1}^{\infty} \left( \sum_{i=1}^{l} |e_j^*(V_n\xi_i)|^p - \sum_{i=1}^{m} |e_j^*(V_n\eta_i)|^p \right) \\
&= \sum_{j=1}^{\infty} \left( \sum_{i=1}^{l} |V_n^* e_j^*(\xi_i)|^p - \sum_{i=1}^{m} |V_n^* e_j^*(\eta_i)|^p \right) \\
&\leq \left( \sum_{j=1}^{\infty} \|V_n^* e_j^*\|^p \right) \max_{s\in K} F(s).
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{j=1}^{\infty} \|V_n^* e_j^*\|^p &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |V_n^* e_j^*(e_k)|^p \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |e_j^*(V_n e_k)|^p \\
&= \sum_{k=1}^{n} \|V_n e_k\|^p \\
&= \sum_{k=1}^{n} 2^{-k} \|T_n e_k\|^p \\
&\leq \left(1 + \frac{1}{n}\right)^p \sum_{k=1}^{\infty} 2^{-k} \\
&= \left(1 + \frac{1}{n}\right)^p.
\end{aligned}$$

Hence

$$\sum_{i=1}^{l} \|V_n\xi_i\|^p - \sum_{i=1}^{m} \|V_n\eta_i\|^p \leq \Big(1 + \frac{1}{n}\Big)^p \max_{s\in K} F(s).$$

If we let $n \to \infty$, the left-hand side converges to $\sum_{i=1}^{l} \|S\xi_i\|^p - \sum_{i=1}^{m} \|S\eta_i\|^p$, and so

$$\sum_{i=1}^{l} \|S\xi_i\|^p - \sum_{i=1}^{m} \|S\eta_i\|^p \leq \max_{s\in K} F(s). \tag{12.2}$$

The set of all $F$ of the form (12.1) forms a linear subspace $\mathcal{V}$ of $\mathcal{C}(K)$. It follows from (12.2) that we can unambiguously define a linear functional $\varphi$ on $\mathcal{V}$ by

$$\varphi(F) = \sum_{i=1}^{l} \|S\xi_i\|^p - \sum_{i=1}^{m} \|S\eta_i\|^p,$$

and that $\varphi(F) \leq \max_{s\in K} F(s)$. By the Hahn–Banach theorem there is a probability measure $\mu$ on $K$ such that

$$\varphi(F) = \int_K F\,d\mu, \qquad F \in \mathcal{V}.$$

Now suppose $x \in E = \cup_{n=1}^{\infty} E_n$. Then $S^{-1}x \in c_{00}$ is well defined, since the sequence $(x_n)_{n=1}^{\infty}$ was chosen to be linearly independent. Define $Ux \in \mathcal{C}(K)$ by

$$Ux(\xi^*) = \xi^*(S^{-1}x).$$

The map $U: E \to \mathcal{C}(K)$ is linear, but we also have

$$\|Ux\|_{L_p(K,\mu)} = \|x\|,$$

so $U$ is an isometry of $E$ into $L_p(K, \mu)$, which extends by density to an isometry of $X$ into $L_p(K, \mu)$. □

**Proposition 12.1.10** **($L_q$-subspaces of $L_p$).**

*(i) For $1 \leq p \leq 2$, $L_q$ embeds in $L_p$ if and only if $p \leq q \leq 2$.*
*(ii) For $2 < p < \infty$, $L_q$ embeds in $L_p$ if and only if $q = 2$ or $q = p$.*

*Moreover, if $L_q$ embeds in $L_p$, then it embeds isometrically.*

*Proof.* Let $1 \leq p, q < \infty$ and suppose that $L_q$ embeds in $L_p$. Then, since $\ell_q$ embeds in $L_q$, it follows that $\ell_q$ embeds in $L_p$. This implies, by Theorem 6.4.18, that either $q = p$, or $q = 2$, or $1 \leq p < q < 2$. It remains to be shown that $L_q$ embeds in $L_p$ for $1 \leq p < q < 2$.

We know that $L_q$ is finitely representable in $\ell_q$ for each $q$ (Proposition 12.1.8) and that $\ell_q$ embeds in $L_p$ for $1 \le p < q < 2$ (Theorem 6.4.18). Hence $L_q$ is finitely representable in $L_p$ if $1 \le p < q < 2$. Since, in turn, $L_p$ is finitely representable in $\ell_p$, it follows that $L_q$ is finitely representable in $\ell_p$ for $1 \le p < q < 2$. By Theorem 12.1.9, $L_q$ is isomorphic to a subspace of $L_p$. □

We will frequently make use of the following lemma:

**Lemma 12.1.11.** *Let $E$ be a finite-dimensional normed space and suppose $(x_j)_{j=1}^N$ is an $\epsilon$-net on the surface of the unit ball $\{e : \|e\| = 1\}$, where $0 < \epsilon < 1$. Suppose $T : E \to X$ is a linear map such that $1 - \epsilon \le \|Tx_j\| \le 1 + \epsilon$ for $1 \le j \le N$. Then for every $e \in E$ we have*

$$\left(\frac{1-3\epsilon}{1-\epsilon}\right)\|e\| \le \|Te\| \le \left(\frac{1+\epsilon}{1-\epsilon}\right)\|e\|.$$

*Proof.* First suppose $\|e\| = 1$. Pick $j$ such that $\|e - x_j\| \le \epsilon$. Then

$$\|Te\| \le \|Te - Tx_j\| + (1+\epsilon),$$

and so

$$\|T\| \le \|T\|\epsilon + (1+\epsilon);$$

i.e.,

$$\|T\| \le \frac{1+\epsilon}{1-\epsilon}.$$

On the other hand, we also have

$$\|Te\| \ge 1 - \epsilon - \|T\|\epsilon \ge \frac{1-3\epsilon}{1-\epsilon}.$$

□

Ultraproducts of Banach spaces (see Appendix J) serve as an appropriate vehicle to study finite representability by infinite-dimensional methods.

**Proposition 12.1.12.** *Let $X$, $Y$ be infinite-dimensional Banach spaces.*

*(i) The ultraproduct $X_{\mathcal{U}}$ is finitely representable in $X$.*
*(ii) Let $\lambda \ge 1$. If $Y$ is separable, then $Y$ is crudely finitely representable in $X$ (with constant $\lambda + \epsilon$ for any $\epsilon > 0$) if and only if $Y$ is isomorphic to a subspace of $X_{\mathcal{U}}$ (and there is an embedding $T\colon Y \to X_{\mathcal{U}}$ such that $\|T^{-1}\|\|T\| \le \lambda$).*
*(iii) If $Y$ is separable, then $Y$ is finitely representable in $X$ if and only if $Y$ is isometrically isomorphic to a subspace of $X_{\mathcal{U}}$.*

*Proof.* $(i)$ Let $E$ be a finite-dimensional subspace of $X_{\mathcal{U}}$ and suppose $\epsilon > 0$. We can (by selecting representatives for a basis in $E$) suppose $E \subset \ell_\infty(X)$ and that $\|\cdot\|_{\mathcal{U}}$ is a norm on $E$. Choose $\nu > 0$ so small that $(1+\nu)(1-3\nu)^{-1} < 1+\epsilon$. Then pick a finite $\nu$-net $\mathcal{N} = \{\xi_1, \dots, \xi_N\}$ in the unit ball of $E$. Thus $B_E \subset \mathcal{N} + \nu B_E$.

There exists $A \in \mathcal{U}$ such that

$$1 - \nu < \|\xi_j(k)\| < 1 + \nu, \qquad k \in A,\ 1 \le j \le N.$$

Pick any fixed $k \in A$ and define $T : E \to X$ by $T\xi = \xi(k)$. Let $T(E) = F$. Then by Lemma 12.1.11, $\|T\|\|T^{-1}\| < 1 + \epsilon$.

$(ii)$ The "only if" part is an easy consequence of $(i)$. To prove the "if" part, let us suppose $(E_n)_{n=1}^\infty$ is an ascending sequence of finite-dimensional subspaces of $Y$ with $E = \cup_{n=1}^\infty E_n$ dense in $Y$, and let $T_n : E_n \to X$ be operators satisfying

$$\|e\| \le \|T_n e\| \le \left(\lambda + \frac{1}{n}\right)\|e\|, \quad e \in E_n,$$

for all $n \in \mathbb{N}$.

We define a map $L\colon E \to \ell_\infty(X)$ by setting $L(y) = \xi$, where

$$\xi(k) = \begin{cases} 0, & y \notin E_k, \\ T_k(y), & y \in E_k. \end{cases}$$

Then $L$ is nonlinear, but since

$$L(x+y) - L(x) - L(y) \in c_{00}(X) \subset c_{0,\mathcal{U}}(X),$$

it induces a linear map $T\colon E \to X_{\mathcal{U}}$. If $y \in E$, it is clear that $\|y\| \le \|T(x)\|_{\mathcal{U}} \le \lambda\|y\|$, whence we infer that $T$ extends to an isomorphism of $Y$ into $X_{\mathcal{U}}$ such that $\|T\| \le \lambda$ and $\|T^{-1}\| \le 1$.

$(iii)$ is an easy consequence of $(ii)$. □

An immediate deduction is the following:

**Proposition 12.1.13.** *Let $(Y, \|\cdot\|)$ be a separable Banach space. The space $Y$ is crudely finitely representable in a Banach space $X$ if and only if there is an equivalent norm $\|\cdot\|_a$ on $Y$ such that $(Y, \|\cdot\|_a)$ is finitely representable in $X$.*

**Corollary 12.1.14.** *A separable Banach space $X$ is finitely crudely representable in $\ell_p$ if and only if it is isomorphic to a subspace of $L_p$.*

The next theorem is an application of the basic idea of an ultraproduct. Note that we prove it only for real scalars; the proof for complex scalars would require some extra work.

**Theorem 12.1.15.** *Let X be a Banach space. Then*

*(i) X fails to have type $p > 1$ if and only if $\ell_1$ is finitely representable in X.*
*(ii) X fails to have cotype $q < \infty$ if and only if $\ell_\infty$ is finitely representable in X.*

*Proof.* We will use Proposition 7.2.5. For $(i)$ it suffices to note that $\alpha_N(X) = \sqrt{N}$ for every $N$. Thus for fixed $N$ and all $n$ we can find $(x_{nk})_{k=1}^N$ such that

$$\left(\sum_{k=1}^N \|x_{nk}\|^2\right)^{1/2} = \sqrt{N},$$

but

$$N - \frac{1}{n} < \left(\mathbb{E}\left\|\sum_{k=1}^N \varepsilon_k x_{nk}\right\|^2\right)^{1/2} \leq \sum_{k=1}^N \|x_{nk}\| \leq N.$$

Consider the elements

$$\xi_k(n) = (x_{nk})_{n=1}^\infty$$

in the ultraproduct $X_{\mathcal{U}}$. Then

$$\left(\sum_{k=1}^N \|\xi_k\|_{\mathcal{U}}^2\right)^{1/2} = \sqrt{N}, \quad \left(\mathbb{E}\left\|\sum_{k=1}^N \varepsilon_k \xi_k\right\|_{\mathcal{U}}^2\right)^{1/2} \geq N, \quad \text{and} \quad \sum_{k=1}^N \|\xi_k\|_{\mathcal{U}} \geq N.$$

Using the Cauchy–Schwarz inequality, we see that the last inequalities are equalities, and we must have $\|\xi_k\|_{\mathcal{U}} = 1$ for all $k$. Furthermore, it follows that

$$\left\|\sum_{k=1}^N \epsilon_k \xi_k\right\|_{\mathcal{U}} = N,$$

whenever $\epsilon_k = \pm 1$.

Now suppose $-1 \leq a_k \leq 1$ and let $\epsilon_k = -1$ if $a_k < 0$ and $\epsilon_k = 1$ if $a_k \geq 0$. Then

$$\left\|\sum_{k=1}^N a_k \xi_k\right\|_{\mathcal{U}} \geq \left\|\sum_{k=1}^N \epsilon_k \xi_k\right\|_{\mathcal{U}} - \left\|\sum_{k=1}^N (\epsilon_k - a_k)\xi_k\right\|_{\mathcal{U}} \geq N - \sum_{k=1}^N (1 - |a_k|) = \sum_{k=1}^N |a_k|.$$

Thus $(\xi_k)_{k=1}^N$ is isometrically equivalent to the canonical basis of $\ell_1^N$, and it follows that $\ell_1$ is finitely representable in $X$.

$(ii)$ is similar, using again Proposition 7.2.5, and we leave the details to the problems. □

A very natural question one can ask is, which properties are inherited by finite representability? For instance, reflexivity it is not, since $\ell_1$ (which is not reflexive) is finitely representable in (the reflexive) space $\ell_2(\ell_1^n)$. This is just one of the ways in which one may reach the notion of superreflexive Banach spaces, introduced by James in 1972 [131, 132].

**Definition 12.1.16.** A Banach space $X$ is said to be *superreflexive* if every Banach space $Y$ that is finitely representable in $X$ is reflexive.

The following lemma is easily deduced from James's criterion of reflexivity (see Problem 12.5).

**Lemma 12.1.17.** *A Banach space $X$ is reflexive if and only if every separable subspace of $X$ is reflexive.*

**Proposition 12.1.18.** *If $1 < p < \infty$, then the space $\ell_p$ is superreflexive.*

*Proof.* Fix $1 < p < \infty$. Let $X$ be a Banach space that is finitely representable in $\ell_p$ and let $Y$ be a separable subspace of $X$. By transitivity, $Y$ is finitely representable in $\ell_p$, so that by Theorem 12.1.9, the space $Y$ is isomorphic to a subspace of $L_p$. Since $L_p$ is reflexive, $Y$ is reflexive, and Lemma 12.1.17 yields that $\ell_p$ is superreflexive. □

**Proposition 12.1.19.** *If a Banach space $Y$ is crudely finitely representable in a superreflexive Banach space $X$, then $Y$ is superreflexive.*

*Proof.* Let $Z$ be a separable subspace of $Y$. By transitivity, $Z$ is crudely finitely representable in $X$. Proposition 12.1.13 yields that $Z$ is finitely representable in $X$ equipped with an equivalent norm; hence $Z$ is reflexive. Using again Lemma 12.1.17, we deduce that $Y$ is superreflexive. □

**Corollary 12.1.20.** *Superreflexivity is an isomorphic property (despite the fact that finite representability is not!).*

**Corollary 12.1.21.** *The space $L_p$ is superreflexive for $1 < p < \infty$.*

*Proof.* We know that $L_p$ is finitely representable in $\ell_p$ and that $\ell_p$ is superreflexive. Now just apply Proposition 12.1.19. □

## 12.2 The Principle of Local Reflexivity

The main result in this section is a very important theorem of Lindenstrauss and Rosenthal from 1969 [198] called the principle of local reflexivity; it asserts that in a local sense, every Banach space is reflexive. More precisely, for every infinite-dimensional Banach space $X$, its second dual $X^{**}$ is finitely representable in $X$. Our proof is based on one given by Stegall [287]; see also [57] for an interpretation of the principle in terms of spaces of operators.

Let $T : X \to Y$ be a bounded operator. If the range $T(X)$ is closed, $T$ is sometimes called *semi-Fredholm*. This is equivalent to the requirement that $T$ factor to an isomorphic embedding on $X/\ker(T)$ (i.e., the canonical induced map $T_0 : X/\ker(T) \to Y$ is an isomorphic embedding), which in turn is equivalent to the statement that for some constant $C$ we have

$$d(x, \ker(T)) \le C\|Tx\|, \qquad x \in X.$$

**Proposition 12.2.1.** *Let $T : X \to Y$ be an operator with closed range. Suppose $y \in Y$ is such that the equation $T^{**}x^{**} = y$ has a solution $x^{**} \in X^{**}$ with $\|x^{**}\| < 1$. Then the equation $Tx = y$ has a solution $x \in X$ with $\|x\| < 1$.*

*Proof.* This is almost immediate. We must show that $y \in T(U_X)$, where $U_X$ is the open unit ball of $X$.

First suppose $y \notin T(X)$. In this case there exists $y^* \in Y^*$ with $T^*(y^*) = 0$ but $y^*(y) = 1$. This is impossible, since $T^{**}x^{**}(y^*) = y^*(y) = 1$.

Next suppose $y \in T(X) \setminus T(U_X)$. By the open mapping theorem, $T(U_X)$ is open relative to $T(X)$, and so, using the Hahn–Banach separation theorem, we can find $y^* \in Y^*$ with $y^*(y) \ge 1$ but $y^*(Tx) < 1$ for $x \in U_X$. Thus $\|T^*y^*\| \le 1$, and so $|x^{**}(T^*y^*)| < 1$, i.e., $|y^*(y)| < 1$, which is a contradiction. □

**Proposition 12.2.2.** *Let $T : X \to Y$ be an operator with closed range and suppose $K : X \to Y$ is a finite-rank operator. Then $T + K$ also has closed range.*

*Proof.* Suppose $T+K$ does not have closed range. Then there is a bounded sequence $(x_n)_{n=1}^\infty$ with $\lim_{n\to\infty}(T + K)(x_n) = 0$ but $d(x_n, \ker(T + K)) \ge 1$ for all $n$. We can pass to a subsequence and assume that $(Kx_n)_{n=1}^\infty$ converges to some $y \in Y$ and hence $\lim_{n\to\infty} Tx_n = -y$. This implies that there exists $x \in X$ with $Tx = -y$ and thus $\lim_{n\to\infty} \|Tx_n - Tx\| = 0$. Hence $\lim_{n\to\infty} d(x_n - x, \ker(T)) = 0$. It follows that $y - Kx \in K(\ker T)$.

Let $y - Kx = Ku$, where $u \in \ker(T)$. Then

$$\lim_{n\to\infty} d(x_n - x - u, \ker(T)) = 0$$

and

$$\lim_{n\to\infty} \|Kx_n - Kx - u\| = 0.$$

Since $K|_{\ker(T)}$ has closed range, this means that

$$\lim_{n\to\infty} d(x_n - x - u, \ker(T) \cap \ker(K)) = 0.$$

But $T(x + u) = -y = -K(x + u)$, so $x + u \in \ker(T + K)$, and therefore

$$\lim_{n\to\infty} d(x_n, \ker(T + K)) = 0,$$

contrary to our assumption. □

**Theorem 12.2.3.** *Let $X$ be a Banach space, $A = (a_{jk})_{j,k=1}^{m,n}$ an $m \times n$ real matrix, and $B = (b_{jk})_{j,k=1}^{p,n}$ a $p \times n$ real matrix. Let $y_1, \dots, y_m \in X$, $y_1^*, \dots, y_p^* \in X^*$, and $\xi_1, \dots, \xi_p \in \mathbb{R}$. Suppose there exist vectors $x_1^{**}, \dots, x_n^{**}$ in $X^{**}$ with* $\max_{1 \le k \le n} \|x_k^{**}\| < 1$ *satisfying the following equations:*

$$\sum_{k=1}^{n} a_{jk} x_k^{**} = y_j, \qquad 1 \le j \le m,$$

*and*

$$y_j^* \left( \sum_{k=1}^{n} b_{jk} x_k^{**} \right) = \xi_j, \qquad 1 \le j \le p.$$

*Then there exist vectors $x_1, \dots, x_n$ in $X$ with* $\max_{1 \le k \le n} \|x_k\| < 1$ *satisfying the (same) equations:*

$$\sum_{k=1}^{n} a_{jk} x_k = y_j, \qquad 1 \le j \le m,$$

*and*

$$y_j^* \left( \sum_{k=1}^{n} b_{jk} x_k \right) = \xi_j, \qquad 1 \le j \le p.$$

*Proof.* Consider the operator $T_0 : \ell_\infty^n(X) \to \ell_\infty^m(X)$ defined by

$$T_0(x_1, \dots, x_n) = \Big( \sum_{k=1}^{n} a_{jk} x_k \Big)_{j=1}^{m}.$$

We claim that $T_0$ has closed range. This is an immediate consequence of the fact that the matrix $A$ can be written in the form $A = PDQ$, where $P$ and $Q$ are nonsingular, and $D$ is in the form

$$D = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where $r$ is the rank of $A$. This allows a factorization of $T_0$ in the form $T_0 = USV$, where $U, V$ are invertible and $S$ is given by the matrix $D$, and therefore trivially it has closed range.

Now define $T : \ell_\infty^n(X) \to \ell_\infty^m(X) \oplus_\infty \ell_\infty^p$ by

$$T(x_1, \dots, x_n) = \Big( T_0(x_1, \dots, x_n), \big( x_j^* ( \sum_{k=1}^{n} b_{jk} x_k ) \big)_{j=1}^{p} \Big).$$

By Proposition 12.2.2 it is clear that $T$ also has closed range. The theorem then follows directly from Proposition 12.2.1. □

**Theorem 12.2.4 (The Principle of Local Reflexivity).** *Let $X$ be a Banach space. Suppose that $F$ is a finite-dimensional subspace of $X^{**}$ and $G$ is a finite-dimensional subspace of $X^*$. Then given $\epsilon > 0$ there exist a subspace $E$ of $X$ containing $F \cap X$ with* $\dim E = \dim F$ *and a linear isomorphism $T : F \to E$ with $\|T\|\|T^{-1}\| < 1 + \epsilon$ such that*

$$Tx = x, \qquad x \in F \cap X,$$

*and*

$$x^*(Tx^{**}) = x^{**}(x^*), \qquad x^* \in G,\ x^{**} \in F.$$

*In particular, $X^{**}$ is finitely representable in $X$.*

*Proof.* Given $\epsilon > 0$ let us take $\nu > 0$ such that $(1+\nu)(1-3\nu)^{-1} < 1+\epsilon$ and pick a $\nu$-net $(x_j^{**})_{j=1}^N$ in $\{x^{**} \in F : \|x^{**}\| = 1\}$. Let $S : \mathbb{R}^N \to F$ be the operator defined by

$$S(\xi_1, \dots, \xi_N) = \sum_{j=1}^N \xi_j x_j^{**}.$$

Let $H = S^{-1}(F \cap X)$ and suppose $(a^{(j)})_{j=1}^m$ is a basis for $H$. Let $S(a^{(j)}) = y_j \in F \cap X$ and define the matrix $A = (a_{jk})_{j=1,k=1}^{m,N}$ by $a^{(j)} = (a_{j1}, \dots, a_{jN})$.

Next pick $x_1^*, \dots, x_N^* \in X^*$ such that $\|x_j^*\| = 1$ and $x_j^{**}(x_j^*) > 1 - \nu$, and finally pick a basis $\{g_1^*, \dots, g_l^*\}$ of $G$.

We consider the following system of equations in $(x_1, \dots, x_N)$:

$$\sum_{k=1}^N a_{jk} x_k = y_j, \qquad j = 1, 2, \dots, m,$$

$$x_j^*(x_j) = x_j^{**}(x_j^*), \qquad j = 1, 2, \dots, N,$$

and

$$g_j^*(x_j) = x_j^{**}(g_j^*), \qquad j = 1, 2, \dots, l.$$

This system has a solution in $X^{**}$, namely $(x_1^{**}, \dots, x_N^{**})$, and $\max_j \|x_j^{**}\| = 1$. It follows from Theorem 12.2.3 that it has a solution $(x_1, \dots, x_N)$ in $X$ with $\max \|x_j\| < 1 + \nu$.

If we define $S_1 : \mathbb{R}^N \to X$ by

$$S_1(\xi_1, \dots, \xi_N) = \sum_{j=1}^N \xi_j x_j,$$

then it is clear from the construction that $S(\xi) = 0$ implies that $S_1(\xi) = 0$, and so $S_1 = TS$ for some operator $T : F \to X$. Let $E = T(F)$. Note that for $1 \le j \le N$ we have

$$1 - \nu < \|x_j\| < 1 + \nu,$$

since $\|x_j\| \ge x_j^*(x_j) > 1 - \nu$. Hence, by Lemma 12.1.11, $\|T\|\|T^{-1}\| < 1 + \epsilon$. The other properties are clear from the construction. □

## 12.3 Krivine's Theorem

In this section we will use the term *sequence space* to denote the completion $\mathcal{X}$ of $c_{00}$ under some norm $\|\cdot\|_{\mathcal{X}}$ such that the basis vectors $(e_n)_{n=1}^{\infty}$ have norm one.

**Definition 12.3.1.** A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space $X$ is *spreading* if it has the property that for all integers $0 < p_1 < p_2 < \cdots < p_n$ and every sequence of scalars $(a_i)_{i=1}^{n}$ we have

$$\Big\| \sum_{j=1}^{n} a_j x_{p_j} \Big\| = \Big\| \sum_{j=1}^{n} a_j x_j \Big\|.$$

Notice that if $(x_n)_{n=1}^{\infty}$ is an unconditional basic sequence in a Banach space $X$, the previous definition means that $(x_n)_{n=1}^{\infty}$ is subsymmetric (Definition 9.2.4).

**Definition 12.3.2.** A sequence space $\mathcal{X}$ is *spreading* if the canonical basis $(e_n)_{n=1}^{\infty}$ of $\mathcal{X}$ is spreading.

**Definition 12.3.3.** Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in a Banach space $X$, and let $(y_n)_{n=1}^{\infty}$ be a bounded sequence in a Banach space $Y$. We will say that $(y_n)_{n=1}^{\infty}$ is *block finitely representable* in $(x_n)_{n=1}^{\infty}$ if given $\epsilon > 0$ and $N \in \mathbb{N}$ there exist a sequence of *blocks* of $(x_n)_{n=1}^{\infty}$,

$$u_j = \sum_{p_{j-1}+1}^{p_j} a_j x_j, \qquad j = 1, 2, \ldots, N,$$

where $(p_j)$ are integers with $0 = p_0 < p_1 < \cdots < p_N$, and $(a_n)$ are scalars, and an operator $T : [y_j]_{j=1}^{N} \to [u_j]_{j=1}^{N}$ with $Ty_j = u_j$ for $1 \le j \le N$ such that $\|T\|\|T^{-1}\| < 1 + \epsilon$.

Note here that we do not assume that $(x_n)_{n=1}^{\infty}$ or $(y_n)_{n=1}^{\infty}$ is a basic sequence, although usually they are.

**Definition 12.3.4.** Let $(x_n)_{n=1}^\infty$ be a bounded sequence in a Banach space $X$. A sequence space $\mathcal{X}$ is said to be *block finitely representable* in $(x_n)_{n=1}^\infty$ if the canonical basis vectors $(e_n)_{n=1}^\infty$ in $\mathcal{X}$ are block finitely representable in $(x_n)_{n=1}^\infty$.

Obviously if $\mathcal{X}$ is block finitely representable in $(x_n)_{n=1}^\infty$, it is also true that $\mathcal{X}$ is finitely representable in $X$. We are thus asking for a strong form of finite representability.

**Definition 12.3.5.** A sequence space $\mathcal{X}$ is said to be *block finitely representable* in another sequence space $\mathcal{Y}$ if it is block finitely representable in the canonical basis of $\mathcal{Y}$.

**Proposition 12.3.6.** *Suppose $(x_n)_{n=1}^\infty$ is a nonconstant spreading sequence in a Banach space $X$.*

*(i) If $(x_n)_{n=1}^\infty$ fails to be weakly Cauchy, then $(x_n)_{n=1}^\infty$ is a basic sequence equivalent to the canonical $\ell_1$-basis.*
*(ii) If $(x_n)_{n=1}^\infty$ is weakly null, then it is an unconditional basic sequence with suppression constant $K_s = 1$.*
*(iii) If $(x_n)_{n=1}^\infty$ is weakly Cauchy, then $(x_{2n-1}-x_{2n})_{n=1}^\infty$ is weakly null and spreading.*

*Proof.* $(i)$ If $(x_n)_{n=1}^\infty$ is not weakly Cauchy, then no subsequence can be weakly Cauchy (by the spreading property), and so by Rosenthal's theorem (Theorem 11.2.1), some subsequence is equivalent to the canonical $\ell_1$-basis; but then again, this means that the entire sequence is equivalent to the $\ell_1$-basis.

$(ii)$ It is enough to show that if $a_1, \dots, a_n \in \mathbb{R}$ and $1 \le m \le n$, then

$$\Big\| \sum_{j<m} a_j x_j + \sum_{m<j\le n} a_j x_j \| \le \| \sum_{j=1}^n a_j e_j \Big\|.$$

Suppose $\epsilon > 0$. By Mazur's theorem we can find $c_j \ge 0$ for $1 \le j \le l$, say, such that $\sum_{j=1}^l c_j = 1$ and

$$\Big\| \sum_{j=1}^l c_j x_j \Big\| < \epsilon.$$

Now consider

$$x = \sum_{j=1}^{m-1} a_j x_j + a_m \sum_{j=m}^{m+l-1} c_{j-m+1} x_j + \sum_{j=m+l}^{m+l-1} a_{j-l+1} x_j.$$

Then

$$x = \sum_{i=1}^l c_i \left( \sum_{j<m} a_j x_j + a_m x_{i+m} + \sum_{j=m+1}^n a_j x_{l+j-1} \right),$$

and so

$$\|x\| \le \Big\| \sum_{j=1}^{n} a_j x_j \Big\|.$$

But

$$\Big\| \sum_{j<m} a_j x_j + \sum_{m<j\le n} a_j x_j \Big\| \le \|x\| + |a_m|\epsilon,$$

and so

$$\Big\| \sum_{j<m} a_j x_j + \sum_{m<j\le n} a_j x_j \Big\| \le \Big\| \sum_{j=1}^{n} a_j x_j \Big\| + |a_m|\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we are done.

(*iii*) This is immediate, since $(x_{2n-1} - x_{2n})_{n=1}^{\infty}$ is weakly null and spreading (obviously, it cannot be constant). □

**Theorem 12.3.7.** *Suppose $(x_n)_{n=1}^{\infty}$ is a normalized sequence in a Banach space $X$ such that $\{x_n\}_{n=1}^{\infty}$ is not relatively compact. Then there is a spreading sequence space that is block finitely representable in $(x_n)_{n=1}^{\infty}$. More precisely, there exist a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ and a spreading sequence space $\mathcal{X}$ such that if we let $M = \{n_k\}_{k=1}^{\infty}$, then*

$$\lim_{\substack{(p_1,\dots,p_r)\in\mathcal{F}_r(M)\\ p_1<\dots<p_r}} \Big\| \sum_{j=1}^{r} a_j x_{p_j} \Big\| = \Big\| \sum_{j=1}^{r} a_j e_j \Big\|_{\mathcal{X}}.$$

*Proof.* This is a neat application of Ramsey's theorem due to Brunel and Sucheston [36]. We first observe that by taking a subsequence, we can assume that $(x_n)_{n=1}^{\infty}$ has no convergent subsequence.

Let us fix some finite sequence of real numbers $(a_j)_{j=1}^{r}$. According to Theorem 11.1.1, given any infinite subset $M$ of $\mathbb{N}$, we can find a further infinite subset $M_1$ such that

$$\lim_{\substack{(p_1,\dots,p_r)\in\mathcal{F}_r(M_1)\\ p_1<\dots<p_r}} \Big\| \sum_{j=1}^{r} a_j x_{p_j} \Big\| \text{ exists.}$$

Let $(a_1^{(k)}, \dots, a_{r_k}^{(k)})_{k=1}^{\infty}$ be an enumeration of all finitely nonzero sequences of rationals, and let us construct a decreasing sequence $(M_k)_{k=1}^{\infty}$ of infinite subsets of $\mathbb{N}$ such that

$$\lim_{\substack{(p_1,\dots,p_r)\in\mathcal{F}_r(M_k)\\ p_1<\dots<p_r}} \Big\| \sum_{j=1}^{r} a_j^{(k)} x_{p_j} \Big\| \text{ exists.}$$

A diagonal procedure allows us to pick an infinite subset $M_\infty$ that is contained in each $M_k$ up to a finite set. It is not difficult to check that

$$\lim_{\substack{(p_1,\dots,p_r)\in\mathcal{F}_r(M_\infty)\\ p_1<\dots<p_r}} \Big\| \sum_{j=1}^{r} a_j x_{p_j} \Big\| \text{ exists}$$

for every finite sequence of reals $(a_j)_{j=1}^r$.

Given $\xi = (\xi(j))_{j=1}^\infty \in c_{00}$, put

$$\|\xi\|_{\mathcal{X}} = \lim_{\substack{(p_1,\dots,p_r)\in\mathcal{F}_r(M_\infty)\\ p_1<\dots<p_r}} \Big\| \sum_{j=1}^{r} \xi(j) x_{p_j} \Big\|.$$

Then $\|\cdot\|_{\mathcal{X}}$ satisfies the spreading property, but we need to check that it is a norm on $c_{00}$ (it obviously is a seminorm). If $\|\xi\|_{\mathcal{X}} = 0$ and $\xi = \sum_{j=1}^r a_j e_j$ with $a_r \neq 0$, then we also have $\| \sum_{j=1}^{r-1} a_j e_j + a_r e_{r+1}\|_{\mathcal{X}} = 0$. Hence

$$\|e_1 - e_2\|_{\mathcal{X}} = \|e_r - e_{r+1}\|_{\mathcal{X}} = 0.$$

Returning to the definition, we see that this implies

$$\lim_{(p_1,p_2)\in\mathcal{F}_2(M_\infty)} \|x_{p_1} - x_{p_2}\| = 0,$$

which can mean only that the subsequence $(x_j)_{j\in M_\infty}$ is convergent, contrary to our construction. □

**Definition 12.3.8.** The spreading sequence space $\mathcal{X}$ introduced in Theorem 12.3.7 is called a *spreading model* for the sequence $(x_n)_{n=1}^\infty$.

We now turn to Krivine's theorem. This result was obtained by Krivine in 1976, and although the main ideas of the proof we include here are the same as in Krivine's original proof, we have used ideas from two subsequent expositions of Krivine's theorem by Rosenthal [274] and Lemberg [186].

Krivine's theorem should be contrasted with Tsirelson space, which we constructed in Section 11.3. The existence of Tsirelson space implies that there is a Banach space with a basis such that no (infinite) block basic sequence can be equivalent to one of the spaces $\ell_p$ and $c_0$. However, if we are content with *finite* block basic sequences, then we can always find a good copy of one of these spaces! This difference in behavior between *infinite* and *arbitrarily large but finite* is a recurrent theme in modern Banach space theory.

**Theorem 12.3.9 (Krivine's Theorem).** *Let $(x_n)_{n=1}^\infty$ be a normalized sequence in a Banach space $X$ such that $\{x_n\}_{n=1}^\infty$ is not relatively compact. Then, either $c_0$ is block finitely representable in $(x_n)_{n=1}^\infty$, or there exists $1 \le p < \infty$ such that $\ell_p$ is block finitely representable in $(x_n)_{n=1}^\infty$.*

In order to simplify the proof of Theorem 12.3.9 let us start by making some observations.

We first claim that it suffices to prove the theorem when $(x_n)_{n=1}^\infty$ is replaced by the canonical basis $(e_n)_{n=1}^\infty$ of a spreading model $\mathcal{X}$; this is a direct consequence of Theorem 12.3.7. We next claim that we can suppose that the canonical basis $(e_n)_{n=1}^\infty$ of the spreading model $\mathcal{X}$ is unconditional with suppression constant $K_s = 1$ (and hence 2-unconditional). Indeed, if the canonical basis of the spreading model fails to be weakly Cauchy, then it is equivalent to the canonical $\ell_1$-basis, and the fact that $\ell_1$ is block finitely representable in $\mathcal{X}$ is simply the content of James's distortion theorem (Theorem 11.3.1). If $(e_n)_{n=1}^\infty$ is weakly Cauchy but not weakly null, we use Proposition 12.3.6 and replace it by the spreading sequence

$$f_k = \frac{e_{2k} - e_{2k+1}}{\|e_{2k} - e_{2k+1}\|}, \qquad k = 1, 2, \ldots.$$

In this way we reduce the proof to showing the result for the canonical basis $(e_n)_{n=1}^\infty$ of some spreading sequence space $\mathcal{X}$.

We also observe at this point that James distortion theorem for $c_0$ (see Problem 11.3) implies that if the spreading model is isomorphic to $c_0$, then $c_0$ is finitely-representable in it. This reduction will be used later.

Now we will introduce some notation. Suppose $\mathcal{X}$ is a spreading sequence space whose canonical basis is unconditional with suppression constant $K_s = 1$. The norm of each $\xi \in \mathcal{X}$ depends only on its nonzero entries and their order of appearance. We shall say that the sequences $\xi$ and $\eta$ in $c_{00}$ are *equivalent* if their nonzero entries and their order of appearance are identical. We will say that $\xi$ and $\eta$ are $\epsilon$*-equivalent* if there exist $u, v \in c_{00}$ such that $u+\xi$ and $v+\eta$ are equivalent and $\|u\|_{\mathcal{X}} + \|v\|_{\mathcal{X}} < \epsilon$.

If $\xi, \eta \in c_{00}$, we define $\xi \oplus \eta$ to be any vector for which nonzero entries of $\xi$ (in correct order) precede the nonzero entries of $\eta$ (in correct order). For example, $\xi \oplus \eta$ could be obtained by writing first the entries of $\xi$ in order and then the nonzero entries of $\eta$ in order. Thus, if $n$ is the largest integer such that $\xi(n) \neq 0$, we could take

$$\xi \oplus \eta = \sum_{j=1}^{n} \xi(j) e_j + \sum_{j=n+1}^{\infty} \eta(j-n) e_j.$$

We will say that $\xi$ is *replaceable* by $\eta$ if

$$\|u \oplus \xi \oplus v\|_{\mathcal{X}} = \|u \oplus \eta \oplus v\|_{\mathcal{X}}, \qquad u, v \in c_{00},$$

and that $\xi$ is $\epsilon$*-replaceable* by $\eta$ if

$$\Big| \|u \oplus \xi \oplus v\|_{\mathcal{X}} - \|u \oplus \eta \oplus v\|_{\mathcal{X}} \Big| < \epsilon, \qquad u, v \in c_{00}.$$

Let us notice that if $\xi$ and $\eta$ are equivalent, then $\xi$ is replaceable by $\eta$. Similarly, if $\xi$ and $\eta$ are $\epsilon$-equivalent, then $\xi$ is $\epsilon$-replaceable by $\eta$.

To complete the proof of Krivine's theorem we will need the following two lemmas.

**Lemma 12.3.10.** *Suppose $\mathcal{X}$ is a spreading sequence space. Then there is a spreading sequence space $\mathcal{Y}$ that is block finitely representable in $\mathcal{X}$ such that the canonical basis of $\mathcal{Y}$ is unconditional with unconditional basis constant $K_u = 1$.*

*Proof.* By the previous remarks we can assume that the canonical basis $(e_n)_{n=1}^{\infty}$ of $\mathcal{X}$ is 2-unconditional, and that $\mathcal{X}$ is not isomorphic to $c_0$. Thus, if we let $y_n = \sum_{j=1}^{n}(-1)^j e_j$, we have $\|y_n\| \to \infty$. For each $k$ let $u_k = y_k/\|y_k\|$. Then $u_k$ is $\epsilon_k$-equivalent to $-u_k$ for $\epsilon_k = 2/\|y_k\|$.

If we take a block basic sequence $(z_n)_{n=1}^{\infty}$ with respect to $(e_n)_{n=1}^{\infty}$, where each $z_n$ is equivalent to $u_k$, we obtain a spreading sequence where $-z_n$ is $\epsilon_k$-replaceable by $z_n$. Define $\mathcal{Y}_k$ by

$$\|\xi\|_{\mathcal{Y}_k} = \Big\| \sum_{j=1}^{\infty} \xi(j) z_j \Big\|_{\mathcal{X}}.$$

We can pass to a subsequence $(k_m)_{m=1}^{\infty}$ in such a way that $\lim_{m\to\infty} \|\xi\|_{\mathcal{Y}_{k_m}}$ exists for all $\xi \in c_{00}$. This is done by a standard diagonal argument for those $\xi$ with rational coefficients, and then extended to all $\xi$ by a routine approximation argument. This formula defines a spreading sequence space, still block finitely representable in $\mathcal{X}$ but such that $e_1$ is replaceable by $-e_1$. This shows that the canonical basis of $\mathcal{Y}$ is 1-unconditional. □

**Lemma 12.3.11.** *Suppose $\mathcal{X}$ is a spreading sequence space whose canonical basis $(e_n)_{n=1}^{\infty}$ is* 1*-unconditional.*

*(i) If $2^{1/p}e_1$ is replaceable by $e_1 + e_2$ for some $1 \le p < \infty$, then the norm on $\mathcal{X}$ is equivalent to the canonical $\ell_p$-norm.*
*(ii) If for some $1 \le p < \infty$, $2^{1/p}e_1$ is replaceable by $e_1 + e_2$, and $3^{1/p}e_1$ is replaceable by $e_1 + e_2 + e_3$, then the norm on $\mathcal{X}$ coincides with the $\ell_p$-norm.*

*Proof.* ($i$) Suppose $(k_j)_{j=1}^{\infty}$ is a sequence of nonnegative integers. If for each $n$ we let $N = \sum_{j=1}^{n} 2^{k_j}$, then we have

$$\Big\| \sum_{j=1}^{n} 2^{k_j/p} e_j \Big\|_{\mathcal{X}} = \Big\| \sum_{j=1}^{N} e_j \Big\|_{\mathcal{X}}.$$

Notice also that

$$\Big\| \sum_{j=1}^{2^r} e_j \Big\|_{\mathcal{X}} = 2^{r/p},$$

and so

$$2^{-1/p}N^{1/p} \leq \Big\| \sum_{j=1}^{N} e_j \Big\|_{\mathcal{X}} \leq 2^{1/p}N^{1/p}.$$

Suppose now that $(a_j)$ are scalars with $\sum_{j=1}^n |a_j|^p = 1$, and let $\alpha$ be the least nonzero value of $|a_j|$. For each $j$ pick a nonnegative integer $k_j$ with $2^{k_j/p} \leq |a_j|\alpha^{-1} \leq 2^{(k_j+1)/p}$. Then, if $N = \sum_{j=1}^n 2^{k_j}$, we have

$$\Big\| \sum_{j=1}^{N} e_j \Big\|_{\mathcal{X}} \leq \alpha^{-1} \Big\| \sum_{j=1}^{n} a_j e_j \Big\|_{\mathcal{X}} \leq \Big\| \sum_{j=1}^{2N} e_j \Big\|_{\mathcal{X}},$$

and so $N\alpha^p \leq 1 \leq 2N\alpha^p$. Thus

$$2^{-1/p}N^{1/p}\alpha \leq \Big\| \sum_{j=1}^{n} a_j e_j \Big\|_{\mathcal{X}} \leq 2^{2/p}N^{1/p}\alpha,$$

which implies

$$2^{-2/p} \leq \Big\| \sum_{j=1}^{n} a_j e_j \Big\|_{\mathcal{X}} \leq 2^{2/p}.$$

The proof of (*ii*) is similar to (*i*). Here we use that the set of real numbers of the form $2^l 3^m$ with $l, m \in \mathbb{Z}$ is *dense* in $(0, +\infty)$, which is a consequence of the fact that $\log 3/\log 2$ is irrational.

If $l, m \geq 0$ and $N = 2^l 3^m$, we have

$$\Big\| \sum_{j=1}^{N} e_j \Big\|_{\mathcal{X}} = N^{1/p}.$$

For any $N$ pick $r, s \in \mathbb{Z}$ such that $N - \epsilon \leq 2^r 3^s \leq N$. Then

$$\Bigg\| \sum_{j=1}^{2^{|r|}3^{|s|}N} e_j \Bigg\|_{\mathcal{X}} = 2^{|r|/p}3^{|s|/p} \Bigg\| \sum_{j=1}^{N} e_j \Bigg\|_{\mathcal{X}} \geq \Bigg\| \sum_{j=1}^{2^{r+|r|}3^{s+|s|}} e_j \Bigg\|_{\mathcal{X}} = 2^{(r+|r|)/p}3^{(s+|s|)/p},$$

and so

$$\Big\| \sum_{j=1}^{N} e_j \Big\|_{\mathcal{X}} \geq 2^{r/p}3^{s/p} \geq (N-\epsilon)^{1/p}.$$

Hence

$$\Big\| \sum_{j=1}^{N} e_j \Big\|_{\mathcal{X}} \geq N^{1/p}.$$

Conversely, we can find $r, s$ in $\mathbb{Z}$ such that $N < 2^r 3^s < N+\epsilon$, and a similar argument yields

$$\Big\| \sum_{j=1}^{N} e_j \Big\|_{\mathcal{X}} \leq N^{1/p}.$$

Thus we obtain

$$\Big\| \sum_{j=1}^{N} e_j \Big\|_{\mathcal{X}} = N^{1/p}, \qquad N = 1, 2, \dots.$$

Suppose $a_1, a_2, \dots, a_n$ are scalars of the form $|a_j| = 2^{l_j/p} 3^{m_j/p}$ for some $l_j, m_j \in \mathbb{Z}$. Pick $L, M \in \mathbb{N}$ such that $L + l_j \geq 0, M + m_j \geq 0$ for all $1 \leq j \leq n$. Then,

$$2^{L/p} 3^{M/p} \Big\| \sum_{j=1}^{n} a_j e_j \Big\|_{\mathcal{X}} = \| \sum_{j=1}^{N} e_j \Big\|_{\mathcal{X}},$$

where

$$N = 2^L 3^M \sum_{j=1}^{n} |a_j|^p.$$

This implies

$$\Big\| \sum_{j=1}^{n} a_j e_j \Big\|_{\mathcal{X}} = \left( \sum_{j=1}^{n} |a_j|^p \right)^{1/p}.$$

A density argument implies the conclusion of the lemma for all sequences of scalars $(a_i)_{i=1}^n$. □

We are almost ready to complete the proof of Theorem 12.3.9. Suppose $\mathcal{X}$ is a 1-unconditional spreading sequence space; we will define a variant of $\mathcal{X}$ modeled on $\mathbb{Q}_0 = \mathbb{Q} \cap [0, 1)$ rather than $\mathbb{N}$.

Consider the space $c_{00}(\mathbb{Q})$ of all finitely nonzero sequences on $\mathbb{Q}$. For $\xi \in c_{00}(\mathbb{Q}_0)$ of the form $\xi = \sum_{j=1}^{n} a_j e_{q_j}$, where $q_1 < q_2 < \cdots < q_n$, we define

$$\Big\| \sum_{j=1}^{n} a_j e_{q_j} \Big\|_{\mathcal{X}(\mathbb{Q}_0)} = \Big\| \sum_{j=1}^{n} a_j e_j \Big\|_{\mathcal{X}}.$$

On $\mathcal{X}(\mathbb{Q}_0)$ we consider two bounded operators given by

$$T_2 e_q = e_{q/2} + e_{(q+1)/2}, \qquad q \in \mathbb{Q}_0,$$

and

$$T_3 e_q = e_{q/3} + e_{(q+1)/3} + e_{(q+2)/3}, \qquad q \in \mathbb{Q}_0.$$

It is clear that $1 \leq \|T_2\| \leq 2$ and $1 \leq \|T_3\| \leq 3$. We consider the spectral radius of $T_2$ and define $0 \leq \theta \leq 1$ by

$$2^\theta = \lim_{n\to\infty} \|T_2^n\|^{\frac{1}{n}}.$$

**Lemma 12.3.12.** *Suppose $\mathcal{X}$ is a 1-unconditional spreading sequence space. Then*

*(i) There exists a sequence $(\xi_n)_{n=1}^\infty$ in $\mathcal{X}(\mathbb{Q}_0)$ with $\|\xi_n\|_{\mathcal{X}(\mathbb{Q}_0)} = 1$ and such that $\lim_{n\to\infty} \|T_2\xi_n - 2^\theta \xi_n\|_{\mathcal{X}(\mathbb{Q}_0)} = 0$.*
*(ii) If the norm on $\mathcal{X}$ is equivalent to the $\ell_p$-norm for some $1 \leq p < \infty$, then $\theta = 1/p$, and there is a sequence $(\xi_n)_{n=1}^\infty$ in $\mathcal{X}(\mathbb{Q}_0)$ such that $\|\xi_n\|_{\mathcal{X}(\mathbb{Q}_0)} = 1$, $\lim_{n\to\infty} \|T_2\xi_n - 2^{1/p}\xi_n\|_{\mathcal{X}(\mathbb{Q}_0)} = 0$, and $\lim_{n\to\infty} \|T_3\xi_n - 3^{1/p}\xi_n\|_{\mathcal{X}(\mathbb{Q}_0)} = 0$.*

*Proof.* ($i$) Let us start by observing that

$$\lim_{n\to\infty} \|T_2^n\|^{\frac{1}{n}} = \inf_n \|T_2^n\|^{\frac{1}{n}}, \tag{12.3}$$

and so

$$\|T_2^n\| \geq 2^{n\theta}, \qquad n = 1, 2, \ldots.$$

Thus

$$\lim_{n\to\infty} \|(n+1)2^{-n\theta}T_2^n\| = \infty,$$

and, by the uniform boundedness principle, we can find $\eta \in \mathcal{X}$ with $\|\eta\| = 1$ such that the sequence $((n+1)2^{-n\theta}T_2^n\eta)_{n=1}^\infty$ is unbounded. Let us note that we can assume that $\eta$ has only nonnegative entries. If we define $|\eta|$ by $|\eta|(q) = |\eta(q)|$, then $T_2^n|\eta|(q) \geq |T_2^n\eta(q)|$ for every $q$. Therefore we assume $\eta \geq 0$, i.e., $\eta(q) \geq 0$ for all $q$.

If $r < 2^{-\theta}$, then $(1 - rT_2)$ is invertible, and we can expand $(I - rT_2)^{-2}$ in its binomial series (which converges). Thus

$$(1 - rT_2)^{-2}(\eta) = \sum_{n=0}^\infty (n+1)r^n T_2^n(\eta).$$

Since $\eta \geq 0$, it is immediate that

$$\lim_{r\to 2^{-\theta}} \|(1 - rT_2)^{-2}\eta\|_{\mathcal{X}(\mathbb{Q}_0)} = \infty.$$

Hence we can find a sequence $(r_n)$ with $r_n \to 2^{-\theta}$ such that either

$$\lim_{n\to\infty} \frac{\|(I - r_nT_2)^{-2}\eta\|_{\mathcal{X}(\mathbb{Q}_0)}}{\|(I - r_nT_2)^{-1}\eta\|_{\mathcal{X}(\mathbb{Q}_0)}} = \infty$$

or

$$\lim_{n\to\infty} \|(I - r_nT_2)^{-1}\eta\|_{\mathcal{X}(\mathbb{Q}_0)} = \infty.$$

In either case we can determine $\xi_n$ with $\|\xi_n\|_{\mathcal{X}(\mathbb{Q}_0)} = 1$ and $\lim_{n\to\infty} \|(I - r_nT_2)\xi_n\|_{\mathcal{X}(\mathbb{Q}_0)} = 0$, which implies $(i)$.

$(ii)$ This is easier. We work in the equivalent $\ell_p$-norm on $\mathcal{X}(\mathbb{Q}_0)$. Then $\|T_2^n\|_{\ell_p(\mathbb{Q}_0)\to\ell_p(\mathbb{Q}_0)} = 2^{n/p}$ and $\|T_3^n\|_{\ell_p(\mathbb{Q}_0)\to\ell_p(\mathbb{Q}_0)} = 3^{n/p}$. Let

$$\xi_n = n^{-2/p} \sum_{j=1}^{n}\sum_{k=1}^{n} 2^{-j/p}3^{-k/p}T_2^jT_3^k e_0, \qquad n = 1, 2, \ldots.$$

Then $\|\xi_n\|_p = 1$, and (since $T_2$ and $T_3$ commute!),

$$\|2^{-1/p}T_2\xi_n - \xi_n\|_p = 2^{\frac{1}{p}}n^{-\frac{1}{p}},$$

$$\|3^{-1/p}T_2\xi_n - \xi_n\|_p = 3^{\frac{1}{p}}n^{-\frac{1}{p}}.$$

Renormalizing in the $\mathcal{X}$-norm gives the result. $\square$

*Conclusion of the Proof of Theorem 12.3.9.* We have reduced the proof to the case that $X$ is a spreading sequence space $\mathcal{X}$ with 1-unconditional canonical basis. Using $(i)$ of Lemma 12.3.12, we can find a sequence $(u_n)$ in $c_{00}$ such that $\|u_n\|_{\mathcal{X}} = 1$ and $2^\theta u_n$ is $\epsilon_n$-equivalent to $u_n \oplus u_n$, where $\epsilon_n \to 0$. Indeed, we may assume that the $\xi_n$ given by the lemma have finite support, and then we simply take $u_n$ to have the same nonzero entries in the same order as $\xi_n$. Then $u_n \oplus u_n$ is, similarly, equivalent to $T_2\xi_n$.

For each $n$ we can define a new spreading sequence space $\mathcal{Y}_n$ by

$$\Big\|\sum_{j=1}^{N} a_je_j\Big\|_{\mathcal{Y}_n} = \|a_1u_n \oplus a_2u_n \oplus \cdots \oplus a_Nu_n\|_{\mathcal{X}},$$

and then passing to a subsequence we can form a limit $\mathcal{Y}$ (as in Lemma 12.3.10). Then $\mathcal{Y}$ is block finitely representable in $\mathcal{X}$, and $2^{\theta}e_1$ is replaceable by $e_1 + e_2$.

If $\theta = 0$, then $\|e_1 + \cdots + e_n\|_{\mathcal{Y}} = 1$, so $\mathcal{Y}$ is isometric to $c_0$, and we are done.

If $\theta > 0$, let $1/p = \theta$ and observe that Lemma 12.3.11 implies that $\mathcal{Y}$ has a norm equivalent to the $\ell_p$-norm. Now use Lemma 12.3.12 (*ii*) and repeat the procedure to produce a spreading sequence space $\mathcal{Z}$ with 1-unconditional canonical basis, still block finitely representable in $\mathcal{X}$ but this time with both the properties that $2^{1/p}e_1$ is replaceable by $e_1 + e_2$ and $3^{1/p}e_1$ is replaceable by $e_1 + e_2 + e_3$. Lemma 12.3.11 ensures that $\mathcal{Z}$ is isometric to $\ell_p$. □

**Theorem 12.3.13 (Dvoretzky's Theorem).** *The space $\ell_2$ is finitely representable in every infinite-dimensional Banach space.*

*Proof.* An immediate conclusion from Krivine's theorem is that some $\ell_p (1 \leq p < \infty)$ or $c_0$ is finitely representable in every infinite-dimensional Banach space $X$. In the case of $c_0$ this implies that $\ell_\infty$ is finitely representable in $X$, and hence so is every separable Banach space. If $\ell_p$ is finitely representable, then so is $L_p$ (Proposition 12.1.8), and since $\ell_2$ is isometric to a subspace of $L_p$ (Theorem 6.4.12), we obtain the theorem.

□

Dvoretzky's theorem is one of the most celebrated results in Banach space theory, but the above proof is not the first or the usual proof. Dvoretzky proved the theorem in 1961 [80], well before the techniques of Krivine's theorem were known. The form we have proved implies a quantitative version. More precisely, given $\epsilon > 0$ and $n \in \mathbb{N}$ there exists $N = N(n, \epsilon)$ such that if $X$ is a Banach space of dimension $N$, then it has a subspace $E$ of dimension $n$ with $d(E, \ell_2^n) < 1+\epsilon$ (see the Problems). However, the *infinite-dimensional* method of proof prevents us from using this approach to gain any information about the function $N(n, \epsilon)$. In the last chapter we will look at quantitative *finite-dimensional* arguments that give more precise information.

There is much more to say about Krivine's theorem. It is of interest, for instance, to determine which $\ell_p$ is obtained in the theorem. For example, if we can find a spreading model $\mathcal{X}$ with 1-unconditional canonical basis $(e_n)_{n=1}^{\infty}$ satisfying a lower estimate

$$\|e_1 + \cdots + e_n\|_{\mathcal{X}} \geq cn^{\frac{1}{p}}, \qquad n = 1, 2, \ldots,$$

for some $p$, then we can show that $\ell_p$ is finitely representable in $\mathcal{X}$ (see the problems). By more delicate considerations, we can obtain the following theorem, essentially due to Maurey and Pisier [215]:

**Theorem 12.3.14.** *Let $X$ be an infinite-dimensional Banach space and suppose $p_X = \sup\{p : X \text{ has type } p\}$ and $q_X = \inf\{q : X \text{ has cotype } q\}$. Then both $\ell_{p_X}$ and $\ell_{q_X}$ are finitely representable in $X$.*

The reader who wishes to know more should consult either the book of Milman and Schechtman [224] or that of Benyamini and Lindenstrauss [23].

## Problems

**12.1.** Prove Theorem 12.1.15 (*ii*).

**12.2.** Suppose $X$ is a Banach space of type $p$ [respectively, cotype $q$]. Show that $X^{**}$ has type $p$ [respectively, cotype $q$] with the same constants.

**12.3.** A Banach space $X$ is said to be *strictly convex* if for every $x, y \in X$ with $\|x\| = \|y\| = 1$ such that $\|x + y\| = 2$ we have $x = y$. A Banach space $X$ is said to be *uniformly convex* if given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x + y\| > 2 - \delta$, then $\|x - y\| < \epsilon$.

Show that a Banach space $X$ is uniformly convex if and only if every Banach space finitely representable in $X$ is strictly convex.

**12.4.** (a) Show that the $L_p$-spaces for $1 < p < \infty$ are strictly convex.

(b) Show that for every $f \in L_p$ with $\|f\|_p = 1$ and $|f(s)| > 0$ a.e. there is an isometric isomorphism $T_f : L_p \to L_p$ with $T_f f = 1$ (the constantly one function).

(c) Show that if $(f_n)_{n=1}^{\infty}$, $(g_n)_{n=1}^{\infty}$ are two sequences in $L_p$ for $1 < p < \infty$ with $f_n + g_n = c_n 1$, where $\lim_{n\to\infty} c_n = 2$, then $\lim_{n\to\infty} \|f_n - g_n\|_p = 0$. [*Hint:* Use reflexivity.]

(d) Combine (a), (b), and (c) to show that the $L_p$-spaces for $1 < p < \infty$ are uniformly convex. Also note that we can deduce this from Problem 12.3.

**12.5 (James Criterion for Reflexivity [130]).**

(a) If $X$ is a nonreflexive Banach space and $0 < \theta < 1$, show that we can find a sequence $(x_n)_{n=1}^{\infty}$ in the unit ball of $X$ such that

$$\|x\| \geq \theta, \qquad x \in \mathrm{co}\{x_j\}_{j=1}^{\infty}, \tag{12.4}$$

and

$$\|y - z\| \geq \theta, \qquad y \in \mathrm{co}\{x_j\}_{j=1}^{n},\ z \in \mathrm{co}\{x_j\}_{j=n+1}^{\infty},\ n = 1, 2, \ldots. \tag{12.5}$$

(b) Show, conversely, that the existence of a sequence in the unit ball satisfying (12.5) implies that $X$ is nonreflexive.

(c) Deduce that a uniformly convex space is reflexive.

**12.6.** (a) Give an example of a reflexive space that is not superreflexive.

(b) Show that $X$ is superreflexive if and only if given $\epsilon > 0$ there exists $N = N(\epsilon)$ such that if $x_j \in B_X$ for $1 \leq j \leq N$, then there exist $1 \leq n \leq N$ and $y \in \mathrm{co}\{x_1, \ldots, x_n\}$, $z \in \mathrm{co}\{x_{n+1}, \ldots, x_N\}$ with $\|y - z\| < \epsilon$.

(c) Show that a uniformly convex space is superreflexive. Deduce from the above that the spaces $L_p$ and $\ell_p$ are superreflexive for $1 < p < \infty$.

It is a result of Enflo [87] and Pisier [254] that superreflexive spaces always have an equivalent uniformly convex norm. The subject of renorming is a topic in itself, and we refer the reader to [60].

**12.7.** Show that a Banach space $X$ has nontrivial type if and only if given $\epsilon > 0$ there exists $N$ such that if $x_j \in B_X$ for $1 \leq j \leq N$ with $\|x_j\| = 1$, then there exist a subset $A$ of $\{1, 2, \ldots, N\}$ and $y \in \text{co}\{x_j\}_{j \in A}$, $z \in \text{co}\{x_j\}_{j \notin A}$ with $\|y - z\| < \epsilon$.

Compare with Problem 12.1.16; this criterion is simply an unordered version of the criterion for superreflexivity. However, James showed the existence of a nonreflexive Banach space with type 2 [134]!

**12.8.** Let $X$ be a separable Banach space such that $X^*$ is separable and has (BAP). Show that $X$ has (BAP) and indeed (MAP) (see Problems 1.9 and 1.10). [*Hint:* The problem here is that there exist finite-rank operators $T_n : X^* \to X^*$ such that $T_n x^* \to x^*$ for $x^* \in X^*$, but the $T_n$ need not be adjoints of operators on $X$. Use the principle of local reflexivity.]

**12.9.** Show that if $X$ is reflexive, then its canonical image in any ultrapower $X_{\mathcal{U}}$ is 1-complemented in $X_{\mathcal{U}}$.

**12.10.** Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Show that if $X$ is superreflexive, then the dual of $X_{\mathcal{U}}$ can be naturally identified with $(X^*)_{\mathcal{U}}$.

**12.11.** Prove the equality (12.3) in Lemma 12.3.12.

**12.12 (Dvoretzky's Theorem (Quantitative Version)).** Prove that given $\epsilon > 0$ and $n \in \mathbb{N}$, there exists $N = N(n, \epsilon)$ such that if $X$ is a Banach space of dimension $N$, then it has a subspace $E$ of dimension $n$ with $d(E, \ell_2^n) < 1 + \epsilon$. [*Hint:* Use an ultraproduct.]

**12.13.** Suppose $X$ is a Banach space with an unconditional basis $(x_n)_{n=1}^\infty$ such that for some $1 \leq p \leq 2$ we have

$$c|A|^{1/p} \leq \Big\| \sum_{j \in A}^{n} x_j \Big\|$$

for every finite subset of $\mathbb{N}$. Show that $\ell_p$ is finitely representable in $X$.

# Chapter 13
# An Introduction to Local Theory

The aim of this chapter is to provide an introduction to the ideas of the local theory and a quantitative proof of Dvoretzky's theorem.

Dvoretzky's theorem asserts that every $n$-dimensional normed space $X$ contains a subspace $X_0$ of dimension $k = k(n, \epsilon)$ with $d(X_0, \ell_2^k) < 1 + \epsilon$, where $k(n, \epsilon) \to \infty$ as $n \to \infty$. Dvoretzky's original paper [80] gave this without the optimal estimates for $k(n, \epsilon)$. We present a proof due to Milman [222] that gives the estimate

$$k(n, \epsilon) \geq c\epsilon^2 |\log \epsilon|^{-1} \log n.$$

This is optimal in dependence on $n$ but not on $\epsilon$. In 1985, Gordon [111] showed that the $|\log \epsilon|$ term can be removed, so that $k(n, \epsilon) \geq c\epsilon^2 \log n$.

The study of finite-dimensional normed spaces is a very rich area, and Dvoretzky's theorem is only the beginning of this subject, which flowered remarkably during the 1980s and early 1990s. Since then there has been an evolution of the area with more emphasis on the geometry of convex sets; nowadays it continues to be an important area.

As a prelude we introduce the John ellipsoid and prove the Kadets–Snobar theorem that every $n$-dimensional subspace of a Banach space is $\sqrt{n}$-complemented.

Finally, we return to the complemented subspace problem and present a complete proof that a Banach space in which every subspace is complemented is a Hilbert space (Lindenstrauss and Tzafriri [200]).

We emphasize that throughout this chapter we treat only *real* scalars, although much of the theory does permit an easy extension to complex scalars. We will rescue a notation that we introduced in Chapter 7 to denote by $d_X$ the Euclidean distance of $X$, i.e., $d_X = d(X, \ell_2^n)$, where $n$ =dim$(X)$. If $X$ is an infinite-dimensional Banach space, then $d_X = d(X, H)$, where $H$ is a Hilbert space of the same density character of $X$. We will be working with different norms on a finite-dimensional Banach space $X$. These will induce different norms on the vector space $\mathcal{B}(X)$ of bounded



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linear operators from $X$ to $X$. We will write $T_{\mathsf{E}\to\mathsf{F}}$ when thinking of $T$ as a bounded linear map from $(X, \|\cdot\|_{\mathsf{E}})$ into $(X, \|\cdot\|_{\mathsf{F}})$ and denote the norm of this operator by $\|T\|_{\mathsf{E}\to\mathsf{F}}$ or $\|T_{\mathsf{E}\to\mathsf{F}}\|$.

## 13.1 The John Ellipsoid

Suppose $(X, \|\cdot\|_X)$ is an $n$-dimensional normed space and let $\|\cdot\|_{\mathsf{E}}$ be a Euclidean norm on $X$, i.e., $\|\cdot\|_{\mathsf{E}}$ is induced by an inner product on $X$. The unit ball of the normed space $(X, \|\cdot\|_{\mathsf{E}})$,

$$\mathcal{E}_n := \{x \in X \colon \|x\|_{\mathsf{E}} \leq 1\},$$

is called an ($n$-dimensional) *ellipsoid* in $X$.

Let $(e_i)_{i=1}^n$ be the unit vector basis in $\ell_2^n$. By choosing an orthonormal basis $(x_i)_{i=1}^n$ with respect to the inner product defining $\|\cdot\|_{\mathsf{E}}$ in $X$, we have a linear transformation

$$S\colon \ell_2^n \to X, \quad e_i \mapsto S(e_i) = x_i \text{ for } i = 1, \dots, n,$$

such that $S(B_{\ell_2^n}) = \mathcal{E}_n$, where $B_{\ell_2^n} = \{\xi = (\xi(i))_{i=1}^n \in \mathbb{R}^n : \sum_{i=1}^n |\xi(i)|^2 \leq 1\}$. Conversely, every one-to-one linear transformation $S\colon \ell_2^n \to X$ carries over the Hilbertian structure of $\ell_2^n$ onto the space $X$ via

$$\langle x, y\rangle = \langle S^{-1}x, S^{-1}y\rangle, \quad x, y \in X.$$

Thus $S(B_{\ell_2^n})$ is an ellipsoid in $(X, \|\cdot\|_X)$.

Our first task in this section will be to prove the existence of an ellipsoid of maximum volume inside $B_X$. To clarify what we mean by that, if $S : \mathbb{R}^n \to X$ is a linear map, the *volume* of a Borel set $A$ in $X$ is naturally measured by

$$\mathrm{vol}_S(A) = |S^{-1}(A)|,$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$. We then say that $\mathrm{vol}_S\colon \mathcal{B}(X) \to [0, \infty]$ is a *Lebesgue measure on* $X$. Of course, the quantity $\mathrm{vol}_S(A)$ changes with $S$, but if instead of computing volumes in $X$ we are merely interested in comparing them, it is enough to observe that for any two Borel sets $A$, $B$ in $X$, the ratio

$$\frac{\mathrm{vol}_S(A)}{\mathrm{vol}_S(B)}$$

is independent of $S$. Quantitatively, given two isomorphisms $S_1\colon \ell_2^n \to X$ and $S_2\colon \ell_2^n \to X$, we have

$$\mathrm{vol}_{S_1}(A) = |\det(S_1 S_2^{-1})| \mathrm{vol}_{S_2}(A), \tag{13.1}$$

for every Borel set $A$ in $X$. In other words, two Lebesgue measures on a finite-dimensional space $X$ differ by a (multiplicative) constant.

**Theorem 13.1.1.** *Suppose $(X, \|\cdot\|_X)$ is an $n$-dimensional normed space. Then there exists an $n$-dimensional ellipsoid of maximum volume contained in $B_X$.*

*Proof.* Pick an ellipsoid $\mathcal{E}_n$ contained in $B_X$. This ellipsoid corresponds to the image $S(B_{\ell_2^n})$ of some linear map $S\colon \ell_2^n \to X$. Let $\|\cdot\|_{\mathsf{E}}$ be the Euclidean norm induced on $X$ by $S$. Any other ellipsoid $\mathcal{E}_n'$ in $X$ is the image of $\mathcal{E}_n$ under some linear isomorphism $T \in \mathcal{B}(X)$. Moreover, $\mathcal{E}_n' \subseteq B_X$ if and only if $\|T\|_{\mathsf{E}\to X} \leq 1$. Since by (13.1),

$$\mathrm{vol}_S(\mathcal{E}_n') = |\det T| \mathrm{vol}_{T\circ S}(\mathcal{E}_n') = |\det T||B_{\ell_2^n}|,$$

we are maximizing the function $T \mapsto |\det T|$ on the unit ball of the Banach space $(\mathcal{B}(X), \|\cdot\|_{\mathsf{E}\to X})$ . The existence of such a maximum follows by continuity and by the compactness of the unit ball of finite-dimensional Banach spaces. □

It is also true but irrelevant to the remainder of the chapter that the ellipsoid of maximum volume contained in the unit ball $B_X$ is *unique*. This ellipsoid is called the *John ellipsoid* and was introduced by Fritz John in 1948 [135].

Once we have agreed on the existence of the John ellipsoid, let us fix as the Euclidean structure on $X$ that induced by $\mathcal{E}_X$. We then denote by $\|\cdot\|_{\mathcal{E}_X}$ the Euclidean norm induced on $X$ by its John ellipsoid. In this way, $X$ has an associated inner product $\langle\ ,\ \rangle$ and corresponding norm $\|\cdot\|_{\mathcal{E}_X}$ such that $\|I\|_{\mathcal{E}_X\to X} \leq 1$, and

$$|\det T| \leq 1 \quad \text{whenever} \quad T \in \mathcal{B}(X) \text{ with } \|T\|_{X_{\mathcal{E}}\to X} \leq 1.$$

Hence if $\dim X = n$, a dilation argument yields

$$|\det T| \leq \|T\|^n_{\mathcal{E}_X\to X} \text{ for all } T \in \mathcal{B}(X). \tag{13.2}$$

Notice that (13.2), when applied to the identity map, gives

$$\|I\|_{\mathcal{E}_X\to X} = 1. \tag{13.3}$$

We will put $(X, \|\cdot\|_{X^*})$ to denote the dual space of $(X, \|\cdot\|_X)$ using the inner product, i.e., for $x \in X$,

$$\|x\|_{X^*} = \sup\{|\langle x, y\rangle| : y \in X,\ \|y\|_X \leq 1\}.$$

Dualizing (13.3) yields $\|I\|_{X^*\to X_{\mathcal{E}}} \leq 1$. Therefore

$$\|x\|_X \leq \|x\|_{\mathcal{E}_X} \leq \|x\|_{X^*}, \quad x \in X.$$

**Lemma 13.1.2.** *Suppose $(X, \|\cdot\|_X)$ is an $n$-dimensional normed space. Let $x \in X$ be such that $\|x\|_X = \|x\|_{\mathcal{E}_X} = 1$. Then $\|x\|_{X^*} = 1$.*

*Proof.* We must see that $\langle x, y\rangle \leq 1$ whenever $\|y\|_X \leq 1$. For $t > 0$ we have

$$\|(1+t)x - ty\|_{\mathcal{E}_X} \geq \|(1+t)x - ty\|_X \geq (1+t)\|x\|_X - t\|y\|_X \geq 1.$$

Consequently,

$$(1+t)^2 + t^2\|y\|_{\mathcal{E}_X}^2 - 2t(1+t)\langle x, y\rangle \geq 1, \quad t > 0.$$

After simplifying we get

$$2 + t + t\|y\|_{\mathcal{E}_X}^2 - 2(1+t)\langle x, y\rangle \geq 0, \quad t > 0.$$

Letting $t \to 0$, we obtain the desired result. □

Next we are going to show that the John ellipsoid has some remarkable and important properties.

**Lemma 13.1.3.** *Suppose that* $(X, \|\cdot\|_X)$ *is an n-dimensional normed space and that* $T \in \mathcal{B}(X)$. *Then there is* $x \in X$ *with* $\|x\|_{\mathcal{E}_X} = \|x\|_X = 1$ *such that*

$$|\operatorname{tr}(T)| \leq n\|T(x)\|_X,$$

*where* $\operatorname{tr}(T)$ *is the trace of* $T$ *and* $\dim X = n$.

*Proof.* For any $t > 0$ pick $x_t$ such that $\|x_t\|_{X_\mathcal{E}} = 1$ and

$$\|x_t + tT(x_t)\|_X = \|I + tT\|_{\mathcal{E}_X \to X}. \tag{13.4}$$

Since $\mathcal{E}_X$ is a compact set, there exist $(t_k)_{k=1}^\infty$ in $(0, \infty)$ and $x \in \mathcal{E}_X$ such that $\lim_k x_{t_k} = x$. Clearly $\|x\|_{\mathcal{E}_X} = 1$ and, by (13.4), $\|x\|_X = \|I\|_{\mathcal{E}_X \to X} = 1$. Applying (13.2) and (13.4) yields

$$\det(I + t_k T) \leq \|x_k + t_k T(x_{t_k})\|_X^n \leq \left(1 + t_k\|T(x_{t_k})\|_X\right)^n, \quad k \in \mathbb{N}.$$

Thus

$$\operatorname{tr}(T) = \lim_{t \to 0^+} \frac{\det(I + tT) - 1}{t} \leq \lim_k \frac{(1 + t_k\|T(x_{t_k})\|_X)^n - 1}{t_k} = n\|T(x)\|_X.$$

□

**Theorem 13.1.4.** *Suppose* $(X, \|\cdot\|_X)$ *is an n-dimensional normed space. Then,*

$$\pi_2(I_{X \to \mathcal{E}_X}) \leq \sqrt{n}.$$

*Proof.* Suppose $x_1, \dots, x_k \in X$. Consider the operator $T\colon X \to X$ given by $T = \sum_{i=1}^k x_i \otimes x_i$, that is,

$$Tu = \sum_{i=1}^k \langle x_i, u\rangle x_i.$$

We note that

$$\operatorname{tr}(T) = \sum_{i=1}^{k} \langle x_i, x_i \rangle = \sum_{i=1}^{k} \|x_i\|_{\mathcal{E}_X}^2.$$

By Lemmas 13.1.3 and 13.1.2, there is $x \in X$ such that $\|x\|_X = \|x\|_{\mathcal{E}_X} = \|x\|_{X^*} = 1$ and $\operatorname{tr}(T) \leq n\|T(x)\|_X$. Let $y \in X$ be such that $\|y\|_{X^*} \leq 1$ and $\|T(x)\|_X = \langle Tx, y \rangle$. Then

$$\begin{aligned}
\sum_{i=1}^{k} \|x_i\|_{\mathcal{E}_X}^2 &\leq n \sum_{i=1}^{k} \langle x_i, x \rangle \langle x_i, y \rangle \\
&\leq n \left( \sum_{i=1}^{k} |\langle x_i, x \rangle|^2 \right)^{1/2} \left( \sum_{i=1}^{k} |\langle x_i, y \rangle|^2 \right)^{1/2} \\
&\leq n \max_{\|u\|_{X^*} \leq 1} \sum_{i=1}^{k} |\langle x_i, u \rangle|^2.
\end{aligned}$$

Taking square roots, we get exactly the statement that $\pi_2(I_{X \to \mathcal{E}_X}) \leq \sqrt{n}$. □

This theorem has immediate applications.

**Theorem 13.1.5 (John).** *If $(X, \|\cdot\|_X)$ is an $n$-dimensional normed space, then $d(X, \ell_2^n) \leq \sqrt{n}$.*

*Proof.* We have $\|I\|_{\mathcal{E}_X \to X} = 1$ and $\|I\|_{X \to \mathcal{E}_X} \leq \pi_2(I_{X \to \mathcal{E}_X}) \leq \sqrt{n}$. Geometrically,

$$\mathcal{E}_X \subset B_X \subset \sqrt{n}\mathcal{E}_X.$$

□

The estimate given by Theorem 13.1.5 is the best possible.

**Proposition 13.1.6.** *If $X = \ell_\infty^n$ (or $X = \ell_1^n$), then $d(X, \ell_2^n) = \sqrt{n}$.*

*Proof.* Let $S \colon \ell_\infty^n \to \ell_2^n$ be a linear isomorphism. Let $D = \|S\|_{\ell_\infty^n \to \ell_2^n}$ and $C = \|S^{-1}\|_{\ell_2^n \to \ell_\infty^n}$. We have

$$C^{-1}\|x\|_\infty \leq \|Sx\|_2 \leq D\|x\|_\infty, \quad x \in \ell_\infty^n.$$

For each choice of signs $(\epsilon_i)_{i=1}^n$, the operator $U_{\epsilon_1,\ldots,\epsilon_n}(x) = (\epsilon_1 x_1, \ldots, \epsilon_n x_n)$ is an isometry on $\ell_\infty^n$, so $SU_{\epsilon_1,\ldots,\epsilon_n}$ satisfies the same bounds as $S$. Considering choices of signs as outcomes of a Rademacher sequence $\varepsilon_1, \ldots, \varepsilon_n$ on some probability space $(\Omega, \mathbb{P})$, we may define $T \colon \ell_\infty^n \to L_2(\Omega, \mathbb{P}; \ell_2^n)$ by

$$Tx(\omega) = SU_{\varepsilon_1(\omega),\ldots,\varepsilon_n(\omega)}x, \quad x \in \ell_\infty^n.$$

Then

$$C^{-1}\|x\|_\infty \le \|Tx\|_{L_2(\mathbb{P})} \le D\|x\|_\infty, \quad x \in \ell_\infty^n. \tag{13.5}$$

But

$$\Big\|T\Big(\sum_{i=1}^k x_i e_i\Big)\Big\|^2 = \mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i S e_i\Big\|^2 = \sum_{i=1}^n |x_i|^2 \|Se_i\|^2.$$

Applying (13.5) to $e_i$ gives

$$C^{-1} \le \|Se_i\|, \quad i = 1, \dots, n.$$

Now, applying (13.5) to $\sum_{i=1}^k e_i$ yields $C^{-1}\sqrt{n} \le D$, i.e., $CD \le \sqrt{n}$. Notice that $\|I\|_{\ell_\infty^n \to \ell_2^n} = \sqrt{n}$ and $\|I\|_{\ell_2^n \to \ell_\infty^n} = 1$, so that the identity map realizes the optimal isomorphism. □

The following result is due to Kadets and Snobar [149].

**Theorem 13.1.7 (The Kadets–Snobar Theorem).** *Let $X$ be a Banach space of dimension $n$. Then for every Banach space $Y$ containing $X$ as a subspace there is a projection $P$ of $Y$ onto $X$ with $\|P\| \le \sqrt{n}$.*

*Proof.* According to Theorem 13.1.4, there is an operator $S : X \to \ell_2^n$, where $n = \dim X$, such that $\|S^{-1}\| = 1$ and $\pi_2(S) \le \sqrt{n}$. Using Theorem 8.2.13, we see that $S$ extends to a bounded operator $T : Y \to \ell_2^n$ with $\pi_2(T) = \pi_2(S)$. Hence $\|T\| \le \sqrt{n}$, and if $P = S^{-1}T$, we have our desired projection. □

This result is not optimal (but very nearly is). We refer to the article [174] for more details. We also mention that the example of Pisier [256] cited in Chapter 8 gives a Banach space $Y$ with the property that there is a constant $c > 0$ such that whenever $X$ is an $n$-dimensional subspace and $P : Y \to X$ is a projection, then $\|P\| \ge c\sqrt{n}$.

## 13.2 The Concentration of Measure Phenomenon

We are now en route to Dvoretzky's theorem, which will be deduced from a principle that has become known as the *concentration of measure phenomenon*. Roughly speaking, this says that a Lipschitz function on the Euclidean sphere in dimension $n$ behaves more and more like a constant as the dimension grows. More precisely, the set on which a Lipschitz function deviates from its average by some fixed $\epsilon$ has measure converging to zero at a very rapid rate. The reader is referred to Definition 14.1.5 for a quick reminder of the concepts of a Lipschitz map and of a Lipschitz constant.

This type of result is usually derived from Lévy's isoperimetric inequality [187]. We follow an alternative approach due to Maurey and Pisier [259] and [224, Appendix V] that has the advantage of using Gaussians.

We shall consider $\mathbb{R}^n$ with its canonical Euclidean norm, $\|\cdot\|_2$.

By $\sigma_n$ we denote the normalized measure on the surface of the sphere $\mathcal{S}^{n-1} = \{\xi = (\xi(j))_{j=1}^n : \|\xi\|_2^2 = \sum_{j=1}^n |\xi_j|^2 = 1\}$. Thus $\sigma_n$ is simply a normalized surface measure, and it is invariant under orthogonal transformations. It can be obtained by the formula

$$\int_{\mathcal{S}^{n-1}} f(\xi) d\sigma_n(\xi) = \int_{\mathcal{O}_n} f(U\xi_0) d\mu(U), \qquad f \in \mathcal{C}(\mathcal{S}^{n-1}),$$

where $\mu$ is the normalized Haar measure on the orthogonal group $\mathcal{O}_n$ and $\xi_0$ is some fixed vector in $\mathcal{S}^{n-1}$.

Let $G$ be an $n$-dimensional Gaussian, i.e., $G = (g_1, \dots, g_n)$, where the $g_i$ are mutually independent Gaussian random variables on some probability space. The distribution of the vector-valued random variable $G$ is given by the density function

$$\frac{1}{(2\pi)^{1/2}} e^{-|\xi_1|^2/2} \cdots \frac{1}{(2\pi)^{1/2}} e^{-|\xi_n|^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\|\xi\|_2^2/2}.$$

This expression yields that $G$ is invariant under orthogonal transformations. Consequently, the distribution of the $\mathcal{S}^{n-1}$-valued random variable $G/\|G\|_2$ is given by the unique orthogonally invariant probability measure on $\mathcal{S}^{n-1}$, that is, $\sigma_n$. Another property of $G$ that we will need is that $\mathbb{E}(\|G\|_2) < \infty$.

**Theorem 13.2.1.** *Let $f$ be a Lipschitz function on $\mathbb{R}^n$ with Lipschitz constant* 1. *Then for each $t > 0$,*

$$\mathbb{P}(|f(G) - \mathbb{E}f(G)| > t) \leq 2e^{-2t^2/\pi^2}.$$

*Proof.* By standard approximation arguments of Lipschitz functions (see Lemmas 14.2.4 and 14.2.5) we can suppose that $f$ is continuously differentiable, in which case $\|\nabla f(\xi)\|_2 \leq 1$ for all $\xi \in \mathbb{R}^n$. Adjusting the constant, we can also assume that $\mathbb{E}f(G) = 0$.

Let us introduce an independent copy $G'$ of $G$ such that $(G, G')$ is a $2n$-dimensional Gaussian. For every $\alpha$ put

$$G_\alpha = G \sin\alpha + G' \cos\alpha$$

and

$$G'_\alpha = G \cos\alpha - G' \sin\alpha.$$

The invariance under orthogonal transformations of $(G,G')$ in $\mathbb{R}^{2n}$ yields that $(G_\alpha, G'_\alpha)$ has the same distribution as $(G,G')$ for every choice of $\alpha$.

Suppose $\lambda > 0$. We note that since $\mathbb{E}(f(G')) = 0$, by Jensen's inequality,

$$\mathbb{E}(\exp(\lambda f(G))) \leq \mathbb{E}(\exp(\lambda f(G)))\,\mathbb{E}(\exp(-\lambda f(G'))) = \mathbb{E}\exp(\lambda(f(G) - f(G'))).$$

Now

$$f(G) - f(G') = \int_0^{\pi/2} \frac{d}{d\alpha} f(G_\alpha)\,d\alpha = \int_0^{\pi/2} \left\langle \nabla f(G_\alpha), G'_\alpha \right\rangle d\alpha.$$

Using Jensen's inequality again, we obtain

$$\begin{aligned}
\mathbb{E}\exp(\lambda(f(G) - f(G'))) &= \mathbb{E}\exp\left(\frac{2}{\pi}\int_0^{\pi/2} \lambda\frac{\pi}{2}\left\langle f(G_\alpha), G'_\alpha\right\rangle d\alpha\right) \\
&\leq \frac{2}{\pi}\int_0^{\pi/2} \mathbb{E}\exp\left(\lambda\frac{\pi}{2}\left\langle \nabla f(G_\alpha), G'_\alpha\right\rangle\right) d\alpha \\
&= \frac{2}{\pi}\int_0^{\pi/2} \mathbb{E}\exp\left(\lambda\frac{\pi}{2}\left\langle \nabla f(G), G'\right\rangle\right) d\alpha \\
&= \mathbb{E}\exp\left(\lambda\frac{\pi}{2}\left\langle \nabla f(G), G'\right\rangle\right) \\
&= \mathbb{E}(F(G)),
\end{aligned}$$

where $F(\xi) = \mathbb{E}\exp\left(\lambda\left\langle \frac{\pi}{2}\nabla f(\xi), G'\right\rangle\right)$. Now, if $g$ a standard scalar Gaussian, then

$$F(\xi) = \mathbb{E}\exp\left(\lambda\frac{\pi}{2}\|\nabla f(\xi)\|_2 g\right) \leq \mathbb{E}\exp\left(\lambda\frac{\pi}{2} g\right) = \exp\left(\frac{\lambda^2\pi^2}{8}\right).$$

Thus

$$\mathbb{E}\left(\exp(\lambda f(G))\right) \leq \exp\left(\frac{\lambda^2\pi^2}{8}\right).$$

Since $\exp(|t|) \leq \exp(t) + \exp(-t)$ and $-G$ has the same distribution as $G$,

$$\mathbb{E}\left(\exp(\lambda|f(G)|)\right) \leq 2\exp\left(\frac{\lambda^2\pi^2}{8}\right);$$

hence, by Chebyshev's inequality,

$$\mathbb{P}(|f(G)| > t) \leq 2\exp\left(\frac{\lambda^2\pi^2 - 8\lambda t}{8}\right).$$

Choosing $\lambda = 4t/\pi^2$, we obtain

$$\mathbb{P}\big(|f(G)| > t\big) \le 2\exp\left(-\frac{2t^2}{\pi^2}\right).$$

□

The following theorem is due to Milman [222] and is generally referred to as the *concentration of measure phenomenon*. The precise constants are irrelevant: the key point is that as $n \to \infty$ the estimate for $\sigma_n(|f - \bar{f}| > t)$ tends to zero very rapidly. In high dimensions, Lipschitz functions on $\mathcal{S}^{n-1}$ are almost constant!

**Theorem 13.2.2 (The Concentration of Measure Phenomenon).** *Let $f$ be a Lipschitz function on $\mathcal{S}^{n-1}$ with Lipschitz constant* 1*. Then for $t > 0$,*

$$\sigma_n(|f - \bar{f}| > t) \le 4e^{-nt^2/72\pi^2},$$

*where*

$$\bar{f} = \int_{\mathcal{S}^{n-1}} f \, d\sigma_n.$$

*Proof.* We shall assume that $\bar{f} = 0$, and so $|f(x)| \le 1$ for all $x \in \mathcal{S}^{n-1}$. Let us first extend $f$ to $\mathbb{R}^n \setminus \{0\}$ by putting

$$f(x) = \|x\|_2 f(x/\|x\|_2), \qquad x \in \mathbb{R}^n.$$

Then if $x, y \in \mathbb{R}^n \setminus \{0\}$,

$$\left\| \frac{x}{\|x\|_2} - \frac{y}{\|y\|_2} \right\|_2 \le 2\frac{\|x - y\|_2}{\|y\|_2}.$$

We therefore have

$$\begin{aligned} |f(x) - f(y)| &\le \|y\|_2 \left| f\left(\frac{x}{\|x\|_2}\right) - f\left(\frac{y}{\|y\|_2}\right) \right| + \Big| \|x\|_2 - \|y\|_2 \Big| \left| f\left(\frac{x}{\|x\|_2}\right) \right| \\ &\le 3\|x - y\|_2. \end{aligned}$$

Thus the extension of $f$ to $\mathbb{R}^n$ has Lipschitz constant at most 3; note that $\mathbb{E} f(G) = 0$.

The result is trivial for $t > 1$. We wish to estimate $\mathbb{P}(|f(G/\|G\|)| > t)$ for $0 < t \le 1$, where $G$ is an $n$-dimensional Gaussian. First note that

$$\mathbb{E}\,\|G\|_2 \ge \frac{1}{\sqrt{n}}\,\mathbb{E} \sum_{j=1}^{n} |g_j| = \sqrt{\frac{2n}{\pi}} > \frac{1}{2}\sqrt{n}.$$

By Theorem 13.2.1,

$$\mathbb{P}\left(\|G\|_2 < \frac{1}{4}\sqrt{n}\right) \le \mathbb{P}\left(\left|\|G\|_2 - \mathbb{E}\,\|G\|_2\right| > \frac{1}{4}\sqrt{n}\right) \le 2\exp\left(-\frac{n}{8\pi^2}\right).$$

On the other hand,

$$\mathbb{P}\Big(|f(G)| > t\sqrt{n}/4\Big) \le 2\exp\left(-\frac{nt^2}{72\pi^2}\right).$$

Thus

$$\mathbb{P}\left(|f(G/\|G\|_2)| > t\right) \le 2\exp\left(-\frac{nt^2}{72\pi^2}\right) + 2\exp\left(-\frac{n}{8\pi^2}\right) \le 4\exp\left(-\frac{nt^2}{72\pi^2}\right).$$

□

## 13.3 Dvoretzky's Theorem

Consider $\mathbb{R}^n$ with its canonical Euclidean norm, $\|\cdot\|_2 = \|\cdot\|_{\ell_2^n}$, and suppose that we are given a second norm $\|\cdot\|_{\mathsf{F}}$ on $\mathbb{R}^n$ such that

$$\|x\|_{\mathsf{F}} \le \|x\|_2, \quad \forall\, x \in \mathbb{R}^n.$$

Notice that

$$|\|x\|_{\mathsf{F}} - \|y\|_{\mathsf{F}}| \le \|x-y\|_{\mathsf{F}} \le \|x-y\|_2, \quad \forall\, x, y \in \mathbb{R}^n.$$

So we can hope to use the results of the previous section for the function $x \mapsto \|x\|_{\mathsf{F}}$, which is 1-Lipschitz. In our study, the following characteristic of the norm plays an important role:

$$\theta_{\|\cdot\|_{\mathsf{F}}} := \theta_{\mathsf{F}} = \int_{\mathcal{S}^{n-1}} \|\xi\|_{\mathsf{F}}\, d\sigma_n(\xi).$$

Using probabilistic terminology, if $G$ is an $n$-dimensional Gaussian, we have

$$\theta_{\mathsf{F}} = \mathbb{E}\left(\left\|\frac{G}{\|G\|_2}\right\|_{\mathsf{F}}\right) = \mathbb{E}\left(\frac{\|G\|_{\mathsf{F}}}{\|G\|_2}\right).$$

We will need the following lemma:

**Lemma 13.3.1.** *Let $(Y_0, \|\cdot\|)$ be an $m$-dimensional Euclidean space. Suppose $\epsilon > 0$. Then there is an $\epsilon$-net $\{x_j\}_{j=1}^N$ for $\{x \in Y_0 \colon \|x\| = 1\}$ with $N \le (1 + \frac{2}{\epsilon})^m$.*

*Proof.* Pick a maximal subset $\{x_j\}_{j=1}^N$ of $\{x \in Y_0 \colon \|x\| = 1\}$ with the property that $\|x_i - x_j\| \geq \epsilon$ whenever $i \neq j$. It is clear that this is an $\epsilon$-net. The open balls $\{x : \|x - x_j\| < \frac{1}{2}\epsilon\}$ are disjoint and contained in $(1 + \frac{1}{2}\epsilon)B_{Y_0}$. Thus, by comparing volumes,

$$N\Big(\frac{\epsilon}{2}\Big)^m \leq \Big(1 + \frac{\epsilon}{2}\Big)^m.$$

This gives the estimate on $N$. □

**Theorem 13.3.2.** *Suppose $\|\cdot\|_{\mathsf{F}}$ is a norm on $\mathbb{R}^n$ with $\|x\|_{\mathsf{F}} \leq \|x\|_2$ for all $x \in \mathbb{R}^n$.*

*(a) Suppose $0 < \epsilon < \frac{1}{3}$. Then there is a k-dimensional subspace $X_0$ of $\mathbb{R}^n$ with*

$$(1 - \epsilon)\theta_{\mathsf{F}}\|x\|_2 \leq \|x\|_{\mathsf{F}} \leq (1 + \epsilon)\theta_{\mathsf{F}}\|x\|_2, \quad x \in X_0, \tag{13.6}$$

*provided*

$$k \leq c\theta_{\mathsf{F}}^2 n \frac{\epsilon^2}{|\log \epsilon|},$$

*where $c > 0$ is a suitable absolute constant.*

*(b) We can find a subspace $X_0$ of $\mathbb{R}^n$ with $\dim X_0 \geq k$ such that $d_{X_0} \leq 1 + \epsilon$, provided $0 < \epsilon < 1/3$ and*

$$k \leq c_1\theta_{\mathsf{F}}^2 n \frac{\epsilon^2}{|\log \epsilon|},$$

*where $c_1$ is an absolute constant.*

*Proof.* (a) Let us fix some $k$-dimensional subspace $Y_0$ of $\mathbb{R}^n$ and use Lemma 13.3.1 to pick an $\epsilon/3$-net $\{x_j\}_{j=1}^N$ for $\{x \in Y_0 \colon \|x\|_2 = 1\}$ with $N \leq (1 + 6/\epsilon)^k$.

Let $\mathcal{O}_n$ denote, as usual, the orthogonal group and $\mu$ its normalized Haar measure. We wish to estimate $\mu(A)$, where $A$ is the set of $U \in \mathcal{O}_n$ such that

$$(1 - \epsilon/3)\theta_{\mathsf{F}} \leq \|Ux_j\|_{\mathsf{F}} \leq (1 + \epsilon/3)\theta_{\mathsf{F}}, \qquad j = 1, 2 \ldots, N.$$

Let $\tilde{A}$ be the complementary set. Then

$$\mu(\tilde{A}) \leq \sum_{j=1}^N \mu\left(\left\{U \in \mathcal{O}_n \colon \big|\, \|Ux_j\|_{\mathsf{F}} - \theta_{\mathsf{F}}\big| > \tfrac{1}{3}\epsilon\theta_{\mathsf{F}}\right\}\right).$$

But,

$$\mu\left(\left\{U \in \mathcal{O}_n \colon |\, \|Ux_j\|_{\mathsf{F}} - \theta_{\mathsf{F}}| > \tfrac{1}{3}\epsilon\theta_{\mathsf{F}}\right\}\right) = \sigma_n\left(\left\{\xi \in \mathcal{S}^{n-1} \colon |\, \|\xi\|_{\mathsf{F}} - \theta_{\mathsf{F}}| > \tfrac{1}{3}\epsilon\theta_{\mathsf{F}}\right\}\right).$$

Hence, by Theorem 13.2.2,

$$\mu(\tilde{A}) \leq 4Ne^{-n\epsilon^2\theta_{\mathsf{F}}^2/648\pi^2}.$$

Now,

$$4N \leq 4^k N < 4^k \left(\frac{7}{\epsilon}\right)^k = e^{k(\log(28)-\log\epsilon)},$$

and so $\mu(\tilde{A}) < 1$, provided

$$k \leq \frac{n\epsilon^2\theta_{\mathsf{F}}^2}{648\pi^2(\log(28) - \log\epsilon)}.$$

We are now in a position to use Lemma 12.1.11, which yields that if $U \in A$, then

$$\frac{1-\epsilon}{1-\epsilon/3}\theta_{\mathsf{F}}\|x\|_2 \leq \|Ux\|_{\mathsf{F}} \leq \frac{1+\epsilon/3}{1-\epsilon/3}\theta_{\mathsf{F}}\|x\|_2, \quad x \in Y_0.$$

Since $(1+\epsilon/3)/(1-\epsilon/3) \leq 1+\epsilon$, taking $X_0 = U(Y_0)$, we obtain (13.6). This implies the theorem for a suitable $c > 0$.

(b) Given $0 < \epsilon < 1/3$, we apply (a) to $\epsilon/3$ instead of $\epsilon$. Then if

$$k \leq \frac{c}{9}\theta_{\mathsf{F}}^2 n \frac{\epsilon^2}{\log 3 - \log\epsilon},$$

there is a subspace $X_0$ with $\dim X_0 = k$ and $d_{X_0} \leq \frac{1+\frac{\epsilon}{3}}{1-\frac{\epsilon}{3}} \leq 1+\epsilon$. □

Notice that in this theorem, $0 < \theta_{\mathsf{F}} \leq 1$. In order to apply Theorem 13.3.2 in a nontrivial way one needs $\theta_{\mathsf{F}}$ large compared with $n^{-1/2}$. Let us establish a lemma that will help us to get lower estimates for $\theta_{\mathsf{F}}$.

**Lemma 13.3.3.** *Let $\|\cdot\|_{\mathsf{F}}$ be a norm on $\mathbb{R}^n$ with $\|x\|_{\mathsf{F}} \leq \|x\|_2$ for all x. Then:*

*(a)* $\theta_{\mathsf{F}} \geq 1/\|I\|_{\mathsf{F}\to\ell_2^n}$.

*(b) If G is an n-dimensional Gaussian, then* $\theta_{\mathsf{F}} \geq n^{-1/2}\,\mathbb{E}(\|G\|_{\mathsf{F}})$.

*Proof.* Part (a) follows from $1 \leq \|I\|_{\mathsf{F}\to\ell_2^n}\|\xi\|_{\mathsf{F}}$ for all $\xi \in \mathcal{S}^{n-1}$.

Since $\|G\|_2$ and $G/\|G\|_2$ are independent random variables,

$$\mathbb{E}(\|G\|_2)\,\theta_{\mathsf{F}} = \mathbb{E}(\|G\|_2)\,\mathbb{E}\left(\left\|\frac{G}{\|G\|_2}\right\|_{\mathsf{F}}\right) = \mathbb{E}\left(\|G\|_2\left\|\frac{G}{\|G\|_{\mathsf{F}}}\right\|_{\mathsf{F}}\right) = \mathbb{E}(\|G\|_{\mathsf{F}}).$$

Notice that if $g$ is a scalar Gaussian, then $\mathbb{E}(g^2) = 1$. By Jensen's inequality,

$$\mathbb{E}(\|G\|_2) \leq \left(\mathbb{E}\,\|G\|_2^2\right)^{1/2} = \left(n\,\mathbb{E}(g^2)\right)^{1/2} = n^{1/2}.$$

Combining, we obtain (b). □

We first use Theorem 13.3.2 to consider finite-dimensional $\ell_p$-spaces. The following result is due to Figiel et al. [95].

**Theorem 13.3.4.** *Suppose* $1 \leq p < \infty$ *and* $n \in \mathbb{N}$. *Then for* $0 < \epsilon < 1/3$*, the space* $\ell_p^n$ *contains a subspace* $X_0$ *with* $\dim X_0 = k$ *and* $d_{X_0} \leq 1 + \epsilon$*, provided:*

- $k \leq cn^{2/p}\epsilon^2 |\log \epsilon|^{-1}$ *if* $p \geq 2$*;*
- $k \leq cn\epsilon^2 |\log \epsilon|^{-1}$ *if* $1 \leq p \leq 2$*,*

*where* $c > 0$ *is an absolute constant.*

*Proof.* We consider $\mathbb{R}^n$ equipped with the norm $\|\cdot\|_{\mathsf{F}} = \|\cdot\|_p$ $(1 \leq p < \infty)$ and denote the corresponding $\theta_{\mathsf{F}}$ simply by $\theta_p$.

If $p > 2$, by Hölder's inequality we have

$$\|x\|_p \leq \|x\|_2 \leq n^{\frac{1}{2}-\frac{1}{p}} \|x\|_p, \quad x \in \mathbb{R}^n,$$

i.e., $\|I\|_{\ell_2^n \to \ell_p^n} \leq 1$ and $\|I\|_{\ell_p^n \to \ell_2^n} \leq n^{1/2-1/p}$. Theorem 13.3.2 and Lemma 13.3.3 (a) give the conclusion.

We do the cases $1 \leq p \leq 2$ simultaneously. Note that

$$\frac{1}{\sqrt{n}} \|x\|_1 \leq \frac{1}{n^{1/p-1/2}} \|x\|_p \leq \|x\|_2. \tag{13.7}$$

We will use the norm $\|\cdot\|_{X_1} = n^{-1/2}\|\cdot\|_1$. Let $G$ be an $n$-dimensional Gaussian and $g$ a scalar Gaussian. By Lemma 13.3.3 (b),

$$\theta_{X_1} \geq n^{-1/2}\, \mathbb{E}(\|G\|_{X_1}) = n^{-1}\, \mathbb{E}(\|G\|_1) = \mathbb{E}(g) = \sqrt{\frac{2}{\pi}}.$$

For $1 < p < 2$, we infer from (13.7) that the estimate $\theta_{X_p} \geq \sqrt{2}/\sqrt{\pi}$ also holds for the norms $\|\cdot\|_{X_p} = n^{1/2-1/p}\|\cdot\|_p$. We are done thanks to Theorem 13.3.2 and the observation that $(\mathbb{R}^n, \|\cdot\|_p)$ and $(\mathbb{R}^n, \|\cdot\|_{X_p})$ are isometric. $\square$

In order to prove Dvoretzky's theorem we need to take an arbitrary $n$-dimensional normed space $(X, \|\cdot\|_X)$ and introduce coordinates or an inner product structure so that Theorem 13.3.2 can be applied. The problem is to find the right inner product structure. The John ellipsoid is a natural place to start. However, the best estimate for $\theta_X$ that we can obtain from Theorem 13.1.4 is

$$\theta_X \geq \frac{1}{\|I\|_{X \to \mathcal{E}_X}} \geq n^{-1/2}.$$

As already remarked, this is insufficient to get any real information from Theorem 13.3.2.

The trick is to use the John ellipsoid and then pass to a smaller subspace. In fact, this technique was originally devised by Dvoretzky and Rogers in their proof of the

Dvoretzky–Rogers theorem in 1950 [81]. We remark that the following proposition is a slightly weaker form of the original lemma that is sufficient for our purposes (we found this version in [224], where it is attributed to Bill Johnson).

**Proposition 13.3.5 (The Dvoretzky–Rogers Lemma).** *Let $(X, \|\cdot\|_X)$ be an $n$-dimensional normed space and suppose that $\|\cdot\|_{\mathcal{E}_X}$ is the norm induced on $X$ by the John ellipsoid. Then there is an orthonormal basis $(e_j)_{j=1}^n$ of $(X, \|\cdot\|_{\mathcal{E}_X})$ with the property that*

$$\|e_j\|_X \geq 2^{-\frac{n}{n-j+1}}, \qquad j = 1, 2, \dots, n.$$

*In particular,*

$$\|e_j\|_X \geq 1/4, \qquad j \leq \frac{n}{2} + 1.$$

*Proof.* We must recall the definition of the John ellipsoid of $X$ as the ellipsoid of maximal volume contained in $B_X$. We pick $(e_j)_{j=1}^n$ inductively such that $\|e_1\|_{\mathcal{E}_X} = \|e_1\|_X = 1$ and, subsequently, we pick $e_j$ in which the maximum

$$t_j := \max\{\|x\|_X \colon \|x\|_{\mathcal{E}_X} = 1, \langle x, e_i\rangle = 0 \text{ for } i < j\}$$

is attained. Thus $(e_j)_{j=1}^n$ is an orthonormal basis and $\|x\|_X \leq t_j\|x\|_{\mathcal{E}_X}$ if $x \in [e_j, \dots, e_n]$.

Fix $1 \leq j \leq n$. For $a, b > 0$ let us consider the ellipsoid $\mathcal{E}_{a,b}$ of all $x$ such that

$$a^{-2}\sum_{i=1}^{j-1} |\langle x, e_i\rangle|^2 + b^{-2}\sum_{i=j}^{n} |\langle x, e_i\rangle|^2 \leq 1.$$

The ellipsoid $\mathcal{E}_{a,b}$ is contained in $B_X$, provided $a + bt_j \leq 1$, and it has volume $a^{j-1}b^{n-j+1}$ relative to the volume of $\mathcal{E}_X$. It follows that if $0 \leq b \leq t_j^{-1}$, then

$$(1 - bt_j)^{j-1}b^{n-j+1} \leq 1.$$

Choosing $b = (2t_j)^{-1}$, we obtain

$$2^n t_j^{-(n-j+1)} \leq 1.$$

This gives the conclusion. □

We will need a lemma on the behavior of the maximum of $m$ Gaussians.

**Lemma 13.3.6.** *There is an absolute constant $c > 0$ such that if $G$ is an $m$-dimensional Gaussian, then*

$$\mathbb{E}(\|G\|_\infty) \geq c(\log m)^{1/2}, \quad m = 2, 3, \dots.$$

*Proof.* If $g$ is a scalar Gaussian and $s > 0$, then

$$\mathbb{P}(|g| > s) = \sqrt{\frac{2}{\pi}} \int_s^\infty e^{-\frac{1}{2}\xi^2}\, d\xi \geq \sqrt{\frac{2}{\pi}} s e^{-2s^2}.$$

Thus

$$\mathbb{P}(\|G\|_\infty > s) \geq 1 - \left(1 - \sqrt{\frac{2}{\pi}} s e^{-2s^2}\right)^m.$$

Pick $t > 0$. Then, for $m \geq 2$,

$$\mathbb{P}\left(\|G\|_\infty > t(\log m)^{1/2}\right) \geq 1 - \left(1 - \sqrt{\frac{2}{\pi}} t(\log m)^{1/2} m^{-2t^2}\right)^m \geq 1 - d,$$

where

$$d := \sup_{m \geq 2} \left(1 - \sqrt{\frac{2}{\pi}} t(\log m)^{1/2} m^{-2t^2}\right)^m.$$

If $0 < t < 1/\sqrt{2}$, then $0 < d < 1$. Indeed, the quantity in the above supremum tends to zero as $m$ tends to infinity. Finally,

$$\mathbb{E}(\|G\|_\infty) \geq t(\log m)^{1/2} \mathbb{P}\left(\|G\|_\infty > t(\log m)^{1/2}\right) \geq (1 - d)t(\log m)^{1/2}.$$

□

We are finally ready to complete the proof of Dvoretzky's theorem, giving quantitative estimates as promised:

**Theorem 13.3.7 (Dvoretzky's Theorem).** *There is an absolute constant $c > 0$ with the following property: If $(X, \|\cdot\|_X)$ is an $n$-dimensional normed space and $0 < \epsilon < 1/3$, then $X$ has a subspace $X_0$ with $\dim X_0 = k$ and $d_{X_0} < 1 + \epsilon$ whenever*

$$k \leq c \log n \frac{\epsilon^2}{|\log \epsilon|}.$$

*Proof.* Let $\|\cdot\|_{\mathcal{E}_X}$ be the norm induced on $X$ by the John ellipsoid. By the Dvoretzky–Rogers lemma, we can pass to a subspace $Y$ of $X$ with $m = \dim Y \geq n/2$, and with the property that $(Y, \|\cdot\|_{\mathcal{E}_X})$ has an orthonormal basis $(e_j)_{j=1}^m$ such that $\|e_j\|_X \geq 1/4$ for $j = 1, \ldots, m$.

Let $G = (g_j)_{j=1}^m$ be an $m$-dimensional Gaussian. Let $(\varepsilon_j)_{j=1}^m$ be a sequence of Radamacher random variables independent of $G$. Consider $G_0 = \sum_{j=1}^m g_j e_j$. Using that every Banach space, in particular $(Y, \|\cdot\|_X)$, has cotype $\infty$ with cotype constant

1 and Lemma 13.3.6, we have

$$\mathbb{E}\,\|G_0\|_X = \mathbb{E}\left\|\sum_{j=1}^{m}\varepsilon_j g_j e_j\right\|_X \geq \mathbb{E}\max_{1\leq j\leq m}\|g_j e_j\|_X \geq \frac{1}{4}\,\mathbb{E}\,\|G\|_\infty \geq \frac{c}{4}(\log m)^{1/2}.$$

Now, we use the standard identification between $(Y, \|\cdot\|_{\varepsilon_X})$ and $(\mathbb{R}^m, \|\cdot\|_2)$ that produces the orthonormal basis $(e_j)_{j=1}^m$ to define $\theta_Y$. By Lemma 13.3.3,

$$\theta_Y \geq m^{-1/2}\,\mathbb{E}\,\|G_0\|_X \geq c_1 m^{-1/2}(\log m)^{1/2}$$

for some absolute constant $c_1 > 0$. If we apply Theorem 13.3.2, we obtain Dvoretzky's theorem. □

Dvoretzky's theorem is, of course, just the beginning of a very rich theory that is still evolving. One of the interesting questions is to decide the precise dimension of the almost Hilbertian subspace of an $n$-dimensional space. The estimate of $\log n$ is, in fact, optimal for arbitrary spaces (see the problems), but we have seen in Theorem 13.3.4 that for special spaces one can expect to do better and perhaps even obtain subspaces of *proportional dimension $cn$* as in the case $\ell_p^n$, where $1 \leq p < 2$. It turns out that this is related to the concept of cotype. Remarkably, the first part of Theorem 13.3.4 holds for every space of cotype two; this is due to Figiel, Lindenstrauss, and Milman [95]. Another remarkable result is Milman's theorem, which, roughly speaking, says that if one can take quotients as well as subspaces, then one can find an almost Hilbertian space of proportional dimension [223]. Let us give the precise statement:

**Theorem 13.3.8 (Milman's Quotient–Subspace Theorem ).** *There is an absolute constant $c$ such that if $0 < \delta < 1$ and $X$ is a finite-dimensional normed space, then there is a quotient $Y$ of a subspace of $X$ with $\dim Y > \theta \dim X$ and*

$$d_Y \leq c(1-\delta)^{-2}\log(1-\delta).$$

The reader interested in this subject should consult the books of Milman and Schechtman [224], Pisier [260], and Tomczak-Jaegermann [295] as a starting point to learn about a rapidly evolving field.

## 13.4 The Complemented Subspace Problem

Armed with Dvoretzky's theorem (which we have proved twice!), we can return to complete the complemented subspace problem, which we solved only partially in Chapter 9. Our proof follows a treatment given by Kadets and Mitjagin [146] (using an observation of Figiel) and not the original proof of Lindenstrauss and Tzafriri [200].

To get the most precise result we will prove a strengthening of Dvoretzky's theorem that is of interest in its own right. Figiel's observation was based on a somewhat easier argument of Milman [221]. However, the proof we present is in the spirit of this chapter, and demonstrates a use of the concentration of measure phenomenon.

Let us start with an elementary lemma, whose proof we omit.

**Lemma 13.4.1.** *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed spaces. Let $\|\cdot\|$ be a seminorm on $E \oplus F$ such that*

- $\|(x,0)\| = \|x\|_E$ *for all* $x \in E$,
- $\|(0,y)\| = \|y\|_F$ *for all* $y \in F$,
- $\|(x,y)\| = \|(x,-y)\|$ *for all* $x \in E$ *and all* $y \in F$.

*Then*

$$\max\{\|x\|_E, \|y\|_F\} \leq \|(x,y)\|$$

*for all $x \in E$ and for all $y \in F$. In particular, $\|\cdot\|$ is a norm.*

**Theorem 13.4.2.** *Let $(X, \|\cdot\|_X)$ be an infinite-dimensional Banach space. Suppose $X_0$ is a finite-dimensional subspace of $X$. Then for every $m \in \mathbb{N}$ there is a norm $\|\cdot\|$ on $Y = X_0 \oplus \ell_2^m$ such that $Y$ is isometric to a subspace of an ultraproduct of $X$ and:*

- $\|(x,0)\| = \|x\|_X$ *for all* $x \in X_0$,
- $\|(0,\xi)\| = \|\xi\|_2$ *for all* $\xi \in \ell_2^m$,
- $\|(x,\xi)\| = \|(x,-\xi)\|$ *for all* $x \in X_0$ *and all* $\xi \in \ell_2^m$.

*Proof.* Let $\nu > 0$ be such that $\nu^{-1}$ is an integer and let $(x_j)_{j=1}^N$ be a $\nu$-net for $B_{X_0}$. We also choose a $\nu$-net $(\xi_j)_{j=1}^M$ for $\mathcal{S}^{m-1}$.

Let $n \in \mathbb{N}$ with $n > m$; we regard $\ell_2^m$ as a subspace of $\ell_2^n$. By Dvoretzky's theorem, there is a linear map $S\colon \ell_2^n \to X$ satisfying $\|S\| \leq 1$ and $\|S^{-1}\| \leq 1/(1-\nu)$.

For $1 \leq j \leq N$ and $1 \leq k \leq \nu^{-1}$, we consider $f_{j,k}\colon \mathcal{S}^{n-1} \to \mathbb{R}$ defined by

$$f_{j,k}(\xi) = \|k\nu S(\xi) + x_j\|_X.$$

Note that each $f_{j,k}$ has Lipschitz constant at most one. Let

$$a_{j,k} = \bar{f}_{j,k} = \int_{\mathcal{S}^{n-1}} f_{j,k}\, d\sigma_n.$$

Using Theorem 13.2.2, we have

$$\sigma_n\left(|f_{j,k} - a_{j,k}| > \nu\right) \leq 4e^{-n\nu^2/72\pi^2}.$$

Thus

$$\sigma_n\left(\max_{1\leq j\leq N}\max_{1\leq k\leq \nu^{-1}} |f_{j,k} - a_{j,k}| > \nu\right) \leq 4N\nu^{-1}e^{-n\nu^2/72\pi^2}.$$

Put

$$A = \{U \in \mathcal{O}_n\colon \max_{1\le i\le M} \max_{1\le j\le N} \max_{1\le k\le \nu^{-1}} |f_{j,k}(U\xi_i) - a_{j,k}| > \nu\},$$

where $\mathcal{O}_n$ is the orthogonal group and $\mu$ its Haar measure. Arguing as in Theorem 13.3.2, we obtain the following estimate for $\mu(A)$:

$$\mu(A) \le 4MN\nu^{-1}e^{-n\nu^2/72\pi^2}.$$

Hence, if $n$ is chosen large enough, $\mu(A) < 1$, and there exists $U \notin A$. Let $T_\nu = SU\colon \ell_2^m \to X$. Then,

$$(1-\nu)\|\xi\|_2 \le \|T_\nu(\xi)\|_X \le \|\xi\|_2, \quad \xi \in \ell_2^n, \tag{13.8}$$

and

$$\Big|\|x_j + k\nu T_\nu(\xi_i)\|_X - a_{j,k}\Big| \le \nu, \quad 1 \le i \le M,\ 1 \le j \le N,\ 1 \le k \le \nu^{-1}.$$

It follows that

$$\Big|\|x_j + k\nu T_\nu(\xi)\|_X - a_{j,k}\Big| \le 2\nu, \quad 1 \le j \le N,\ 1 \le k \le \nu^{-1}, \xi \in \mathcal{S}^{m-1},$$

and so for $1 \le j \le N, 1 \le k \le \nu^{-1}$, and $\xi \in \mathcal{S}^{m-1}$,

$$\Big|\|x_j + k\nu T_\nu(\xi)\|_X - \|x_j - k\nu T_\nu(\xi)\|_X\Big| \le 4\nu.$$

Hence, approximating $\xi$ by $k\nu\xi/\|\xi\|_2$, where $k \in \mathbb{N}$ is such that $(k-1)\nu < \|\xi\|_2 \le k\nu$, and using (13.8), we have

$$\Big|\|x_j - T_\nu(\xi)\|_X - \|x_j + T_\nu(\xi)\|_X\Big| \le 6\nu, \quad 1 \le j \le N,\ \|\xi\|_2 \le 1.$$

This, in turn, implies that

$$\Big|\|x - T_\nu(\xi)\|_X - \|x + T_\nu(\xi)\|_X\Big| \le 8\nu, \quad x \in X_0,\ \|x\|_X \le 1,\ \|\xi\|_2 \le 1.$$

Since, by the triangle law and (13.8),

$$\Big|\|x - T_\nu(\xi)\|_X + \|x + T_\nu(\xi)\|_X\Big| \ge 2(1-\nu), \quad x \in X,\ \max\{\|x\|_X, \|\xi\|_2\} = 1,$$

this yields

$$\|x - T_\nu(\xi)\|_X \le \frac{1+3\nu}{1-5\nu}\|x + T_\nu(\xi)\|_X, \quad x \in X_0,\ \max\{\|x\|_X, \|\xi\|_2\} = 1. \tag{13.9}$$

By homogeneity, this inequality extends to every $x \in X_0$ and $\xi \in \ell_2^m$. Using (13.8), we can define a bounded linear operator $L: X_0 \oplus \ell_2^m \to \ell_\infty(X)$ by

$$L(x, \xi) = (x + T_{1/i}(\xi))_{i=1}^\infty.$$

Appealing to (13.8) and (13.9), we claim that the mapping $Q: X_0 \oplus \ell_2^m \to X_\mathcal{U}$ induced by $L$ satisfies

$$\|Q(x, 0)\|_\mathcal{U} = \|x\|, \ \|Q(0, \xi)\|_\mathcal{U} = \|\xi\|_2, \ \|Q(x, \xi)\|_\mathcal{U} = \|Q(x, -\xi)\|_\mathcal{U}$$

for all $x \in X_0$ and for all $\xi \in \ell_2^m$. We define a seminorm on $X_0 \oplus \ell_2^m$ by putting $\|(x, \xi)\| = \|Q(x, \xi)\|_\mathcal{U}$. Taking into account Lemma 13.4.1, it is clear that $\|\cdot\|$ possesses the desired properties. ◻

**Theorem 13.4.3.** *Let $(X, \|\cdot\|_X)$ be an infinite-dimensional Banach space with the property that there exists $\lambda \geq 1$ such that for every finite-dimensional subspace $X_0$ of $X$ there is a projection $P : X \to X_0$ with $\|P\| \leq \lambda$. Then $X$ is isomorphic to a Hilbert space.*

*Proof.* Taking into account Theorem 12.1.6, it suffices to show that $d_{X_0} \leq 4\lambda^2$ for all finite-dimensional subspaces $X_0$ of $X$. Let $n = \dim X_0$. Using Theorem 13.4.2, Lemma 13.4.1, and Proposition 12.1.12, we may find a space $Y = X_0 \oplus \ell_2^n$ isometric to a subspace of a space finitely representable in $X$, so that the norm $\|\cdot\|$ on $X_0 \oplus \ell_2^n$ satisfies

$$\|(x, 0)\| = \|x\|_X, \quad \|(0, \xi)\| = \|\xi\|_2, \quad \max(\|x\|_X, \|\xi\|_2) \leq \|(x, \xi)\| \tag{13.10}$$

for all $x \in X$ and all $\xi \in \ell_2^n$. The space $Y$ has the property that each of its subspaces is $(\lambda + \epsilon)$-complemented for all $\epsilon > 0$. Then, by a compactness argument, the space $Y$ must also have the property that each of its subspaces is $\lambda$-complemented.

Let $\rho^2 = d_{X_0}$, and choose an invertible operator $S: X_0 \to \ell_2^n$ such that $\|S\| \leq \rho$ and $\|S^{-1}\| \leq \rho$. We define a subspace of $Y$ by taking

$$Z = \{(x, Sx): x \in X_0\}.$$

Let $R: Y \to Z$ be a projection with $\|R\| \leq \lambda$.

We now define a second operator $T: X_0 \to \ell_2^n$ by

$$R(x, 0) = (S^{-1}Tx, Tx), \quad x \in X.$$

Appealing to (13.10), we obtain $\|T\| \leq \lambda$.

Then we introduce an operator $V : X_0 \to \ell_2^{2n} = \ell_2^n \oplus \ell_2^n$ given by $Vx = (\lambda S(x), \rho T(x))$. Let us estimate $\|V\|$. Clearly,

$$\|Vx\|_2^2 = \lambda^2\|S(x)\|_2^2 + \rho^2\|T(x)\|_2^2 \leq 2\lambda^2\rho^2\|x\|_2^2,$$

that is, $\|V\| \leq \sqrt{2}\lambda\rho$.

If $x \in X_0$, we have

$$\begin{aligned} R(0, S(x)) &= R(x, S(x)) - R(x, 0) \\ &= (x, S(x)) - R(x, 0) \\ &= (x - S^{-1}T(x), S(x) - T(x)), \end{aligned}$$

and so, taking into account (13.10),

$$\|x - S^{-1}T(x)\|_X \le \lambda \|S(x)\|_2, \quad x \in X_0.$$

Hence

$$\begin{aligned} \|x\|_X &\le \|x - S^{-1}T(x)\|_X + \|S^{-1}T(x)\|_X \\ &\le \lambda\|S(x)\|_2 + \rho\|T(x)\|_2 \\ &\le \sqrt{2}(\lambda^2\|Sx\|_2^2 + \rho^2\|T(x)\|_2^2)^{1/2} \\ &= \sqrt{2}\|V(x)\|_2. \end{aligned}$$

This yields that $V$ is an isomorphism onto its range, and that $\|V^{-1}\| \le \sqrt{2}$. Thus $\|V\|\|V^{-1}\| \le 2\lambda\rho$. But by our choice of $\rho$, this implies that $\rho^2 \le 2\lambda\rho$, i.e., $\rho \le 2\lambda$, or equivalently, $d_{X_0} \le 4\lambda^2$. □

**Lemma 13.4.4.** *Suppose $E$ is a finite-dimensional subspace of $X$ and let $\nu > 0$. Then there is a finite-codimensional subspace $Y$ of $E$ such that*

$$\|e + y\| \ge (1 - \nu)\|e\|, \quad e \in E,\ y \in Y.$$

*Proof.* Let $\{x_i\}_{i=1}^N$ be a $\nu$-net for $S = \{x \in E: \|x\| = 1\}$. We use the Hahn–Banach theorem to construct $\{x_i^*\}_{i=1}^N$ in $X^*$ such that $\|x_i^*\| = 1$ and $x_i^*(x_i) = 1$. Consider

$$Y = \{x \in X: x_i^*(x) = 0 \text{ for } i = 1, \ldots, N\}.$$

Let $x \in S$ and $y \in Y$. Choose $i$ such that $\|x - x_i\| \le \nu$. Then

$$\|x + y\| \ge x_i^*(x + y) = x_i^*(x) = 1 + x_i^*(x - x_i) \ge 1 - \nu.$$

From here the result follows by homogeneity. □

**Theorem 13.4.5 (Lindenstrauss and Tzafriri [200]).** *Suppose $X$ is an infinite-dimensional Banach space in which every closed subspace is complemented. Then $X$ is isomorphic to a Hilbert space.*

*Proof.* By Theorem 13.4.3 we must prove that there is $\lambda \ge 1$ such that every finite-dimensional subspace of $X$ is $\lambda$-complemented. Suppose that this is not the case and

for a finite-dimensional subspace $E$ of $X$, set,

$$\lambda(E) = \inf\{\|P\| \colon P \text{ projection on } E\}.$$

We first argue that, for every subspace $X_0$ of finite-codimension we would have

$$\sup\{\lambda(E) \colon \dim E < \infty,\ E \subset X_0\} = \infty. \tag{13.11}$$

Indeed, suppose $\sup\{\lambda(E) \colon \dim E < \infty,\ E \subset X_0\} = \alpha < \infty$. Let $k$ be the codimension of $X_0$. Then suppose $E$ is any finite-dimensional subspace of $X$. Let $E_0 = E \cap X_0$ and let $P_0$ be a projection of $X$ onto $E_0$ with $\|P_0\| \leq \alpha + 1$. Let $F_0 = \{x \in E \colon P_0 x = 0\}$. Then $\dim F_0 = \dim E/E_0 \leq \dim X/X_0 \leq k$, so by Theorem 13.1.7, there is a projection $Q_0$ of $X$ onto $F_0$ with $\|Q_0\| \leq \sqrt{k}$. Let $P = P_0 + Q_0 - Q_0 P_0$; then $P$ is a projection onto $E$ with $\|P\| \leq \alpha + 1 + \sqrt{k} + (\alpha + 1)\sqrt{k}$. This contradiction establishes (13.11).

Combining Lemma 13.4.4 with (13.11), we construct, by induction, a sequence of finite-dimensional subspaces $(E_n)_{n=1}^{\infty}$ and finite-codimensional subspaces $(X_n)_{n=1}^{\infty}$ such that

- $\lambda(E_n) > n, \quad n \in \mathbb{N}$,
- $\|e\| \leq 2\|e + x\|, \quad e \in \sum_{i=1}^{n} E_i,\ x \in X_n$,
- $E_{n+1} \subset \cap_{i=1}^{n} X_i, \quad n \in \mathbb{N}$.

Let $Y$ be the closed linear span of $\cup_{n=1}^{\infty} E_n$. If $e_j \in E_j$ for $j = 1, 2, \ldots, N$, and $1 \leq n \leq N$, since $e_{n+1} + \cdots + e_N \in X_n$, we have

$$\|e_1 + \cdots + e_n\| \leq 2\|e_1 + \cdots + e_N\|.$$

Hence,

$$\|e_n\| \leq 4\|e_1 + \cdots + e_N\|,$$

from which it follows that each $E_n$ is 4-complemented in $Y$. Since, by assumption, $Y$ is complemented, this implies $\sup_n \lambda(E_n) < \infty$, and we have reached a contradiction. □

## Problems

**13.1 (Auerbach's Lemma).**
Let $X$ be an $n$-dimensional normed space. Show that $X$ has a basis $(x_j)_{j=1}^{n}$ with biorthogonal functionals $(x_j^*)_{j=1}^{n}$ such that $\|x_j\| = \|x_j^*\| = 1$ for $1 \leq j \leq n$. This basis is called an *Auerbach basis* [*Hint*: Maximize the volume of the parallelepiped generated by $n$ vectors $y_1, \ldots, y_n$ in the unit ball.]

**13.2.** Let $E$ be a subspace of $\ell_1^n$ of dimension $k$ and suppose $E$ is complemented by a projection of norm $\alpha$. Show that $k \le K_G \alpha^2 d_E^2$, where $K_G$ is Grothendieck's constant.

**13.3.** Suppose $1 \le p < 2$. Let $E$ be a subspace of $\ell_p^n$ of dimension $k$ and suppose $E$ is complemented by a projection of norm $\alpha$. By considering $E$ a subspace of $\ell_1^n$, show that

$$k \le K_G \alpha^2 n^{2-2/p} d_E^2,$$

where $K_G$ is Grothendieck's constant.

**13.4.** Suppose $d > 1$ and $2 < p < \infty$. Show that there is a constant $C = C(d,p)$ such that if $E$ is a subspace of $\ell_2^n$ with $d_E \le d$, then $k \le Cn^{2/p}$. [*Hint*: Use the fact that $\ell_p^n$ is of type 2, and duality.] This shows that Theorem 13.3.4 is (in a certain sense) best possible.

**13.5.** Show that the volume of a ball of radius $R$ in $\mathbb{R}^n$ is

$$\operatorname{vol} B_{\mathbb{R}^n}(R) = \frac{\pi^{n/2}}{(n/2)!} R^n.$$

(We need to recall that the factorial of a nonnegative real number is defined using Euler's gamma function $\Gamma(x)$ as $x! = \Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt$. This function has the property that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$, and $\Gamma(1/2) = \sqrt{\pi}/2$.)

**13.6.** Let $X$ be an $n$-dimensional normed space. Suppose $(x_j)_{j=1}^N$ is a set of points in $X$ such that $\partial B_X \subset \cup_{j=1}^N (x_j + \nu B_X)$. Show that $B_X$ is covered by the sets $A_{jk} = k\nu x_j + 2\nu B_X$ for $1 \le j \le N$ and $1 \le k \le [\nu^{-1}]$. Deduce that

$$N \ge 2^{-n} \nu^{1-n}.$$

**13.7.** Let $H$ be a Hilbert space and suppose $x \in H$ with $\|x\| = 1$ is written as a convex combination

$$x = \sum_{j=1}^n c_j y_j,$$

where $c_1, \ldots, c_n \ge 0$, $c_1 + \cdots + c_n = 1$, and $\|y_j\| \le \alpha$ for $1 \le j \le n$. Show that there exists $j$ such that

$$\|x - y_j\|^2 \le \alpha^2 - 1.$$

**13.8.** Let $H$ be a $k$-dimensional Hilbert space and suppose $T : H \to \ell_\infty^N$ is a linear operator satisfying

$$\|x\| \le \|Tx\| \le (1+\epsilon)\|x\|, \qquad x \in H.$$

(a) By considering the adjoint, show that

$$2N \geq 2^{-k}((1+\epsilon)^2 - 1)^{-(k-1)/2}.$$

(b) Deduce that if $\ell_\infty^N$ contains a $k$-dimensional subspace $E$ with $d_E < 11/10$, then $k \leq C \log N$, where $C$ is some absolute constant.

**13.9.** Prove the Dvoretzky–Rogers theorem directly from Proposition 13.3.5.

**13.10 (Lozanovskii Factorization).**
Let $\|\cdot\|_X$ be a norm on $\mathbb{R}^n$ for which the canonical basis $(e_j)_{j=1}^n$ is 1-unconditional. Show that for every $u = (u(j))_{j=1}^n$ with $u(j) \geq 0$ and $\sum_{j=1}^n u(j) = 1$ we can find $\xi, \eta \in \mathbb{R}^n$ such that $\xi(j), \eta(j) \geq 0$ for $1 \leq j \leq n$, $\|\xi\|_X = \|\eta\|_{X^*} = 1$ and

$$\xi(j)\eta(j) = u(j), \qquad 1 \leq j \leq n.$$

[*Hint:* Maximize $\sum_{j=1}^n u(j) \log |\xi(j)|$ for $\|\xi\|_X \leq 1$.]

This result and infinite-dimensional generalizations are due to Lozanovskii [209]; see also [104].

**13.11 (Figiel et al. [95]).** Let $X$ be an infinite-dimensional Banach space of cotype $q < \infty$. Show that if $\epsilon > 0$, then every $n$-dimensional subspace $F$ of $X$ contains a subspace $E$ with $\dim E \geq cn^{2/q}$ and $d_E < 1 + \epsilon$, where $c = c(\epsilon, X)$.

# Chapter 14
# Nonlinear Geometry of Banach Spaces

A Banach space is, by its nature, also a metric space. When we identify a Banach space with its underlying metric space, we choose to forget its linear structure. The fundamental question of nonlinear geometry is to determine to what extent the metric structure of a Banach space already determines its linear structure. What can be said of two Banach spaces that are Lipschitz-isomorphic? uniformly homeomorphic? or coarsely Lipschitz-isomorphic in the spirit of M. Gromov's geometric theory of groups? These questions are still partly open, and investigating them requires some quite technical tools. In this chapter, we will consider only basic techniques whose purpose is to produce specific linear maps from nonlinear ones (typically, Lipschitz maps).

In the first section we discuss some earlier work on homeomorphisms and isometries, which may be regarded as the two extremes for the type of problem we are interested in. We also describe the general nature of embeddings and isomorphisms between metric spaces that are of concern to us. In the second section we discuss the classical approach to Lipschitz embedding problems for Banach spaces via differentiability theorems. The results are now almost classical and very well covered in [23]; nevertheless, the ideas are so central that we feel that it is important for the reader to appreciate what can be done through this approach. The last paragraph of the section shows that the local structure of Banach spaces is stable under coarse Lipschitz embeddings [268]. This time we need a weak* version of Rademacher's theorem on the existence of weak* derivatives of Lipschitz maps defined on a finite-dimensional space taking values in dual spaces. Somewhat surprisingly, *large* Banach spaces turn out to be useful for handling finite-dimensional ones. Indeed, ultrapowers lead to simpler proofs, and we use them. In the third section we address one of the main problems: when are two Lipschitz-isomorphic Banach spaces (linearly) isomorphic? This section gives a very good illustration of how the techniques of Section 14.2 are put to work in the theorem of Heinrich and Mankiewicz [124] from 1982 that the Lipschitz structure of the spaces $\ell_p$ and $L_p$, $1 < p < \infty$, determines its linear structure. The fourth section deals



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with isometric embeddings, Figiel's extension [93] of Mazur–Ulam's theorem, and the lifting property from [105], and uses the existence of points of differentiability of convex functions on a finite-dimensional space. In Section 14.5 we touch the uniform homeomorphism problem for Banach spaces. Here the methods employed lean heavily on the theory of coarse Lispchitz maps to show the theorem of Johnson et al. [141] that the uniform structure of the spaces $\ell_p$ for $1 < p < \infty$, $p \neq 2$, determines their linear structure. Finally, Section 14.6 shows that the modulus of asymptotic uniform smoothness is another Lipschitz invariant. Our main tools are the Gorelik principle [141] and the definition of a specific equivalent norm through the *rate of change* of the Lipschitz map [106, 107]. Hence, although actual smoothness results are in general not available for the spaces to which these techniques apply, differentiation still casts a shadow on our arguments.

## 14.1 Various Categories of Nonlinear Maps

Linear operators between Banach spaces have been studied and used since the early days of the theory, as indicated by the very title of Banach's fundamental book [18]. However, many natural maps between finite- or infinite-dimensional Banach spaces fail to be linear, and investigating such maps is a relatively recent and important field of research in modern functional analysis. Thus the *arrows* we consider between Banach spaces are not linear, although they still enjoy some regularity properties. To start this section we put ourselves in two extreme situations. In one case we are given only minimal information about the metric structure of a Banach space, and in the other case we have maximal information. In the former of the two we ask, how much can we determine about a Banach space $X$ if all we know is the homeomorphic class of $X$ considered as a topological space? Thus we consider the following problem raised by Fréchet in 1928:

**Problem 14.1.1.** *When are two separable Banach spaces homeomorphic as topological spaces?*

This question was the subject of intense research in the years after the Second World War and was resolved beautifully by Kadets (announced in 1965) [145]:

**Theorem 14.1.2.** *All separable infinite-dimensional Banach spaces are homeomorphic.*

Some years later, Toruńczyk completed Kadets's result by showing that all Banach spaces of the same density character are homeomorphic [296], where the density character of an infinite-dimensional space is the smallest possible cardinality of a dense subset. This is essentially the end of the story in the study of the homeomorphic theory of Banach spaces, but there is still something else to mention. If we leave the realm of locally convex spaces, things become quite mysterious. There is a very remarkable but perhaps not so well known result of Cauty that there is a separable nonlocally convex $\mathsf{F}$-space (complete metric linear space) that is

*not* homeomorphic to a separable Banach space [47]! Perhaps one should contrast this result of Cauty with another more recent result. In 2001, Cauty [48] showed that every compact convex subset of an $\mathsf{F}$-space (even without local convexity) has the Schauder fixed point property. This problem had been open since 1930, when Schauder proved the original fixed point theorem. It does not seem to be known whether an infinite-dimensional compact convex set is necessarily homeomorphic to the Hilbert cube (for subsets of a Banach space this is essentially a result of Keller [168] dating back to 1931). Thus the homeomorphic theory of nonlocally convex $\mathsf{F}$-spaces seems to be a very rich and interesting area for future research.

The opposite end of the spectrum is to assume that our map $\Phi$ between the Banach spaces $X$ and $Y$ is an *isometry*, i.e., $\|\Phi(x)-\Phi(y)\| = \|x-y\|$ for all $x, y \in X$, so that $\Phi$ carries complete information about the metric space associated to $X$. In this situation it was shown by Mazur and Ulam in [217] that one can recapture the real linear structure. The proof we present below is due to Väisälä [299], which in turn is inspired by ideas of Vogt [301] and uses reflections in points. Given a point $z$ in a normed space $Z$, the *reflection of $Z$ in $z$* is the map

$$R_z\colon Z \to Z, \quad x \mapsto R_z(x) = 2z - x.$$

This map trivially satisfies the following properties:

- $R_z \circ R_z = I_Z$, hence $R_z$ is bijective with inverse $R_z^{-1} = R_z$.
- $R_z$ is an isometry.
- $z$ is the only fixed point of $R_z$.
- The equations

$$\|R_z(x) - z\| = \|x - z\|, \qquad \|R_z(x) - x\| = 2\|x - z\| \tag{14.1}$$

  hold for all $x \in Z$.

**Theorem 14.1.3 (Mazur and Ulam [217]).** *Let $X$ and $Y$ be real normed spaces and suppose that $\Phi$ is an isometry mapping $X$ onto $Y$ that carries $0$ into $0$. Then $\Phi$ is a bounded linear operator. In particular, if two real Banach spaces $X$ and $Y$ are isometric as Banach spaces, then they are also linearly isometric.*

*Proof.* It suffices to show that $\Phi$ preserves midpoints of line segments, i.e.,

$$\Phi\left(\frac{x+y}{2}\right) = \frac{\Phi(x) + \Phi(y)}{2}, \tag{14.2}$$

for all $x, y \in X$. Indeed, if this is the case, setting $y = 0$ and using the assumption that $\Phi(0) = 0$, we get that

$$\Phi\left(\frac{x}{2}\right) = \frac{\Phi(x)}{2}, \quad x \in X. \tag{14.3}$$

Applying this to equation (14.2) gives the first property of linearity,

$$\Phi(x+y) = \Phi(x) + \Phi(y), \quad x, y \in X.$$

Next, using induction in (14.3), we obtain

$$\Phi\left(\frac{x}{2^n}\right) = \frac{1}{2^n}\Phi(x), \quad n \in \mathbb{N}.$$

Hence,

$$\Phi\left(\frac{k}{2^n}x\right) = \frac{k}{2^n}\Phi(x), \quad k, n \in \mathbb{N}.$$

Since dyadic rational numbers are dense in $\mathbb{R}$, and an isometry is continuous, we get

$$\Phi(rx) = r\Phi(x), \qquad r \in \mathbb{R}^+.$$

Now (14.2) and what has already been proved imply $\Phi(-x) = -\Phi(x)$, and $\Phi$ is linear.

Take $x$ and $y$ any pair of points in $X$ and set $z = (x+y)/2$. Let $\mathcal{I}_{x,y}$ be the family of all onto isometries $g: X \to X$ such that $g(x) = x$ and $g(y) = y$. We will see that if $g$ belongs to $\mathcal{I}_{x,y}$, then $g$ also fixes their midpoint $z$. To that end, put

$$s = \sup\{\|g(z) - z\| : g \in \mathcal{I}_{x,y}\} \in [0, \infty].$$

Every $g \in \mathcal{I}_{x,y}$ satisfies $\|g(z) - x\| = \|g(z) - g(x)\| = \|z - x\|$, so

$$\|g(z) - z\| \leq \|g(z) - x\| + \|x - z\| = 2\|x - z\|,$$

which yields $s < \infty$.

Let $R_z$ be the reflection of $X$ in $z$. Note that if $g \in \mathcal{I}_{x,y}$, then the map $\tilde{g} = R_z g^{-1} R_z g$ is also a member of $\mathcal{I}_{x,y}$, and therefore $\|\tilde{g}(z) - z\| \leq s$. Since $g^{-1}$ is an isometry, this fact combined with (14.1) implies that for all $g \in \mathcal{I}_{x,y}$,

$$\begin{aligned} 2\|g(z) - z\| &= \|R_z g(z) - g(z)\| \\ &= \|g^{-1} R_z g(z) - z\| \\ &= \|R_z g^{-1} R_z g(z) - z\| \\ &= \|\tilde{g}(z) - z\| \leq s, \end{aligned}$$

which yields $2s \leq s$. Thus $s = 0$, which means that $g(z) = z$.

Set $z' = (\Phi(x) + \Phi(y))/2$. To conclude the proof let us see that $\Phi(z) = z'$. Let $R_{z'}$ be the reflection of $Y$ in $z'$. Note that the map $\psi = R_z \Phi^{-1} R_{z'} \Phi$ belongs to $\mathcal{I}_{x,y}$; hence $\psi(z) = z$. This implies that $R_z \Phi(z) = \Phi(z)$. But we know that $z'$ is the only fixed point of $R_{z'}$, so it must be $\Phi(z) = z'$. □

*Remark 14.1.4.* If we drop the hypothesis $\Phi(0) = 0$, the above proof yields that an onto isometry $\Phi: X \to Y$ between real normed spaces is *affine*, i.e.,

$$\Phi((1-t)x + ty) = (1-t)\Phi(x) + t\Phi(y)$$

for all $x, y \in X$ and for all $t \in [0, 1]$. Then the map $\tilde{\Phi}: X \to Y$ defined by $\tilde{\Phi}(x) = \Phi(x) - \Phi(0)$ is linear. In general, into isometries need not be affine (see Problem 14.1). However, there are very important cases (such as when the target space $Y$ is $L_p$ or $\ell_p$ for $1 < p < \infty$) in which every into isometry $\Phi: X \to Y$ is affine (see Problem 14.2).

In view of the above, it is only natural to wonder whether the complex version of the Mazur–Ulam theorem is valid. This question reduces to a problem in the linear theory; precisely we are asking whether two complex Banach spaces that are linearly isometric as real Banach spaces are additionally linearly isometric (or even linearly isomorphic) in the complex sense. This problem aroused some interest in the 1980s. For every complex space $X$ we have a natural *conjugate space* $\overline{X}$, where complex multiplication is defined using conjugates, i.e., we have a new complex multiplication $\times$ given by

$$\alpha \times x := \overline{\alpha} x, \qquad \alpha \in \mathbb{C},\ x \in X.$$

Clearly, $X$ and $\overline{X}$ are real linearly isometric, but must they be complex isometric? This question was answered by Bourgain [28] negatively. He showed the existence of a space $X$ such that $X$ and $\overline{X}$ are not even linearly isomorphic. His methods were probabilistic (see also [290]). Kalton gave an explicit nonprobabilistic construction of such a space in [155]. This space is an example of a *twisted Hilbert space*, i.e., a Banach space $X$ with a closed subspace $Z$ such that both $Z$ and $X/Z$ are isometrically Hilbert spaces. The importance of such examples to the nonlinear theory is that they show that we must confine ourselves to the category of real Banach spaces in order to have a viable theory. There is no real chance of reconstructing the complex structure of a complex Banach space from the metric structure.

Our goal in this chapter is to study how much information about the different structures that coexist in a Banach space $X$ is transferred into a Banach space $Y$ via a map $f: X \to Y$ that lies somewhere between those two extreme cases and that may enjoy some of the familiar properties that we will recall next in the more general context of metric spaces.

**Definition 14.1.5.** Let $X$ and $Y$ be metric spaces. A map $f: X \to Y$ is *Lipschitz* with constant $K$ (also $K$-Lipschitz) if

$$d(f(x), f(y)) \leq K d(x, y), \qquad \forall\, x,\ y \in X. \tag{14.4}$$

The least constant $K$ in (14.4) will be denoted by $\mathrm{Lip}(f)$, i.e.,

$$\mathrm{Lip}(f) = \sup\left\{ \frac{d(f(x), f(y))}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

Such a mapping $f$ is a *Lipschitz embedding* if $f$ is one-to-one and Lipschitz, and $f^{-1}: f(X) \to X$ is also Lipschitz, i.e., there exist constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 d(x,y) \leq d(f(x), f(y)) \leq c_2 d(x,y), \quad \forall\, x,y \in X.$$

The *distortion constant* of a Lipschitz embedding $f$ is $\operatorname{dist}(f) = \operatorname{Lip}(f)\operatorname{Lip}(f^{-1})$. Obviously, $f$ is a *Lipschitz isomorphism* if it is a surjective Lipschitz embedding. Thus we may define two metric spaces $X$ and $Y$ to be *Lipschitz isomorphic* if there is a Lipschitz isomorphism $f: X \to Y$.

Given Banach spaces $X$ and $Y$, we will write $\operatorname{Lip}_0(X,Y)$ for the Banach space of all Lipschitz maps from $X$ into $Y$ that send 0 to 0 with the norm

$$\operatorname{Lip}(f) = \sup\left\{ \frac{\|f(x) - f(y)\|}{\|x - y\|} : (x,y) \in X^2,\ x \neq y \right\}.$$

If $Y = \mathbb{R}$, we call $\operatorname{Lip}_0(X,\mathbb{R})$ the *Lipschitz dual of* $X$ and denote it by $\operatorname{Lip}_0(X)$.

It is easily checked that $\mathcal{B}(X,Y)$ is a subspace of $\operatorname{Lip}_0(X,Y)$, and in particular, the dual space of $X$ is a linear subspace of $\operatorname{Lip}_0(X)$ with $\|x^*\| = \operatorname{Lip}(x^*)$ for all $x^* \in X^*$.

**Definition 14.1.6.** A map $f: X \to Y$ is *uniformly continuous* if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$d(f(x), f(y)) < \epsilon$$

whenever $0 < d(x,y) < \delta$. If $f$ is one-to-one and both $f$ and $f^{-1}: f(X) \to X$ are uniformly continuous, then $f$ is called a *uniform embedding*. The mapping $f$ is a *uniform homeomorphism* if it is an onto uniform embedding. The spaces $X$ and $Y$ are *uniformly homeomorphic* if there is a uniform homeomorphism $f: X \to Y$.

To quantify the uniform continuity of a map $f: X \to Y$, it is useful to introduce the *modulus of continuity* of $f$,

$$\omega_f(s) = \sup\{d(f(x), f(y)) : x,y \in X, d(x,y) \leq s\}, \quad s > 0.$$

For general $f$ one has $0 \leq \omega_f(s) \leq \infty$, and $f: X \to Y$ is uniformly continuous if and only if $\omega_f(s) \to 0$ when $s \to 0^+$. When $X$ is a convex subset of a Banach space, or more generally when $X$ is a metrically convex space, and $f: X \to Y$ is any mapping, then by the triangle inequality we have subadditivity of the function $\omega_f$, i.e.,

$$\omega_f(s + t) \leq \omega_f(s) + \omega_f(t), \qquad \forall\, s,t > 0.$$

Recall that a metric space $(X,d)$ is *metrically convex* if given any $x,y$ in $X$ and $0 < \lambda < 1$ there exists $z_\lambda \in X$ with

$$d(x, z_\lambda) = \lambda d(x,y) \quad \text{and} \quad d(y, z_\lambda) = (1 - \lambda) d(x,y).$$

The following lemma gathers other basic facts about the modulus of continuity.

**Lemma 14.1.7.** *Let $f\colon X \to Y$ be a map between two metric spaces.*

*(i) Suppose $\omega\colon [0,\infty) \to [0,\infty]$ is a function such that $d(f(x),f(y)) \le \omega(d(x,y))$ for every $x,y \in X$, and $\omega(s) \to 0$ as $s \to 0^+$. Then $f$ is uniformly continuous and $\omega \ge \omega_f$.*
*(ii) $f$ is $K$-Lipschitz if and only if $\omega_f(s) \le Ks$ for all $s > 0$.*
*(iii) If $f$ is uniformly continuous and $X$ is metrically convex, then $\omega_f(s) < \infty$ for all $s > 0$.*

*Proof.* We do (*iii*) and leave the other statements as an exercise. We need to show that for $s > 0$ there is $C_s > 0$ such that $d(f(x),f(y)) \le C_s$ whenever $d(x,y) \le s$. From the definition of uniform continuity there exists $\delta_1 > 0$ such that if $0 < d(a,b) < \delta_1$, then $d(f(a),f(b)) < 1$. Let $N = N_s \in \mathbb{N}$ be such that $s/N < \delta_1$. By the metric convexity of $X$ one can find points $x = x_0, x_1, \dots, x_N = y$ in $X$ such that $d(x_j, x_{j+1}) < d(x,y)/N < s/N < \delta_1$ for $0 \le j \le N-1$. Therefore, by the triangle inequality,

$$d(f(x),f(y)) \le \sum_{j=0}^{N-1} d(f(x_j),f(x_{j+1})) \le N,$$

and our claim holds with $C_s = N_s$. □

By Lemma 14.1.7 (ii), the modulus of continuity of a Lipschitz map is controlled by a linear function. Roughly speaking, one could interpret this by saying that the Lipschitz behavior of a map is closer to a linear behavior than the uniform behavior; hence it seems natural to attempt to *Lipschitz-ize* a uniformly continuous map. The next result [53] does this in an explicit manner.

**Proposition 14.1.8 (Corson and Klee [53]).** *Let $f\colon X \to Y$ be a uniformly continuous map. If $X$ is metrically convex, then for every $\theta > 0$ there exists a constant $K_\theta > 0$ such that $d(f(x),f(y)) \le K_\theta d(x,y)$ whenever $d(x,y) \ge \theta$.*

*Proof.* Fix $\theta > 0$. Given $x,y$ in $X$ with $d(x,y) \ge \theta$, let $m$ be the smallest integer such that $d(x,y)/m < \theta$. By the metric convexity of $X$ we may choose points $x = x_0, x_1, \dots, x_m = y$ in $X$ with $d(x_j, x_{j+1}) < \theta$. The triangle inequality, Lemma 14.1.7(iii), and our choice of $m$ yield

$$d(f(x),f(y)) \le \sum_{j=0}^{m-1} df(x_j),f(x_{j+1})) \le m\omega_f(\theta) \le \frac{2\omega_f(\theta)}{\theta} d(x,y).$$

□

**Definition 14.1.9.** Given a map $f\colon X \to Y$ between two metric spaces $X$ and $Y$, for $\theta > 0$ let us define the (possibly infinite) number

$$\mathrm{Lip}_\theta(f) = \sup\left\{ \frac{d(f(x),f(y))}{d(x,y)} \colon x,y \in X, d(x,y) \ge \theta \right\}.$$

Obviously, $\mathrm{Lip}_\theta(f)$ decreases as $\theta$ increases. Put

$$\mathrm{Lip}_\infty(f) := \inf_{\theta>0} \mathrm{Lip}_\theta(f) = \lim_{\theta\to\infty} \mathrm{Lip}_\theta(f)$$

to denote the (possibly zero) *asymptotic Lipschitz constant* of $f$. The map $f$ is *coarse Lipschitz* if $\mathrm{Lip}_\infty(f) < \infty$, i.e., there is $\theta > 0$ for which $\mathrm{Lip}_\theta(f) < \infty$. In this case it is said that $f$ satisfies a *Lipschitz condition for large distances*. When $\mathrm{Lip}_\theta(f) < \infty$ for all $\theta > 0$, we say that $f$ is *Lipschitz at large distances*.

**Definition 14.1.10.** A map $f\colon X \to Y$ between metric spaces is a *coarse Lispchitz embedding* if there exist constants $0 < c_1 < c_2$ and $\theta > 0$ such that

$$c_1\, d(x,y) \le d(f(x), f(y)) \le c_2\, d(x,y), \quad \forall\, x, y \in X \text{ with } d(x,y) \ge \theta. \tag{14.5}$$

Note that a coarse Lipschitz embedding need not be injective.

Coarse Lipschitz embeddings are the large-scale analogue of Lipschitz embeddings. Of course, this notion is of interest only for unbounded metric spaces.

**Definition 14.1.11.** A metric space $(X, d)$ is said to be *bounded* if there exists $r > 0$ such that $d(x,y) \le r$ for all $x, y \in X$. Otherwise, it is called *unbounded*.

A coarse Lipschitz embedding does not observe the fine structure of a metric space in a neighborhood of a point, since it need not be continuous; it captures only the macroscopic structure of the space, i.e., where distances are large. This is in stark contrast to a uniform homeomorphism, which observes only the local structure of the space.

*Example 14.1.12.* Consider $\mathbb{R}$ with its standard metric $d$ induced by the absolute value, and define new metrics on $\mathbb{R}$ by

$$\rho(x,y) = \min\{|x-y|, 1\}, \quad \forall\ (x,y) \in \mathbb{R}^2, \qquad \sigma(x,y) = \begin{cases} |x-y| + 1 & x \ne y, \\ 0 & x = y. \end{cases}$$

Then the identity map $\mathrm{id}\colon (\mathbb{R}, d) \to (\mathbb{R}, \rho)$ is a uniform homeomorphism but not a coarse Lipschitz embedding, whereas $\mathrm{id}\colon (\mathbb{R}, d) \to (\mathbb{R}, \sigma)$ is a coarse Lipschitz embedding that is not a uniform homeomorphism.

*Remark 14.1.13.* In geometric group theory it is customary to define a *quasi-isometric embedding* as an embedding $f\colon X \to Y$ such that for all $x, y \in X$ the inequalities

$$\frac{1}{A} d(x,y) - B \le d(f(x), f(y)) \le A d(x,y) + B$$

hold for some constants $A \ge 1$ and $B \ge 0$. Lemma 14.1.15 states that when the domain space is metrically convex, then an embedding is quasi-isometric if and

only if it is a coarse Lipschitz embedding. This terminology will be preferred in this book, since in nonlinear Banach space theory a *quasi-isometry* means that for every $\epsilon > 0$ there exists a Lipschitz isomorphism with distortion constant $1 + \epsilon$.

Let us record an elementary fact that will be handy.

**Lemma 14.1.14.** *Let $X$ be an unbounded metrically convex space. Given $x \in X$ and $r > 0$ there is $y \in X$ such that $d(x, y) = r$.*

*Proof.* Since $X$ is unbounded, there is $z \in X$ such that $d(x, z) = R > r$. Let $t = r/R \in (0, 1)$. The metric convexity of $X$ yields $y \in X$ such that $d(x, y) = td(x, z) = r$. □

**Lemma 14.1.15.** *Let $f: X \to Y$ be a map between two unbounded metric spaces.*

*(i) Suppose that for some constants $A \geq 1$ and $B \geq 0$ the inequalities*

$$\frac{1}{A}d(x, y) - B \leq d(f(x), f(y)) \leq Ad(x, y) + B$$

*hold for all $x, y \in X$. Then $f$ is a coarse Lipschitz embedding.*

*(ii) If $X$ is metrically convex and $f$ is a coarse Lipschitz embedding, then there exist constants $A \geq 1$ and $B \geq 0$ such that*

$$\frac{1}{A}d(x, y) - B \leq d(f(x), f(y)) \leq Ad(x, y) + B, \qquad \forall\, x, y \in X. \tag{14.6}$$

*Proof.* (i) Suppose $d(x, y) > \theta$ for a given $\theta > 0$. Then,

$$d(f(x), f(y)) \leq Ad(x, y) + B \leq Ad(x, y) + \frac{B}{\theta}d(x, y) = \Big(A + \frac{B}{\theta}\Big)d(x, y).$$

If, on the other hand, we impose that $d(x, y) > \theta > AB$, we also have

$$d(f(x), f(y)) \geq \frac{1}{A}d(x, y) - B \geq \frac{1}{A}d(x, y) - \frac{B}{\theta}d(x, y) = \Big(\frac{1}{A} - \frac{B}{\theta}\Big)d(x, y).$$

Summing up, for all $x, y \in X$ such that $d(x, y) > \theta > AB$,

$$\Big(\frac{1}{A} - \frac{B}{\theta}\Big)d(x, y) \leq d(f(x), f(y)) \leq \Big(A + \frac{B}{\theta}\Big)d(x, y).$$

(ii) Now assume (14.5) holds for some constants $0 < c_1 < c_2$ and some $\theta > 0$. If $d(x, y) \geq \theta$, then (14.6) is trivially satisfied for every $B \geq 0$, so suppose $d(x, y) < \theta$. By Lemma 14.1.14 we can pick $z \in X$ with $d(y, z) = 2\theta$, and so $d(x, z) > \theta$. Then,

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f(z)) + d(f(z), f(y)) \\ &\leq c_2 d(x, z) + c_2 d(z, y) \end{aligned}$$

$$\leq c_2 d(x,y) + 2c_2 d(z,y)$$
$$= c_2 d(x,y) + 4c_2\theta.$$

Similarly, to estimate $d(f(x), f(y))$ from below we use the reverse triangle inequality,

$$\begin{aligned} d(f(x),f(y)) &\geq d(f(x),f(z)) - d(f(z),f(y)) \\ &\geq c_1 d(x,z) - c_2 d(z,y) \\ &\geq c_1 d(x,y) - (c_1+c_2) d(z,y) \\ &= c_1 d(x,y) - 2(c_1+c_2)\theta. \end{aligned}$$

□

To understand the real meaning of a coarse Lipschitz embedding it is useful to introduce the ideas of a *separated net* and *net-equivalence*.

**Definition 14.1.16.** If $X$ is a metric space, a collection of elements $\mathcal{N}$ of $X$ is called a *separated net* (or *skeleton*) if there exist $0 < \alpha < \beta < \infty$ such that

$$\inf\left\{d(s_1,s_2)\colon (s_1,s_2) \in \mathcal{N}^2,\ s_1 \neq s_2\right\} \geq \alpha \quad \text{and} \quad \sup\{d(x,\mathcal{N})\colon x \in X\} \leq \beta.$$

That is, every two points in $\mathcal{N}$ are separated by a distance larger than $\alpha$, and every point in $X$ is of distance less than $\beta$ from some point of $\mathcal{N}$. We will usually say that $\mathcal{N}$ is an $\alpha$*-separated* $\beta$*-net* for $X$.

**Lemma 14.1.17.** *A metric space $X$ contains an $\alpha$-separated $\alpha$-net for every $\alpha > 0$.*

*Proof.* For $\alpha > 0$, let $\mathcal{I} = \{\mathcal{N} \subseteq X\colon d(s_1,s_2) \geq \alpha$ for all $(s_1,s_2) \in \mathcal{N} \times \mathcal{N}$, $s_1 \neq s_2\}$. The set $\mathcal{I}$ is nonempty, since it contains $\emptyset$, and it is partially ordered by inclusion. Besides, if $\{\mathcal{N}_\gamma\}_{\gamma\in\Gamma}$ is a totally ordered subset of $\mathcal{I}$, then $\cup_{\gamma\in\Gamma}\mathcal{N}_\gamma$ is an upper bound and belongs to $\mathcal{I}$. By Zorn's lemma there is a maximal element $\overline{\mathcal{N}}$ in $\mathcal{I}$. It is now obvious that whenever $x \in X\setminus\overline{\mathcal{N}}$, there is $s \in \overline{\mathcal{N}}$ such that $d(s,x) < \alpha$. □

It was observed by Lindenstrauss, Matoušková, and Preiss in [206] that if $X$ is an infinite-dimensional normed space, then every two separated nets in $X$ are Lipschitz isomorphic. This motivates our following definition.

**Definition 14.1.18.** Two unbounded metric spaces $X$ and $Y$ are *net equivalent* if there exists a separated net $\mathcal{N}_X$ in $X$ that is Lipschitz isomorphic to a separated net $\mathcal{N}_Y$ in $Y$, i.e., there is a bijection $\psi\colon \mathcal{N}_X \to \mathcal{N}_Y$ such that for some constants $0 < c_1 < c_2 < \infty$,

$$c_1 d(s_1,s_2) \leq d(\psi(s_1),\psi(s_2)) \leq c_2 d(s_1,s_2), \qquad \forall s_1,s_2 \in \mathcal{N}_X. \tag{14.7}$$

**Proposition 14.1.19.** *Let $X$ and $Y$ be unbounded metrically convex spaces. Then the following are equivalent:*

*(a) $X$ and $Y$ are net equivalent.*
*(b) There is a coarse Lipschitz embedding $f\colon X \to Y$ with* $\sup_{y\in Y} d(y, f(X)) < \infty$.

*Proof.* (a) $\Rightarrow$ (b) We know that for some positive constants $\alpha, \beta, \delta, \gamma$ there exist an $\alpha$-separated $\beta$-net $\mathcal{N}_X$ in $X$, a $\delta$-separated $\gamma$-net $\mathcal{N}_Y$ in $Y$ and a bijection $\psi\colon \mathcal{N}_X \to \mathcal{N}_Y$ such that (14.7) holds. By the axiom of choice there is a map $\varphi\colon X \to \mathcal{N}_X$ that assigns to each $x \in X$ a point $\varphi(x) = s_x$ in $\mathcal{N}_X$ with $d(x, s_x) \le \beta$. Let us see that $f\colon X \to \mathcal{N}_Y \subseteq Y$ given by $f(x) = \psi \circ \varphi(x)$ is a coarse Lipschitz embedding. Indeed, given any $x, y \in X$ let $s_x = \varphi(x)$ and $s_y = \varphi(y)$ in $\mathcal{N}_X$. Using the triangle inequality twice yields

$$d(f(x), f(y)) = d(\psi(s_x), \psi(s_y)) \le c_2 d(s_x, s_y) \le c_2 \left(d(x, y) + 2\beta\right).$$

Analogously, using the reverse triangle inequality twice we also have

$$d(f(x), f(y)) \ge c_1 d(s_x, s_y) \ge c_1 \left(d(x, y) - 2\beta\right),$$

and we are done by an appeal to Lemma 14.1.15 (i). The second condition in (b) comes for free, because $f(X) = \mathcal{N}_Y$ and $d(y, \mathcal{N}_Y) \le \gamma$ for all $y \in Y$.

(b) $\Rightarrow$ (a) Suppose $f\colon X \to Y$ satisfies (14.5). Take $\mathcal{N}_X$ a $\theta$-separated $\theta$-net in $X$, whose existence is guaranteed by Lemma 14.1.17. Let $g = f|_{\mathcal{N}_X}\colon \mathcal{N}_X \to Y$ and put $\mathcal{N}_Y = g(\mathcal{N}_X)$. It then follows from the inequalities (14.5) that $g$ is a Lipschitz isomorphism from $\mathcal{N}_X$ onto $\mathcal{N}_Y$. It remains to show that $\mathcal{N}_Y$ is a separated net in $Y$. It is clear from the left-hand-side inequality of (14.5) that every two points $f(s_1), f(s_2) \in \mathcal{N}_Y$ are separated by a distance at least $c_1\theta$. Let us see that a given point $y_0$ in $Y$ is never too far from some point in $\mathcal{N}_Y$. Take $R > \sup_{y\in Y} d(y, f(X))$. By the second statement of (b), there is $x \in X$ with $d(y_0, f(x)) \le R$. Pick $s \in \mathcal{N}_X$ such that $d(x, s) \le \theta$ in this way, by the triangle inequality and a last-minute crucial contribution of Lemma 14.1.15 (ii) we have

$$d(y_0, f(s)) \le d(y_0, f(x)) + d(f(x), f(s)) \le R + Ad(x, s) + B \le R + \theta A + B,$$

and we are done. □

In fact, this proof kills two birds with one stone, since (with the obvious modifications) it also shows the following:

**Proposition 14.1.20.** *Let $X$ and $Y$ be unbounded metrically convex spaces. Then the following are equivalent:*

*(a) The space $X$ coarse Lipschitz embeds into $Y$.*
*(b) There exists a separated net $\mathcal{N}_X$ in $X$ that Lipschitz embeds into $Y$.*

**Proposition 14.1.21.** *Let $X$ and $Y$ be normed spaces and suppose $f: X \to Y$ is a uniform homeomorphism. Then $f$ is a (surjective) coarse Lipschitz embedding.*

*Proof.* Since $f$ is uniformly continuous, Proposition 14.1.8 yields for every $\theta > 0$ a constant $K_\theta$ such that $\|f(x)-f(y)\| \le K_\theta \|x-y\|$ for every $x, y \in X$ with $\|x-y\| \ge \theta$. Since $f^{-1}$ is also uniformly continuous and $f(X) = Y$, there exists $\delta$ such that $\|x - y\| \ge \theta$ implies $\|f(x)-f(y)\| \ge \delta$. Now we use the fact that $f^{-1}: Y \to X$ is Lipschitz for large distances: given $\delta$, there exists $K_\delta$ such that $\|x-y\| \le K_\delta \|f(x)-f(y)\|$ whenever $\|f(x)-f(y)\| \ge \delta$. Combining, we obtain

$$K_\delta^{-1}\|x-y\| \le \|f(x)-f(y)\| \le K_\theta \|x-y\|$$

for every $x, y \in X$ with $\|x-y\| \ge \theta$. □

**Corollary 14.1.22.** *If two normed spaces $X$ and $Y$ are uniformly homeomorphic, then they are net equivalent.*

*Remark 14.1.23.* Let us observe that problems associated with coarse Lipschitz and uniform *embeddings* of one Banach space into another are quite different in nature, because we do not require the image of the embedding to be metrically convex, and hence we cannot guarantee that their inverses are Lipschitz for large distances as well. Thus the arguments we have given to reach Proposition 14.1.21 break down completely, and in particular, we cannot infer that a uniform embedding $f$ is a coarse Lipschitz embedding unless of course $f$ is onto.

There is an immediate application of the idea of net equivalence due to Johnson, Lindenstrauss, and Schechtman [141] using ultraproducts (see Appendix J) as the key ingredient.

**Theorem 14.1.24 (Johnson et al. [141]).** *If $X$ and $Y$ are net equivalent Banach spaces, then they have Lipschitz isomorphic ultrapowers.*

*Proof.* Suppose $\mathcal{N}_X$ is a $b$-separated $b$-net in $X$, that $\mathcal{N}_Y$ is a $c$-separated $c$-net in $Y$, and that $f: \mathcal{N}_X \to \mathcal{N}_Y$ is a Lipschitz isomorphism. Given two sequences $(x_n)_{n=1}^\infty$ and $(\bar{x}_n)_{n=1}^\infty$ in $X$ we may pick two (not necessarily unique) corresponding sequences $(s_n)_{n=1}^\infty$ and $(\bar{s}_n)_{n=1}^\infty$ in $\mathcal{N}_X$ with

$$\|nx_n - s_n\| \le b \quad \text{and} \quad \|n\bar{x}_n - \bar{s}_n\| \le b, \qquad n = 1, 2, \dots. \tag{14.8}$$

We then have

$$\|f(s_n)-f(\bar{s}_n)\| \le \mathrm{Lip}(f)\|s_n - \bar{s}_n\| \le \mathrm{Lip}(f)(n\|x_n - \bar{x}_n\| + 2b),$$

so that

$$\left\|\frac{f(s_n)}{n} - \frac{f(\bar{s}_n)}{n}\right\| \le \mathrm{Lip}(f)\left(\|x_n - \bar{x}_n\| + \frac{2b}{n}\right). \tag{14.9}$$

Now suppose $(x_n)_{n=1}^{\infty}$ is a bounded sequence in $X$. We choose $(\bar{x}_n)_{n=1}^{\infty}$ in $X$ such that $\bar{x}_n = 0$ for all $n$ and $(\bar{s}_n)_{n=1}^{\infty}$ in $\mathcal{N}_X$ with $\bar{s}_n = \bar{s}$ for all $n$. In this particular case (14.9) takes the form

$$\left\| \frac{f(s_n)}{n} - \frac{f(\bar{s})}{n} \right\| \leq \mathrm{Lip}(f) \left( \|x_n\| + \frac{2b}{n} \right),$$

which implies

$$\left\| \frac{f(s_n)}{n} \right\| \leq \mathrm{Lip}(f) \left( \|x_n\| + \frac{2b}{n} \right) + \left\| \frac{f(\bar{s})}{n} \right\|.$$

Thus $(f(s_n)/n)_{n=1}^{\infty}$ is bounded in $Y$.

Let $\mathcal{U}$ be a free ultrafilter on the natural numbers. If $(x_n)_{n=1}^{\infty}$ and $(\bar{x}_n)_{n=1}^{\infty}$ are bounded sequences in $X$ with $\lim_{\mathcal{U}} \|x_n - \bar{x}_n\| = 0$, then

$$\lim_{\mathcal{U}} \left\| \frac{f(s_n)}{n} - \frac{f(\bar{s}_n)}{n} \right\| = 0,$$

so that the map

$$F\colon X_{\mathcal{U}} \to Y_{\mathcal{U}}, \qquad (x_n)_{\mathcal{U}} \mapsto \left( \frac{f(s_n)}{n} \right)_{\mathcal{U}}$$

is well defined. Taking limits through the ultrafilter in (14.9) gives

$$\|F((x_n)_{\mathcal{U}}) - F((\bar{x}_n)_{\mathcal{U}})\| \leq \mathrm{Lip}(f) \|(x_n)_{\mathcal{U}} - (\bar{x}_n)_{\mathcal{U}}\|.$$

Thus $F$ is Lipschitz.

Let $g\colon \mathcal{N}_Y \to \mathcal{N}_X$ be the inverse of $f$. Swapping the roles of $X$ and $Y$, we obtain a Lipschitz map $G\colon Y_{\mathcal{U}} \to X_{\mathcal{U}}$. Let us check that $F$ and $G$ are inverses of each other.

Given $(x_n)_{\mathcal{U}}$ in $X_{\mathcal{U}}$ we pick $(\bar{s}_n)_{n=1}^{\infty}$ in $\mathcal{N}_X$ satisfying (14.8). The trivial inequality

$$\left\| n\frac{f(s_n)}{n} - f(s_n) \right\| = 0 \leq c$$

yields that the sequences $y_n = f(s_n)/n$ in $Y$ and $t_n = f(s_n)$ in $\mathcal{N}_Y$ satisfy an estimate analogous to (14.8) in the space $Y$, i.e.,

$$\|ny_n - t_n\| \leq c, \qquad n = 1, 2, \ldots.$$

Then

$$G \circ F((x_n)_{\mathcal{U}}) = \left( \frac{g \circ f(s_n)}{n} \right)_{\mathcal{U}} = \left( \frac{s_n}{n} \right)_{\mathcal{U}} = (x_n)_{\mathcal{U}}.$$

Swapping roles of $X$, $Y$ and doing the analogous thing, we see that $F \circ G = I_{Y_{\mathcal{U}}}$. □

The same reasoning would yield the following variation of Theorem 14.1.24.

**Theorem 14.1.25.** *Suppose a Banach space $X$ coarse Lipschitz embeds into a Banach space $Y$. Then the ultrapower $X_{\mathcal{U}}$ Lipschitz embeds into the ultrapower $Y_{\mathcal{U}}$ for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$.*

**Corollary 14.1.26.** *Let $X$ and $Y$ be infinite-dimensional Banach spaces. Suppose $X$ coarse Lipschitz embeds into $Y$. Then $X$ Lipschitz embeds into the ultrapower $Y_{\mathcal{U}}$ of $Y$ for every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$.*

A deduction from Proposition 14.1.21 and Theorem 14.1.24 is that uniformly homeomorphic Banach spaces have Lipschitz isomorphic ultraproducts. This provides an instant link between uniform homeomorphisms and problems concerning Lipschitz isomorphisms, which explains why so much effort has been devoted to understanding Lipschitz isomorphisms between Banach spaces.

## 14.2 The Lipschitz Embedding Problem

A basic (yet fundamental) question that we shall ask ourselves repeatedly throughout the chapter is, when does the existence of a Lipschitz map between Banach spaces with some added features such as, for instance, being an isometry, a quotient map, an isomorphism, or a projection, ensure the existence of a corresponding linear map of the same kind? The added feature we shall concentrate on in this section is being an embedding.

The classical approach to Lipschitz isomorphisms and embedding problems is based on differentiability results that date back to the 1970s and were independently discovered by Christensen [49], Mankiewicz [210], and Aronszajn [15]. This is a powerful approach, but it has some distinct limitations, as we shall see. For instance, these methods seem to yield only (not necessarily surjective) linear embeddings even if the original map was an (onto) Lipschitz isomorphism. We shall see how to fix this in some special cases in Section 14.3.

The idea here is simple. Suppose we have a Lipschitz embedding $f: X \to Y$ between Banach spaces, namely for some $0 < c_1 < c_2 < \infty$,

$$c_1\|x_1 - x_2\| \le \|f(x_1) - f(x_2)\| \le c_2\|x_1 - x_2\|, \qquad \forall\, x_1, x_2 \in X. \tag{14.10}$$

Now suppose that we can introduce some notion of differentiability so that $f$ is differentiable with derivative $T$ at some point $x_0 \in X$. Then, given any $0 \neq t \in \mathbb{R}$ and $u \in X$, plugging $x_2 = x_0$ and $x_1 = x_0 + tu$ in (14.10) yields

$$c_1\|tu\| \le \|f(x_0 + tu) - f(x_0)\| \le c_2\|tu\|.$$

We divide the inequalities by $|t|$ and make $t \to 0$ to obtain

$$c_1\|u\| \le \|T(u)\| \le c_2\|u\|, \qquad \forall\, u \in X, \tag{14.11}$$

so that $T$ is a linear embedding of $X$ into $Y$.

To make this precise we will need to define a suitable notion of differentiability. There are two basic notions of derivative available for functions $f$ mapping from an open set of a Banach space $X$ into a Banach space $Y$.

**Definition 14.2.1.** (a) The function $f$ is said to be *Gâteaux differentiable* at a point $x \in X$ if there is a bounded linear operator $T$ from $X$ to $Y$ such that for every $u \in X$,

$$\lim_{t\to 0} \frac{f(x+tu) - f(x)}{t} = T(u). \tag{14.12}$$

The uniquely determined operator $T$ is called the *Gâteaux derivative* of $f$ at $x$ and is denoted by $D_f(x)$. The set of points in $X$ where $f$ is Gâteaux differentiable will be denoted by $\Omega_f$.

(b) If for some fixed $u \in X$ the limit

$$\partial_u f(x) := \lim_{t\to 0} \frac{f(x+tu) - f(x)}{t}$$

exists, we say that $f$ has a *directional derivative at $x$ in the direction $u$*. Thus $f$ is Gâteaux differentiable at $x$ if and only if all the directional derivatives $\partial_u f(x)$ exist and the correspondence $u \mapsto \partial_u f(x)$ defines a bounded linear operator from $X$ to $Y$. In this case $\partial_u f(x) = D_f(x)(u)$.

(c) If the limit in (14.12) holds uniformly for $u$ in the unit ball (or the unit sphere), we say that $f$ is *Fréchet differentiable* at $x \in X$, and the operator $T$ is then called the *Fréchet derivative* of $f$ at $x$. Equivalently, $f$ is Fréchet differentiable at $x$ if there is a bounded linear operator $T : X \to Y$ such that

$$f(x+u) = f(x) + T(u) + o(\|u\|) \quad \text{as} \quad \|u\| \to 0.$$

Of course, Fréchet differentiability always implies Gâteaux differentiability. If $f$ is Lipschitz and $\dim(X) < \infty$, then the two notions of differentiability coincide.

**Proposition 14.2.2.** *Let $X$ be a finite-dimensional Banach space, and let $f$ be a Lipschitz map from an open set in $X$ to a (possibly infinite-dimensional) Banach space $Y$. If $f$ is Gâteaux differentiable at a point $x$, then $f$ is Fréchet differentiable at $x$.*

*Proof.* Given $\epsilon > 0$, by compactness of the unit sphere of $X$ we may choose an $\epsilon$-net $\{u_i\}_{i=1}^N$ in $S_X$. Let $\delta > 0$ be such that $\|f(x+tu_i) - f(x) - tD_f(x)(u_i)\| \le \epsilon|t|$ for all $|t| \le \delta$ and for all $1 \le i \le N$. If for $u \in S_X$ we choose $1 \le i \le N$ such that $\|u - u_i\| < \epsilon$, we obtain

$$\|f(x+tu) - f(x) - tD_f(x)(u)\| \le (\mathrm{Lip}(f) + \|D_f(x)\| + 1)\,\epsilon|t|$$

for all $|t| \le \delta$. $\square$

Another well known fact from calculus that easily extends to vector-valued functions is that if $X$ is finite-dimensional and $f$ is a map from $X$ into a (possibly infinite-dimensional) Banach space $Y$ such that the directional derivatives $\partial_{u_i} f(x)$ exist for all $x$ in an open set $U \subset X$ in all directions $(u_i)_{i=1}^n$ of a basis of $X$, and the mappings $\partial_{u_i} f: U \to Y$ are continuous, then $f$ is Fréchet differentiable at every $x \in U$.

The situation is known to be completely different if $\dim(X) = \infty$. In this case there are reasonably satisfactory results on the existence of Gâteaux derivatives of Lipschitz maps, while results on existence of Fréchet derivatives are rare and usually very hard to prove. See Problems 14.4–14.8 for some basic properties of derivatives. We refer to [263] for a survey on differentiability in infinite-dimensional spaces.

### 14.2.1 Existence of Derivatives of Lipschitz Maps

Our goal here is to present a proof of the main theorem in this area: a Lipschitz function from an infinite-dimensional Banach space into a Banach space with (RNP) is Gâteaux differentiable outside a *negligible* set, in a sense that will be explained below. First of all, we show that the result is true for Lipschitz functions on finite-dimensional spaces. Rademacher proved it for $Y = \mathbb{R}^m$, but the generalization is only formal. To show this we will rely on standard approximation techniques using convolutions and bump functions. In the sequel, by a Lebesgue measure on a $k$-dimensional Banach space $E$ we shall mean any measure, denoted by $\lambda$, on $E$ that is an image of the Lebesgue measure on $\mathbb{R}^k$ via a linear isomorphism between $E$ and $\mathbb{R}^k$. Since all Lebesgue measures on $E$ are equal up to a constant, the class of sets with Lebesgue measure zero in a finite-dimensional Banach space is well defined.

**Lemma 14.2.3.** *Let $f: E \to Y$ be a bounded map from a $k$-dimensional normed space $E$ into a Banach space $Y$. Take a bump function $\varphi \in \mathcal{C}^{(\infty)}(E, \mathbb{R})$ that is everywhere positive, compactly supported, and has integral* 1. *Define*

$$g_n(x) = 2^{nk} \int_E f(x - \xi)\varphi(2^n \xi)\, d\lambda(\xi), \quad x \in E, n \in \mathbb{N}.$$

*Then* $\lim_n g_n(x) = f(x)$ *at each Lebesgue point $x$ of $f$. In particular,* $\lim_n g_n(x) = f(x)$ *a.e. $x \in E$ and* $\lim_n g_n(x) = f(x)$ *at every point point of continuity $x$ of $f$.*

We will also appeal to well known results on summability kernels. With an eye to their application also in Section 14.3, we formulate their vector-valued versions. As before, the integrals involved must be interpreted in the sense of Bochner (see Appendix K).

**Lemma 14.2.4.** *Let $f: X \to Y$ be a Lipschitz map between Banach spaces. Suppose $E$ is a finite-dimensional subspace of $X$. Define $g: X \to Y$ by*

$$g(x) = \int_E f(x - \xi)\varphi(\xi)\, d\lambda(\xi), \quad x \in X,$$

*where* $\varphi \in \mathcal{C}^{(\infty)}(E, \mathbb{R})$ *is a nonnegative function with compact support,* $\int_E \varphi \, d\lambda = 1$, *and* $\lambda$ *is a Lebesgue measure on* $E$. *Then:*

*(i) The map* $g$ *is Lipschitz with* $\operatorname{Lip}(g) \leq \operatorname{Lip}(f)$.
*(ii) For every* $x \in X$ *and each* $u \in E$ *the directional derivative* $\partial_u g(x)$ *exists and* $\partial_u g : X \to Y$ *is continuous.*

*Proof.* Using the properties of the Bochner integral, we have

$$\begin{aligned}\|g(y) - g(y)\| &= \left\| \int_E (f(x-\xi) - f(y-\xi))\varphi(\xi)\, d\lambda(\xi) \right\| \\ &\leq \int_E \|f(x-\xi) - f(y-\xi)\| \varphi(\xi)\, d\lambda(\xi) \\ &\leq \int_E \operatorname{Lip}(f)\|x-y\|\varphi(\xi)\, d\lambda(\xi) \\ &= \operatorname{Lip}(f)\|x-y\|,\end{aligned}$$

and so (i) follows. In order to prove (ii), observe that by the invariance under translations of the Lebesgue measure,

$$g(x+y) = \int_E f(x-\xi)\varphi(y+\xi)\, d\lambda(\xi), \quad x \in X,\ y \in E.$$

Pick $x \in E$ and $u \in E$. For $t \in \mathbb{R} \setminus \{0\}$ we have

$$\frac{g(x+tu) - g(x)}{t} = \int_E h(\xi, t)\, d\lambda(\xi),$$

where

$$h(\xi, t) = f(x-\xi)\frac{\varphi(tu+\xi) - \varphi(\xi)}{t}.$$

Let $M = \sup_{y \in E} |D_\varphi(y)(u)|$ and $K = \operatorname{supp} \varphi + [-1,1]u$. Notice that $M < \infty$ and that $K$ is compact. For every $\xi \in E$ and $t \in [-1,1] \setminus \{0\}$ we have

$$\|h(\xi, t)\| \leq g(\xi) := M\|f(x-\xi)\|\chi_K(\xi).$$

Moreover, $\int_E g(\xi)\, d\lambda(\xi) < \infty$. By the dominated convergence theorem the partial derivative $\partial_u g(x)$ exists and

$$\partial_u g(x) = \int_E \lim_{t \to 0} h(\xi, t)\, d\lambda(\xi) = \int_E f(x-\xi) D_\varphi(\xi)(u)\, d\lambda(\xi).$$

Applying (i), we obtain that $\partial_u g : X \to Y$ is Lipschitz, hence continuous. □

**Lemma 14.2.5.** *Let $f\colon X \to Y$ be a Lipschitz map between Banach spaces. Suppose $E$ is a $k$-dimensional subspace of $X$. Define maps $(g_n)_{n=1}^{\infty}$ from $X$ to $Y$ by*

$$g_n(x) = 2^{nk} \int_E f(x-\xi)\varphi(2^n\xi)\, d\lambda(\xi), \quad x \in X, n \in \mathbb{N},$$

*where $\varphi \in \mathcal{C}^{(\infty)}(E,\mathbb{R})$ is a nonnegative function with compact support such that $\int_E \varphi\, d\lambda = 1$, and $\lambda$ is a Lebesgue measure on $E$. Then for all $x \in X$,*

$$\|g_n(x) - f(x)\| \le \kappa\, 2^{-n}\, \mathrm{Lip}(f),$$

*where $\kappa = \int_E \|\xi\|\varphi(\xi) d\lambda(\xi)$. In particular, $\lim_n g_n = f$ uniformly.*

*Proof.* Write $f(x) = 2^{nk}\int_E f(x)\varphi(2^n\xi)\, d\xi$. Then

$$\begin{aligned}\|g_n(x) - f(x)\| &= 2^{nk}\left\| \int_E (f(x-\xi) - f(x))\varphi(2^n\xi)\, d\lambda(\xi)\right\| \\ &\le 2^{nk}\int_E \|f(x-\xi) - f(x)\|\varphi(2^n\xi)\, d\lambda(\xi) \\ &\le 2^{nk}\,\mathrm{Lip}(f)\int_E \|\xi\|\varphi(2^n\xi)\, d\lambda(\xi) \\ &\le \kappa 2^{-n}\,\mathrm{Lip}(f).\end{aligned}$$

□

At the turn of the twentieth century, H. Lebesgue proved that a Lipschitz map $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere. Rademacher's theorem is the natural extension to finite-dimensional spaces of Lebesgue's differentiation theorem.

**Theorem 14.2.6 (Rademacher's theorem).** *Every Lipschitz map $f$ from a finite-dimensional normed space $E$ into a Banach space $Y$ with the Radon–Nikodym property is differentiable almost everywhere.*

*Proof.* We claim that for $u \in E$ the subset $\Omega_u$ of points $x \in E$ where $\partial_u f(x)$ exists is the complement of a zero-measure set. To see this, using a linear transformation we assume that $E = \mathbb{R}^k$, where $k = \dim E$, and $u = e_1$, the first vector of the canonical basis of $\mathbb{R}^k$. Since $Y$ has (RNP),

$$|\{\xi_1 \in \mathbb{R}\colon \nexists \partial_{e_1} f(\xi_1, \xi_2, \dots, \xi_k)\}| = 0, \quad \text{for all } (\xi_2, \dots, \xi_k) \in \mathbb{R}^{k-1},$$

whence by Fubini's theorem,

$$\left|\left\{\xi \in \mathbb{R}^k\colon \nexists\, \partial_{e_1} f(\xi)\right\}\right| = 0.$$

For each $t \neq 0$ and each $u \in E$, the map

$$E \to Y, \qquad x \mapsto \frac{f(x+tu)-f(x)}{t},$$

is strongly measurable and is bounded by $\mathrm{Lip}(f)\|u\|$. Hence for each $u \in E$, there is a strongly measurable function $S(\cdot)(u) \colon E \to X$ such that

$$S(x)(u) = \partial_u f(x), \quad \forall\, u \in E, \forall\, x \in \Omega_u, \tag{14.13}$$

and

$$\|S(x)(u)\| \leq \mathrm{Lip}(f)\|u\|, \quad \forall\, u, x \in E. \tag{14.14}$$

Let $\lambda$ be a Lebesgue measure on $E$ and let $\varphi$ be a nonnegative smooth function of compact support such that $\int_E \phi(\xi)\, d\lambda(\xi) = 1$. For $n \in \mathbb{N}$ define

$$g_n(x) = 2^{nk} \int_E f(x-\xi)\varphi(2^n \xi)\, d\lambda(\xi), \quad x \in E.$$

The map $g_n$ is continuously differentiable on $E$ by Lemma 14.2.4. Consequently, for every $s, t \in \mathbb{R}$, $x, u, v \in E$, and $n \in \mathbb{N}$,

$$D_{g_n}(x)(su+tv) = sD_{g_n}(x)(u) + tD_{g_n}(x)(v). \tag{14.15}$$

Pick $u$ and $x$ in $E$. Using (14.13) and the translation-invariance of the Lebesgue measure, we have

$$\lim_{t\to 0} \frac{f(x+tu-\xi)-f(x-\xi)}{t} = S(x-\xi)(u), \quad \text{a.e. } \xi \in E.$$

Hence, by the dominated convergence theorem,

$$\begin{aligned} D_{g_n}(x)(u) &= \lim_{t\to 0} 2^{nk} \int_E \frac{f(x+tu-\xi)-f(x-\xi)}{t} \varphi(2^n\xi)\, d\lambda(\xi) \\ &= 2^{nk} \int_E S(x-\xi)(u)\, \varphi(2^n\xi)\, d\lambda(\xi). \end{aligned}$$

By Lebesgue's differentiation theorem for the Bochner integral (see Appendix K) the set of Lebesgue points $L_u$ of $S(\cdot)(u)$ is the complement of a zero-measure set. Appealing to Lemma 14.2.3, we get

$$\lim_{n\to\infty} D_{g_n}(x)(u) = S(x)(u), \quad u \in E, x \in L_u. \tag{14.16}$$

Choose a basis for $E$ and let $G$ be the set of vectors having rational coordinates with respect to that basis. Since $G$ is countable, each of the sets

$$\Omega = \bigcap_{u \in G} \Omega_u \quad \text{and} \quad L = \bigcap_{u \in G} L_u$$

is the complement of a zero-measure set. Taking into account that $G$ is a vector space over $\mathbb{Q}$, and combining (14.15) with (14.16), we obtain

$$S(x)(su + tv) = sS(x)(u) + tS(x)(v), \quad x \in L,\ s, t \in \mathbb{Q}, u, v \in G. \tag{14.17}$$

Pick $x \in L$. From the identities (14.14) and (14.17), and the density of $G$ in $E$ we infer that $S(x)(\cdot)\colon G \to Y$ extends univocally to a linear map $T(x)(\cdot)\colon E \to Y$ whose norm is bounded by $\mathrm{Lip}(f)$. Let us prove that $\partial_u f(x) = T(x)(u)$ for all $u \in E$ and all $x$ in $L \cap \Omega$.

Given $\epsilon > 0$ there is $v \in G$ such that $\|u - v\| \leq (\mathrm{Lip}(f))^{-1}\epsilon/6$. By (14.13) there is $\delta > 0$ such that $0 < |t| \leq \delta$ implies

$$\left\| \frac{f(x + tv) - f(x)}{t} - T(x)(v) \right\| \leq \frac{\epsilon}{3}.$$

Then, for $0 < |t| \leq \delta$,

$$\begin{aligned} &\left\| \frac{f(x + tu) - f(x)}{t} - T(x)(u) \right\| \\ &\quad \leq \frac{\epsilon}{3} + \left\| \frac{f(x + tu) - f(x + tv)}{t} + T(x)(v - u) \right\| \\ &\quad \leq \frac{\epsilon}{3} + 2\,\mathrm{Lip}(f)\|u - v\| \leq \epsilon, \end{aligned}$$

as desired. Since $L \cap \Omega$ is the complement of a zero-measure set, we are done. □

The question is, can we prove an analogous result to Theorem 14.2.6 for Lipschitz maps defined on infinite-dimensional Banach spaces? The first problem we meet is to decide what sets of measure zero would look like in infinite-dimensional spaces, where there is no underlying measure. One is, of course, tempted to think in terms of Baire category to define a notion of small set, but in the infinite-dimensional case this is not the correct idea for differentiability theorems. There are several solutions to this problem (see [23]), but we will follow the approach of Christensen [49] by considering Haar-null sets.

**Definition 14.2.7.** A Borel subset $A$ of a separable Banach space $X$ is called *Haar-null* if there exists a probability measure $\mu$ on the $\sigma$-algebra $\mathcal{B}(X)$ of Borel subsets of $X$ such that $\mu(A + x) = 0$ for all $x \in X$.

It is important to observe that not every subset of $X$ is Haar-null, since $X$ itself is not. To better understand this notion we will introduce the companion concept of a Haar-null function.

**Definition 14.2.8.** A Borel-measurable function $h: X \to [0, \infty]$ on a separable Banach space $X$ is said to be *Haar-null* if there exists a probability measure $\mu$ on $(X, \mathcal{B}(X))$ such that

$$\int_X h(x + \xi)\, d\mu(\xi) = 0, \qquad \forall x \in X.$$

Of course, a set $A$ is Haar-null if and only if $\chi_A$ is a Haar-null map. Using the language of probability theory, a map $h$ is Haar-null if and only if there exist a probability measure space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $\eta: \Omega \to X$ such that

$$\mathbb{E}(h(x + \eta)) = \int_\Omega h(x + \eta(\omega))\, d\mathbb{P}(\omega) = 0, \qquad \forall x \in X.$$

If $X$ is finite-dimensional, then a Borel subset $A$ of $X$ is Haar-null if and only if its Lebesgue measure is zero. This will be a consequence of the following criterion, which we use to decide when a function (or a set) is Haar-null.

**Lemma 14.2.9.** *Let $h: X \to [0, \infty]$ be a measurable function. Suppose there exists a finite-dimensional subspace $E$ of $X$ such that $h(x + \xi) = 0$ a.e. $\xi \in E$ for all $x \in X$, i.e., $\lambda(\{\xi \in E: h(x + \xi) \neq 0\}) = 0$. Then $h$ is Haar-null.*

*Proof.* Pick a probability measure $\mu$ on $(X, \mathcal{B}(X))$ such that $\mu(X \setminus E) = 0$, and $\mu(A) = 0$ if and only if $A \cap E$ has Lebesgue measure zero. Then we have $h(x + \xi) = 0$ $\mu$-a.e $\xi \in X$, or in other words,

$$\int_X h(x + \xi)\, d\mu(\xi) = 0, \qquad \forall\, x \in X.$$

□

**Proposition 14.2.10.** *A measurable function $h: \mathbb{R}^n \to [0, \infty]$ is Haar-null if and only if $h = 0$ almost everywhere, i.e., $|\{\xi \in \mathbb{R}^n: h(\xi) \neq 0\}| = 0$.*

*Proof.* Assume $h: \mathbb{R}^n \to [0, \infty]$ is Haar-null, i.e., for some probability measure $\mu$ on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} h(x + \xi)\, d\mu(\xi) = 0, \qquad \forall x \in X.$$

Using Fubini's theorem and the translation-invariance of the Lebesgue measure $\lambda$ on $\mathbb{R}^n$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} h(x)\, d\lambda(x) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(x + \xi)\, d\lambda(\xi)\, d\mu(\xi) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(x + \xi)\, d\mu(\xi)\, d\lambda(\xi) = 0. \end{aligned}$$

Then $h = 0$ almost everywhere.

The converse is a consequence of Lemma 14.2.9 and the translation-invariance of the Lebesgue measure. □

Our next auxiliary result provides an improvement on the condition of being a Haar-null set.

**Lemma 14.2.11.** *Let $h$ be a Haar-null map on a separable infinite-dimensional Banach space $X$. For every $\epsilon > 0$ there exist a probability measure space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $\eta: \Omega \to X$ with $\|\eta\| < \epsilon$ almost everywhere such that*

$$\mathbb{E}(h(x + \eta)) = 0, \qquad \forall x \in X.$$

*Proof.* We know that there exist a probability space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $\eta: \Omega \to X$ such that $\mathbb{E}(h(x + \eta)) = 0$ for all $x \in X$. Since $X$ is separable, we can decompose $\Omega$ into countably many measurable sets $\Omega_n$ such that for each $n$ there exists $x_n \in X$ with $\|\eta(\omega) - x_n\| < \epsilon$ for $\omega \in \Omega_n$. Replace $\Omega$ by $\Omega' = \Omega_n$ with $\mathbb{P}(\Omega_n) > 0$ and define the probability measure $\mathbb{P}'(A) = \mathbb{P}(A)/\mathbb{P}(\Omega_n)$ on $\Omega_n$. Finally, let $\eta' = \eta - x_n$ on $\Omega'$. □

The key result we need is the following:

**Lemma 14.2.12.** *Let $X$ be a separable infinite-dimensional Banach space.*

*(i) If $(h_n)_{n=1}^{\infty}$ is a sequence of Haar-null maps on $X$, then $h = \sum_{n=1}^{\infty} h_n$ is Haar-null.*
*(ii) If $(A_n)_{n=1}^{\infty}$ is a sequence of Haar-null sets, then $A = \cup_{n=1}^{\infty} A_n$ is also Haar-null.*

*Proof.* For each $n$ choose a probability measure space $(\Omega_n, \mathbb{P}_n)$ and a random variable $\eta_n: \Omega_n \to X$ such that $\mathbb{E}(h_n(x + \eta_n)) = 0$ for all $x \in X$ and (by Lemma 14.2.11) $\|\eta_n\| < 2^{-n}$ almost everywhere. Further, construct another sequence of independent random variables $(\eta'_n)_{n=1}^{\infty}$ on a common probability space $(\Omega, \mathbb{P})$ maintaining the same properties as $(\eta_n)_{n=1}^{\infty}$. Thus the random variable $\eta = \sum_{k=1}^{\infty} \eta'_k$ is norm convergent, hence well-defined on $\Omega$. For each $n \in \mathbb{N}$, using independence,

$$\mathbb{E}\left(h_n\left(x + \eta\right)\right) = \mathbb{E}\, h_n\left(\left(x + \sum_{k \neq n} \eta'_k\right) + \eta'_n\right) = 0, \qquad \forall\, x \in X.$$

Summing over $n$, we obtain

$$\mathbb{E}\left(\sum_{n=1}^{\infty} h_n\left(x + \eta\right)\right) = 0, \qquad \forall\, x \in X.$$

This proves (i) and also yields (ii) by the fact that $\chi_{\cup_{n=1}^{\infty} A_n} \leq \sum_{n=1}^{\infty} \chi_{A_n}$. □

We state and prove next the main existence theorem for Gâteaux derivatives on infinite-dimensional spaces. The requirement that $Y$ have the Radon–Nikodym property is essential for this type of result, since if $Y$ fails (RNP), then there exists a nowhere-differentiable Lipschitz map $f: \mathbb{R} \to Y$ (see Section 5.5).

**Theorem 14.2.13 (Infinite-dimensional Rademacher Theorem).** *Let $X$ be a separable Banach space and suppose that $Y$ is a Banach space with the Radon–Nikodym property. Let $f: X \to Y$ be a Lipschitz map. Then the set of points at which $f$ fails to be Gâteaux differentiable is Haar-null.*

*Proof.* Let $(E_n)_{n=1}^{\infty}$ be an increasing sequence of finite-dimensional spaces whose union is dense in $X$. We then consider for each $n$ the set $D_n \subset X$ of all points $x \in X$ such that there is a linear operator $T_n: E_n \to Y$ with

$$\lim_{t\to 0}\frac{f(x+tu)-f(x)}{t} = T_n(u), \qquad \forall\, u \in E_n. \tag{14.18}$$

Of course, when the limit exists, $T_n(u)$ is nothing other than the directional derivative of $f$ at $x$ in the direction of $u \in E_n$. Put $A_n = X \setminus D_n$.

Given $z$ in $X$, let $f_z: X \to Y$ be the Lipschitz map defined by $f_z(x) = f(x-z)$. Then for $n \in \mathbb{N}$,

$$(z + D_n) \cap E_n = \{x \in E_n : f_z|_{E_n} \text{ is Gâteaux differentiable at } x\}.$$

Hence the set $(z + A_n) \cap E_n$ has Lebesgue measure zero by an application of Theorem 14.2.6. By Lemma 14.2.9 applied to $h = \chi_{A_n}$, $A_n$ is a Haar-null subset of $X$, and so $\cup_{n=1}^{\infty} A_n$ is Haar-null by Lemma 14.2.12. Let us see that the set of points where $f$ is Gâteaux differentiable is exactly its complementary set, $\cap_{n=1}^{\infty} D_n$.

Trivially, if $f$ is Gâteaux differentiable at a point $x \in X$, then $x$ must belong to $\cap_{n=1}^{\infty} D_n$. Conversely, pick $x \in \cap_{n=1}^{\infty} D_n$. We know that for each $n$ there exists $T_n: E_n \to Y$ fulfilling (14.18). Moreover, $\|T_n\| \le \operatorname{Lip}(f)$ for all $n$, and by definition, $T_{n+1}$ extends $T_n$ to $E_{n+1}$. Therefore by continuity we can define a linear operator $T: X \to Y$ whose norm is bounded by $\operatorname{Lip}(f)$, which will be the Gâteaux derivative of $f$ at $x$. Indeed, given $u \in X$ and $\epsilon > 0$ we choose $v \in \cup_{n=1}^{\infty} E_n$ with $\|u - v\| < \operatorname{Lip}(f)^{-1}\epsilon/6$. At the same time, there exists $\delta > 0$ such that

$$\left\|\frac{f(x+tv)-f(x)}{t} - T(v)\right\| < \frac{\epsilon}{3},$$

whenever $|t| < \delta$. Thus if $|t| < \delta$,

$$\left\|\frac{f(x+tu)-f(x)}{t} - T(u)\right\| \le \epsilon.$$

□

*Remark 14.2.14.* Theorem 14.2.13 fails dramatically if we want Fréchet derivatives instead of Gâteaux derivatives. For example, the Lipschitz map $f: \ell_2 \to \ell_2$ defined by $f(x_1, x_2, \dots) = (|x_1|, |x_2|, \dots)$ is nowhere Fréchet differentiable. See also Problem 14.7.

### 14.2.2 Linearization of Lipschitz Embeddings

Next we will apply the previous differentiability results to get linear embeddings from Lipschitz embeddings in some special cases.

**Theorem 14.2.15 (Heinrich and Mankiewicz [124]).** *Let $X$ and $Y$ be Banach spaces and suppose $f\colon X \to Y$ is a Lipschitz embedding. Assume $X$ is separable and that $Y$ has the Radon–Nikodym property. Then $X$ is linearly isomorphic to a subspace of $Y$. More specifically, if $x_0 \in X$ is a point of Gâteaux differentiability of $f$, then $D_f(x_0)$ is a linear embedding of $X$ into $Y$, and the isomorphism constant of $D_f(x_0)$ (i.e., the product of the norm of $D_f(x_0)$ and the norm of its inverse) is bounded by the Lipschitz distortion constant of $f$.*

*Proof.* By Theorem 14.2.13 there exists at least one point $x_0 \in X$ where $f$ is Gâteaux differentiable. Then, by the discussion leading to the estimate (14.11), $D_f(x_0)$ is an isomorphic embedding of $X$ into $Y$ and $\|D_f(x_0)\|\|(D_f(x_0))^{-1}\| \leq \mathrm{Lip}(f)$. □

*Remark 14.2.16.* It is an open question whether given a Lipschitz isomorphism $f\colon X \to Y$ between separable reflexive Banach spaces there exists an $x_0 \in X$ such that the Gâteaux derivative of $f$ at $x_0$ is surjective. If the answer were positive, then Lipschitz isomorphic separable reflexive Banach spaces would be linearly isomorphic. In [206] examples are presented of Lipschitz isomorphisms $f\colon \ell_2 \to \ell_2$ such that for a *large* set (i.e., a non-Haar null set) of points $x$, $D_f(x)$ exists but $D_f(x)(\ell_2)$ is a proper subspace of $\ell_2$.

Let us see what Theorem 14.2.15 tells about classical spaces. If $Y$ is a Hilbert space, then we deduce immediately that $X$ must also be a Hilbert space, up to isomorphism. We also observe that if $1 \leq p < \infty$ and $X$ Lipschitz embeds into $\ell_p$ then $X$ can be linearly embedded into $\ell_p$ (in the case $p = 1$, $\ell_1$ has (RNP)). The same results extend to the function spaces $L_p[0,1]$ for $p > 1$. Thus, for these Banach spaces the linear theory immediately yields:

(i) If $1 \leq p \neq q < \infty$, then neither of the spaces $\ell_p$ and $c_0$ Lipschitz embed into $\ell_q$.
(ii) Unless $p = q = 2$, the space $L_q$ does not Lipschitz embed into $\ell_p$.
(iii) If $1 \leq q < \infty$ then $L_p$ does not Lipschitz embed into $L_q$ unless $1 \leq q \leq p \leq 2$ or $p = q$.

Note that the case of $L_1$ as a potential target space of a Lipschitz embedding is special, because $L_1$ does not have the Radon–Nikodym property (see Example 5.5.3). Nevertheless, it still holds that $L_1$ does not contain any subset Lipschitz isomorphic to $L_p$ for $p > 2$ because of a cotype obstruction imposed by Corollary 14.2.29: the space $L_1$ has cotype 2 versus the cotype of $L_p$ for $p > 2$, which is only $p$.

Theorem 14.2.13 yields that Lipschitz maps into separable dual spaces have derivatives; hence by Theorem 14.2.15, it follows that if a separable space is Lipschitz embeddable into a separable dual space $Y$, then it is isomorphic to a subspace of $Y$. However, if the assumption on separability of $Y$ is lifted, then $Y$ may

fail (RNP), and so Lipschitz embeddings into $Y$ need not be differentiable anywhere. To circumvent this obstruction we shall define a weaker notion of a derivative, the $w^*$-derivative of Lipschitz functions mapping in dual spaces, and it will turn out that $w^*$-derivatives of a Lipschitz embedding induce linear embeddings almost everywhere.

**Definition 14.2.17.** A map $f$ from a Banach space $X$ to a dual space $Y = Z^*$ is said to be *weak* differentiable* at $x \in X$ if there exists a bounded linear operator $T\colon X \to Z^*$ such that for every $u \in X$ and every $z \in Z$ the following limit exists:

$$T(u)(z) := \lim_{t\to 0} \frac{(f(x+tu) - f(x))(z)}{t}.$$

In this case $T$ is called the *weak* derivative of $f$ at $x$* (or $w^*$-derivative) and is denoted by $D_f^*(x)$. Note that no uniformity on $u$ or $z$ is required in the above limit.

The next theorem could be called a *weak* Rademacher theorem* [124].

**Theorem 14.2.18.** *Let $f\colon E \to Z^*$ be a Lipschitz map, with $E$ a finite-dimensional normed space and $Z$ a separable Banach space. Then*

*(a) $f$ is weak* differentiable almost everywhere.*
*(b) $\|D_f^*(x)\| \le \mathrm{Lip}(f)$ whenever $D_f^*(x)$ exists.*
*(c) If, in addition, $f$ is a Lipschitz embedding, then $D_f^*(x)\colon E \to Z^*$ is a linear embedding for almost every $x \in E$. More precisely, if*

$$a\|x_1 - x_2\| \le \|f(x_1) - f(x_2)\| \le b\|x_1 - x_2\|, \qquad \forall x_1, x_2 \in E,$$

*then $a\|u\| \le \|D_f^*(x)(u)\| \le b\|u\|$ for all $u \in E$ and almost every $x \in E$.*

*Proof.* Let $(z_n)_{n=1}^{\infty}$ be a dense sequence in the unit sphere of $Z$. For each $n$, we put $\phi_n(x) = f(x)(z_n)$. Each $\phi_n$ is a Lipschitz map from $E$ to $\mathbb{R}$ with $\mathrm{Lip}(\phi_n) \le \mathrm{Lip}(f)$, and thus by Theorem 14.2.6 in the case $Y = \mathbb{R}$, the set

$$\Omega = \{x \in E\colon D_{\phi_n}(x) \text{ exists for all } n \in \mathbb{N}\}$$

is of full measure (that is, the measure of its complement is 0). Since the sequence $(z_n)_{n=1}^{\infty}$ is dense in the sphere of $Z$, it follows easily that $f$ is weak* differentiable for every $x_0 \in \Omega$, and that the norm of its weak* derivative $D_f^*(x_0)$ is at most equal to the Lipschitz norm of $f$. This takes care of statements (a) and (b).

To show (c) we can suppose without loss of generality that $a = 1$. Note that when we write $(D_f^*(x))^{-1}$, we assume only that it is defined on the range of $D_f^*(x)$. Let $\Omega_{w^*}$ be the set of weak* differentiability points of $f$ (which is of full measure by part (a)). We claim that for every $u \in E$ with $\|u\| = 1$, the set

$$M_u = \{x \in \Omega_{w^*}\colon \|D_f^*(x)(u)\| \ge 1\}$$

is of full measure. Assume it is not. Then, there is $\delta > 0$ such that

$$N := \{x \in \Omega_{w^*}: \|D_f^*(x)(u)\| \leq 1-\delta\}$$

is of positive measure. Then, Fubini's theorem gives $x_0 \in E$ such that the (one-dimensional) Lebesgue measure of the set $U = \{t \in \mathbb{R}: x_0 + tu \in N\}$ is strictly positive. Let $t_0 \in U$ be a Lebesgue point of $\chi_U$, i.e.,

$$\lim_{h\to 0^+} \frac{|U \cap (t_0 - h/2, t_0 + h/2)|}{h} = 1.$$

Choosing $h$ small enough and letting $x = x_0 + (t_0 - h/2)u$, we get that the set

$$A = \{t \in (0,h): x + tu \in N\}$$

has Lebesgue measure $|A| > h(1 - \delta/(2b))$.

Since $\|f(x+hu) - f(x)\| \geq h$, there is a norm-one vector $z \in Z$ such that

$$\phi(h) - \phi(0) > h(1 - \delta/2),$$

where $\phi$ is defined by $\phi(t) = f(x+tu)(z)$. The map $\phi$ is $b$-Lipschitz and

$$\phi'(t) = D_f^*(x+tu)(u)(z), \qquad t \in A.$$

Hence $\phi'(t) \leq b$ a.e. $t \in [0,h]$ and $\phi'(t) \leq 1-\delta$ for $t \in A$. Since $\phi$ is in particular absolutely continuous, we can write

$$\begin{aligned}
\phi(h) - \phi(0) &= \int_0^h \phi'(t)dt \\
&= \int_A \phi'(t)dt + \int_{[0,h]\setminus A} \phi'(t)dt \\
&\leq h(1-\delta) + \frac{h\delta}{2} \\
&= h\left(1 - \frac{\delta}{2}\right).
\end{aligned}$$

This contradiction shows that $M_u$ is of full measure for every $u$ in the unit sphere of $E$. Since this sphere is separable, it follows that for almost every $x \in \Omega_{w^*}$ we have $\|D_f^*(x)(u)\| \geq 1$ for every $u \in E$ of norm 1, and this concludes the proof. $\square$

And next we can deduce the infinite-dimensional version of Theorem 14.2.18.

**Theorem 14.2.19 (Infinite-dimensional weak$^*$ Rademacher Theorem).** *Let $X$ and $Z$ be separable Banach spaces and let $f: X \to Z^*$ be a Lipschitz map. Then*

*(a)* $f$ *is weak* differentiable outside a Haar-null set.*
*(b)* $\|D_f^*(x)\| \leq \mathrm{Lip}(f)$ *whenever* $D_f^*(x)$ *exists.*
*(c) If, in addition,* $f$ *is a Lipschitz embedding, then* $D_f^*(x): X \to Z^*$ *is an isomorphic embedding for every* $x$ *outside a Haar-null subset of* $X$*, with* $\|(D_f^*(x))^{-1}\| \leq \mathrm{Lip}(f^{-1})$.

*Proof.* To show part (c) let $a$ be the inverse of $\mathrm{Lip}(f^{-1})$. Choose an increasing sequence $(E_n)_{n=1}^\infty$ of finite-dimensional subspaces of $X$ whose union is dense. For each $n$, consider the set $D_n$ of all points $x \in X$ for which there is a linear operator $T_n: E_n \to Z^*$ such that

$$\lim_{t\to 0} \frac{f(x+tu)(z) - f(x)(z)}{t} = T_n(u)(z), \quad u \in E_n,\ z \in Z,$$

and

$$a\|u\| \leq \|T_n(u)\|, \quad u \in E_n.$$

From the proof of Theorem 14.2.18, the conclusion follows using arguments similar to those in the proof of Theorem 14.2.13.

Parts (a) and (b) can be obtained by taking $a = 0$ and mimicking the proof of (c). □

Our next proposition is a purely linear result that will allow us to dispense with the assumption of separability of $Z$ in Theorem 14.2.19. The original proof from [124] leaned on ultraproducts on uncountable sets (a technique that is not in this book's toolbox) and used a Banach space version of the Loewenheim–Skolem theorem (cf. [124, Theorem 1.6]) . We have chosen an alternative proof taken from [282] that does not use ultraproducts (it does use ultrafilters, though). Instead it relies on the following result of Lindenstrauss [193], whose proof we skip in order not to lead the reader away from the expositional flow.

**Lemma 14.2.20.** *Let* $F_0$ *be a finite-dimensional subspace of a Banach space* $Z$ *and let* $k \in \mathbb{N}$ *and* $\epsilon > 0$ *be given. Then there exists a finite-dimensional subspace* $F_0 \subseteq F_1 \subseteq Z$ *such that for all subspaces* $F_0 \subseteq E \subseteq Z$ *with* $\dim(E/F_0) \leq k$ *there is a linear mapping* $S: E \to F_1$ *with* $\|S\| \leq 1+\epsilon$ *and* $S|_{F_0} = I_{F_0}$.

**Proposition 14.2.21 (Existence of linear Hahn–Banach extension operators).** *Let* $Z$ *be a Banach space, and let* $V$ *be a separable subspace of* $Z$*. There exist a separable space* $W$ *with* $V \subseteq W \subseteq Z$ *and a linear map* $T: W^* \to Z^*$ *such that* $\|T\| \leq 1$ *and* $Q \circ T = I_{W^*}$*, where* $Q: Z^* \to W^*$ *is the canonical quotient map, given by* $Q(z^*) = z^*|_W$ *for* $z^* \in Z^*$.

*Proof.* Certainly, if $Z$ is separable, then the result is clear, since it suffices to take $W = Z$. Having disposed of this case, we may assume that $Z$ is nonseparable. Then, in particular, $Z$ is infinite-dimensional and $V$ can be assumed to be also infinite-dimensional. Pick a sequence $(x_n)_{n=1}^\infty$ whose closed linear span is $V$. Construct

recursively an increasing sequence $(W_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of $Z$ in the following fashion: let $W_1 = 0$ and for the inductive step let $W_{n+1} = F_1$, the space provided by Lemma 14.2.20 with $F_0 = W_n + \langle x_n \rangle$, $k = n$, and $\varepsilon = 1/n$. Let $W$ be the closure in $Z$ of $\cup_{n=1}^{\infty} W_n$. The space $W$ is separable and $V \subseteq W$.

Given a finite-dimensional subspace $E \subseteq Z$, denote by $\Gamma_E$ the set of positive integers $n$ such that $W_n \subseteq E$ and $\dim(E/W_n) \le n$. Let

$$\mathcal{I} = \{E : \Gamma_E \neq \emptyset\}.$$

Note that $1 \in \Gamma_{\langle z \rangle}$ for every $z \in Z$, so that $\langle z \rangle \in \mathcal{I}$. Also, for $n \in \mathbb{N}$ we have $n \in \Gamma_{W_n}$, and so $W_n \in \mathcal{I}$. We claim that given $E, F \in \mathcal{I}$ (or simply $E$ and $F$ finite-dimensional subspaces of $Z$) there is $G \in \mathcal{I}$ such that $E \cup F \subseteq G$. Indeed, if we let $j = \dim(E)$ and $k = \dim(F)$ and choose $G = W_{j+k} + E + F$, then

$$\dim(G/W_{j+k}) \le \dim(E + F) \le j + k,$$

so that $j + k \in \Gamma_G$. In other words, $(\mathcal{I}, \subseteq)$ is a directed set. Consequently, there is an ultrafilter $\mathcal{U}$ on $\mathcal{I}$ such that $\mathcal{I}_E := \{F \in \mathcal{I} : E \subseteq F\} \in \mathcal{U}$ for all $E \in \mathcal{I}$. Since $\lim_n \dim(W_n) = \infty$, for every finite-dimensional subspace $E \subseteq Z$ the set $\Delta_E := \{n \in \mathbb{N} : W_n \subseteq E\}$ is finite. Moreover, $\Gamma_E \subseteq \Delta_E$. This way for $E \in \mathcal{I}$ we can safely define $j_E = \max \Gamma_E$ and $k_E = \max \Delta_E$. We claim that $j_E = k_E$. Indeed, it is obvious that $j = j_E \le k = k_E$. But then,

$$\dim(E/W_k) \le \dim(E/W_j) \le j \le k,$$

and we get $k \le j$. Therefore, for every $n \in \mathbb{N}$,

$$\mathcal{A}_n := \{E \in \mathcal{I} : n \le j_E\} = \mathcal{I}_{W_n} \in \mathcal{U}.$$

For every $z \in Z$ we also have

$$\mathcal{B}_z := \{E \in \mathcal{I} : z \in E\} = \mathcal{I}_{\langle z \rangle} \in \mathcal{U}.$$

Using the axiom of choice, for every $E \in \mathcal{I}$ we define a linear map $S_E : E \to W_{j+1}$ such that $S_E|_{W_j} = I_{W_j}$ and $\|S_E\| \le 1 + 1/j$, where $j = j_E$. Let $T_E : W^* \to \mathbb{R}^Z$ be the linear map given by

$$T_E(f)(z) = \begin{cases} f(S_E(z)) & \text{if } z \in E, \\ 0 & \text{otherwise,} \end{cases}$$

for $f \in W^*$ and $z \in Z$. We have

$$|T_E(f)(z)| \le \left(1 + \frac{1}{j_E}\right) \|f\| \, \|z\| \le 2\|f\| \, \|z\|, \quad f \in W^*, z \in Z. \tag{14.19}$$

Thus we may and do define a linear map $T: W^* \to \mathbb{R}^Z$ by

$$T(f)(z) = \lim_{\mathcal{U}} T_E(f)(z), \quad f \in W^*, z \in Z.$$

Let $f \in W^*$ and $z \in Z$. Fix $n \in \mathbb{N}$. Appealing to (14.19), we obtain

$$|T_E(f)(z)| \leq \left(1 + \frac{1}{n}\right)\|f\| \, \|z\|, \quad E \in \mathcal{A}_n.$$

Therefore

$$|T(f)(z)| \leq \left(1 + \frac{1}{n}\right)\|f\| \, \|z\|.$$

Since $n$ is arbitrary, we obtain $|T(f)(z)| \leq \|f\| \, \|z\|$.

Let $f \in W^*$, $y, z \in Z$, and $\lambda, \mu \in \mathbb{R}$. We have

$$T_E(f)(\lambda y + \mu z) = \lambda T_E(f)(y) + \mu T_E(f)(z), \quad E \in \mathcal{B}_y \cap \mathcal{B}_z.$$

Consequently,

$$T(f)(\lambda y + \mu z) = \lambda T(f)(y) + \mu T(f)(z).$$

This proves that $T(f)$ is linear and therefore continuous, i.e., $T(f) \in Z^*$.

Let $f \in W^*$ and $z \in \cup_{n=1}^{\infty} W_n$. There is $n \in \mathbb{N}$ such that $z \in W_n$. We have

$$T_E(f)(z) = f(S_E(z)) = f(z), \quad E \in \mathcal{A}_n.$$

Hence $T(f)(z) = f(z)$. Appealing to the continuity of $T(f)$, we get $T(f)(z) = f(z)$ for all $f \in W^*$ and $z \in W$. □

**Corollary 14.2.22.** *Let $Y$ be a separable subspace of a dual space $Z^*$. Then there exists a separable Banach space $W$ such that $Y$ embeds linearly isometrically into $W^*$ and $W^*$ embeds linearly isometrically into $Z^*$.*

*Proof.* Choose a separable subspace $V \subseteq Z$ that is isometrically norming for $Y$. By Proposition 14.2.21, there exist $V \subseteq W \subseteq Z$ and a linear isometric embedding $T: W^* \to Z^*$ such that $Q \circ T = I_{W^*}$. Since $W$ is also isometrically norming for $Y$, we infer that $Q|_Y$ is a linear embedding of $Y$ into $W^*$. □

**Theorem 14.2.23.** *If a separable Banach space $X$ Lipschitz embeds into the dual $Z^*$ of a Banach space $Z$, then $X$ is isomorphic to a subspace of $Z^*$. Quantitatively, if $f: X \to Z^*$ is a Lipschitz embedding, then there is a linear embedding $T: X \to Z^*$ such that $\|T\|\|T^{-1}\| \leq \operatorname{dist}(f)$.*

*Proof.* Assume $f: X \to Z^*$ is a Lipschitz embedding. Since $X$ is separable, $f(X)$ is contained in a separable subspace $Y$ of $Z^*$. By Corollary 14.2.22 we can find

a separable Banach space $W$ and isometric linear embeddings $S_0: Y \to W^*$ and $S_1: W^* \to Z^*$. Put $g = S_0 \circ f: X \to Y$. By Theorem 14.2.19, there exists a point $x \in X$ such that the weak* derivative $D^*_g(x)$ is a linear embedding of $X$ into $Y$ with $\|D^*_g(x)\| \le \mathrm{Lip}(g)$ and $\|(D^*_g(x))^{-1}\| \le \mathrm{Lip}(g^{-1})$. Thus $T = S_1 \circ D^*_g(x)$ is the desired linear embedding of $X$ into $Z^*$. □

**Corollary 14.2.24.** *If a separable Banach space $X$ Lipschitz embeds in a Banach space $Y$ such that $Y^*$ is separable, then $X$ can be linearly embedded into $Y^{**}$. Quantitatively, if $f: X \to Y$ is a Lipschitz embedding, then there is a linear embedding $T: X \to Y^{**}$ such that $\|T\|\|T^{-1}\| \le \mathrm{dist}(f)$.*

These techniques give us no information about Banach spaces that Lipschitz embed into $c_0$. Indeed, since $c_0$ fails the Radon–Nikodym property (see Example 5.5.3), we can only use Corollary 14.2.24, and unfortunately $c_0^{**} = \ell_\infty$ contains a linear isometric copy of every separable Banach space.

The next result shows that there is a very good reason why $c_0$ is a special case and also exemplifies that without the assumption that $Y$ has (RNP) in Theorem 14.2.15 the Lipschitz and linear structures of a space can be very different. Indeed, while $c_0$ is considered a *small* space in the linear theory (for example, it does not contain reflexive subspaces or any of the other classical Banach spaces), Aharoni showed that it is *universal* for separable spaces in the Lipschitz category. The proof we include below is not the original one by Aharoni (see [1]). We use a simple argument that strips the construction of the embedding to the bone at the cost of relaxing the distortion constant.

**Theorem 14.2.25 (Aharoni [1]).** *Every separable metric space $(X, d)$ is Lipschitz isomorphic to a subset of the Banach space $c_0$. Thus for some constant $K$, there is a map $f: X \to c_0$ that satisfies the inequalities*

$$d(x, y) \le \|f(x) - f(y)\|_\infty \le K d(x, y), \quad \forall\, x, y \in X. \tag{14.20}$$

*Proof.* Let $(x_j)_{j=1}^\infty$ be a countable dense set in $X$. Pick $0 < r < 1$ such that

$$a := 1 - \frac{2r}{1 - r} > 0.$$

For $x \in X$ and $j \in \mathbb{N}$ put $d_j(x) = d(x_j, x)$ and define $f = (f_n)_{n=1}^\infty: X \to \mathbb{R}^{\mathbb{N}}$ by

$$f_n(x) = \min\left\{ \frac{1}{ra} d_1(x), \dots, \frac{1}{ra} d_{n-1}(x), \frac{1}{a} d_n(x) \right\}.$$

The simple estimate $d_j(x) - d_j(y) \le d(x, y)$ yields

$$|f_n(x) - f_n(y)| \le \frac{1}{ar} d(x, y), \quad x, y \in X.$$

We have that $f_n(x_j) = 0$ for $j \leq n$. Therefore $f(x_j) \in c_0$ for all $j \in \mathbb{N}$. Hence, appealing to the density of $(x_j)_{j=1}^{\infty}$ in $X$ and the continuity of $f$, we obtain that $f(x) \in c_0$ for all $x \in X$. If we show that

$$\delta(i,j) := d(x_i, x_j) \leq \|f(x_i) - f(x_j)\|, \quad i,j \in \mathbb{N}, \tag{14.21}$$

then, again by an approximation argument, we would have

$$d(x,y) \leq \|f(x) - f(y)\|_\infty, \quad x,y \in X,$$

and we will have proved (14.20) with $K = 1/ar$.

In order to prove (14.21), assume that $j < i$ and let

$$i(k) = \begin{cases} i & \text{if } k \text{ is odd,} \\ j & \text{if } k \text{ is even.} \end{cases}$$

Choose recursively $j = j_0 > j_1 > \cdots > j_k > \cdots > j_m \geq 1$ such that

$$\delta(j_k, i(k)) < r\,\delta(j_{k-1}, i(k)), \quad k = 1, \ldots, m, \tag{14.22}$$

and

$$\delta(j, i(m+1)) \geq r\,\delta(j_m, i(m+1)), \qquad 1 \leq j < j_m. \tag{14.23}$$

Note that neither the case $m = 0$ nor the case $j_m = 1$ is excluded. Combining (14.22) with the triangle inequality, we get

$$\delta(j_k, i(k)) < r\,\delta(i,j) + r\,\delta(j_{k-1}, i(k-1)), \quad k = 1, \ldots, m.$$

Consequently,

$$\delta(j_m, i(m)) \leq (r + \cdots + r^m)\,\delta(i,j) \leq \frac{r}{1-r}\text{ă}\delta(i,j). \tag{14.24}$$

By (14.23),

$$\delta(j_m, i(m+1)) = a f_{j_m}(x_{i(m+1)}).$$

Therefore,

$$\begin{aligned} \delta(j_m, i(m+1)) - \delta(j_m, i(m)) &\leq a(f_{j_m}(x_{i(m+1)}) - f_{j_m}(x_{i(m)})) \\ &\leq a\|f(x_i) - f(x_j)\|. \end{aligned}$$

Combining this inequality with (14.24) and with the triangle inequality yields,

$$\begin{aligned}\delta(i,j) &\leq \delta(j_m, i(m+1)) + \delta(j_m, i(m)) \\ &\leq a\|f(x_i) - f(x_j)\| + 2\,\delta(j_m, i(m)) \\ &\leq a\|f(x_i) - f(x_j)\| + \frac{2r}{1-r}\delta(i,j).\end{aligned}$$

Now (14.21) follows with a trivial manipulation. □

There is a curious side-plot here with regard to the best result concerning the distortion of a Lipschitz embedding of a metric space $X$ into $c_0$. The constant obtained in the above proof for $0 < r < 1/3$ is $K = (1-r)(1-3r)^{-1}r^{-1}$. The best choice is $r = 1 - \sqrt{2/3}$, in which case $9.8 < K < 9.9$. Aharoni showed that an embedding $f: X \to c_0$ can be achieved with distortion $6 + \epsilon$ for any $\epsilon$, i.e.,

$$d(x,y) \leq \|f(x) - f(y)\| \leq (6+\epsilon)d(x,y), \quad x, y \in X.$$

He noted that if $X = \ell_1$, we can establish a lower bound of 2 for the distortion. Shortly afterward, Assouad [16] improved $6+\epsilon$ to $3+\epsilon$; later still, Pelant [238] improved the constant to 3. Each of these authors actually found embeddings into the positive cone $c_0^+$, and for this setting, 3 is indeed the optimal constant. However, in 2008, Lancien and Kalton [162] finally showed that 2 is the optimal distortion constant for embeddings into $c_0$. The paper [162] contains other results in this direction; one of them is that for $1 \leq p < \infty$ the space $\ell_p$ Lipschitz embeds into $c_0$ with (best) distortion constant $2^{1/p}$.

In view of the fact that a Banach space $X$ is Lipschitz universal for all separable metric spaces if and only if $c_0$ Lipschitz embeds into $X$, a natural unsolved problem remains:

**Problem 14.2.26.** *If $c_0$ Lipschitz embeds into a Banach space $X$, does $c_0$ linearly embed into $X$?*

### *14.2.3 Invariance of the Local Structure Under Coarse Lipschitz Embeddings*

We are ready to prove an important result on finite representability, namely that the *local* structure of a Banach space (the collection of its finite-dimensional subspaces) is stable under coarse Lipschitz embeddings. We will see, for instance, that a Banach space that coarse Lipschitz embeds into a Hilbert space is itself a Hilbert space. Our linearization arguments will again be differentiation techniques, as could be expected. The ideas leading to the fact that local properties are preserved under coarse Lipschitz embeddings go back to Ribe [268].

**Theorem 14.2.27 (Ribe [268]).** *If there is a coarse Lipschitz embedding of a Banach space $X$ into a Banach space $Y$, then $X$ is crudely finitely representable in $Y$.*

*Proof.* Suppose $X$ coarse Lipschitz embeds into $Y$. By Corollary 14.1.26 there is a Lipschitz embedding $f: X \to Y_{\mathcal{U}}$ for some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Since $[Y_{\mathcal{U}}]^{**}$ is finitely representable in $Y_{\mathcal{U}}$ (Theorem 12.2.4) and $Y_{\mathcal{U}}$ is finitely representable in $Y$ (Proposition 12.1.12), it suffices to show that $X$ is crudely finitely representable in $[Y_{\mathcal{U}}]^{**}$. Let $E$ be a finite-dimensional subspace of $X$. By Corollary 14.2.24 there exists a subspace $F$ of $Y_{\mathcal{U}}^{**}$ such that $d(E, F) \leq \operatorname{dist}(f)$. Thus we obtain that $X$ is crudely finitely representable in $Y$ with constant $\operatorname{dist}(f) + \epsilon$ for every $\epsilon > 0$. □

*Remark 14.2.28.* It should be mentioned that ultrapowers can be totally avoided in the proof of Theorem 14.2.27 if we replace the coarse Lipschitz embedding hypothesis by the assumption of having instead a Lipschitz embedding with values in spaces with separable dual (e.g., in separable reflexive spaces), since then, the conclusion follows directly from Theorem 14.2.18 and the principle of local reflexivity (see Section 12.2).

Theorem 14.2.27 has many corollaries; next we state a few of them.

**Corollary 14.2.29.** *Let $X$ and $Y$ be Banach spaces such that there exists a coarse Lipschitz embedding from $X$ into $Y$. If $Y$ has type $p$ [respectively, cotype $q$], then $X$ has type $p$ [respectively, cotype $q$].*

*Proof.* Use Theorem 14.2.27 and the definitions of type and cotype. □

**Corollary 14.2.30.** *If a Banach space $X$ coarse Lipschitz embeds into a Hilbert space, then it is isomorphic to a Hilbert space.*

*Proof.* This follows directly from Theorem 14.2.27 and Theorem 12.1.6. □

The last one is left as an easy exercise for the reader (see Problem 14.12).

**Corollary 14.2.31.** *Suppose a Banach space $X$ coarse Lipschitz embeds into a superreflexive Banach space $Y$. Then $X$ is superreflexive.*

## 14.3 Lipschitz Isomorphisms Between Banach Spaces

In this section we consider one of the central problems in the field and show that under some general conditions, Lipschitz isomorphic Banach spaces are, in fact, linearly isomorphic. In particular, the methods serve the purpose to show that the isomorphic structure of the spaces $L_p$ and $\ell_p$ for $1 < p < \infty$ is completely determined by their Lipschitz structure.

**Problem 14.3.1.** *If $X$ and $Y$ are separable Banach spaces that are Lipschitz isomorphic, are $X$ and $Y$ linearly isomorphic?*

To understand this problem we must first explain why it is necessary to restrict to the separable case. If we allow nonseparable spaces, then counterexamples have been known for some time. The first example of two nonseparable, nonreflexive Lipschitz isomorphic nonisomorphic Banach spaces was given by Aharoni and Lindenstrauss [2]. We will describe this example, which is based on the fact that the quotient space $\ell_\infty/c_0$ contains a nonseparable $c_0(\Gamma)$.

*Example 14.3.2.* Let $q\colon \ell_\infty \to \ell_\infty/c_0$ be the canonical quotient map. We start from the existence of a continuum of infinite subsets $(\mathbb{A}_i)_{i\in\mathcal{I}}$ of $\mathbb{N}$ with the property that $\mathbb{A}_i \cap \mathbb{A}_j$ is finite when $i \neq j$ (see Lemma 2.5.3). Let $\xi_i(k) = 1$ if $k \in \mathbb{A}_i$ and zero otherwise. Then the vectors $(q(\xi_i)_{i\in\mathcal{I}})$ are isometrically equivalent to the canonical basis vectors of $c_0(\mathcal{I})$. Let $E = [q(\xi_i)]_{i\in\mathcal{I}}$. We will define a Lipschitz map $f\colon E \to \ell_\infty$ with the property that $q \circ f = I_E$.

If $x = \sum_{n=1}^\infty a_n q(\xi_{i_n})$, where $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$, we let

$$f(x)(k) = \begin{cases} a_1 & \text{if } \ k \in \mathbb{A}_{i_1}, \\ a_n & \text{if } \ k \in \mathbb{A}_{i_n} \setminus \cup_{r=1}^{n-1} \mathbb{A}_{i_r}, \ n \geq 2, \\ 0 & \text{if } \ k \notin \cup_{r=1}^\infty \mathbb{A}_{i_r}. \end{cases}$$

For general $x$ (where not all the coefficients are nonnegative) we split $x = x^+ - x^-$, where $x^+$ and $x^-$ have disjoint supports and nonnegative coefficients. Then we set $f(x) = f(x^+) - f(x^-)$. Of course, in this definition one must verify that $f$ is defined unambiguously (it is!) and that $f$ is Lipschitz (with constant 2).

If $x \in q^{-1}(E)$, we define $h(x) = (x - f \circ q(x), q(x)) \in c_0 \oplus E$. Then $h$ is a Lipschitz isomorphism from $q^{-1}(E)$ onto $c_0 \oplus c_0(\mathcal{I})$, which is a nonseparable $c_0(\Gamma)$-space and hence cannot be linearly isomorphic to a subspace of $c_0$.

This argument was later refined by Deville, Godefroy, and Zizler in [59, 60] to show that $\mathcal{C}(K)$ is Lipschitz isomorphic to $c_0(\Gamma)$ provided $K$ is a compact Hausdorff space such that the $n$-derived set $K^{(n)}$ is empty for some finite $n$. Unfortunately, in the separable case (i.e., $\Gamma$ countable) this does not give anything significant, since these conditions imply that $\mathcal{C}(K)$ is linearly isomorphic to $c_0$. However, in the uncountable case there are examples (as shown above) in which $\mathcal{C}(K)$ is not linearly isomorphic to a $c_0(\Gamma)$.

We have presented the classical Example 14.3.2 in some detail to highlight its nature. Let $X$ be a Banach space and let $Z$ be a closed subspace. Let $q\colon X \to X/Z$ be the quotient mapping. Suppose we can find a *Lipschitz lifting* of $q$, i.e., a Lipschitz map $f\colon X/Z \to X$ such that $q \circ f = I_{X/Z}$. Then, as before, the map

$$X \to Z \oplus X/Z, \quad x \mapsto (x - f \circ q(x), q(x)),$$

is a Lipschitz isomorphism of $X$ onto $Z \oplus X/Z$. Of course, if there is a linear lifting $T\colon X/Z \to X$ of $q$, then $T(X/Z)$ is complemented and $X$ must be linearly isomorphic to $Z \oplus X/Z$. Thus the Aharoni–Lindenstrauss approach requires the existence of a quotient map that has a Lipschitz lifting but no linear lifting.

And unfortunately, this approach cannot work in separable Banach spaces, as Theorem 14.3.3 and Corollary 14.3.4 will show, which explains why Problem 14.3.1 is open for separable spaces!

**Theorem 14.3.3 (Godefroy and Kalton [105]).** *Let $Q : Y \to X$ be a continuous linear map between a Banach space $Y$ and a separable Banach space $X$. Suppose there exists a Lipschitz map $g\colon X \to Y$ such that $Q \circ g = I_X$. Then there exists a continuous linear operator $S\colon X \to Y$ such that $Q \circ S = I_X$ and $\|S\| \le \mathrm{Lip}(g)$.*

In this theorem the assumption of separability is not a matter of convenience, since the conclusion fails to hold for some nonseparable Banach spaces $X$. The strategy of the proof is to consider a convolution of the Lipschitz lifting $g$ with a probability measure and to differentiate the resulting map. Although such arguments can be displayed in an infinite-dimensional setting (see [105]) our proof uses only finite-dimensional considerations.

*Proof.* We will construct a suitable linear projection $U$ from $\mathrm{Lip}_0(X,Y)$ onto the Banach space $\mathcal{B}(X,Y)$ of bounded linear operators from $X$ to $Y$. To that end, we begin with the case $Y = \mathbb{R}$. There exists a linearly independent sequence $(x_i)_{i=1}^\infty$ of vectors in $X$ such that $E := \langle x_i\colon i \in \mathbb{N}\rangle$ is dense in $X$ and $\sum_{i=1}^\infty \|x_i\| < \infty$. Let $E_k = \langle x_i\colon 1 \le i \le k\rangle$ and consider the linear bijection $T_k\colon \mathbb{R}^k \to E_k$ determined by $T_k(e_i) = x_i$ for $1 \le i \le k$. Let $f \in \mathrm{Lip}_0(X)$. By Rademacher's theorem (Theorem 14.2.6) in the case $Y = \mathbb{R}$, the map $f|_{E_k}$ is differentiable almost everywhere. So, we can safely define a linear operator $R_k\colon E_k \to \mathrm{Lip}_0(X)^*$ by

$$R_k(u)(f) = \int_{[0,1]^k} D_f(T_k(\xi))(u)\, d\xi, \quad u \in E_k, f \in \mathrm{Lip}_0(X).$$

We have

$$|R_k(u)(f)| \le \mathrm{Lip}(f)\|u\|, \quad u \in E_k, f \in \mathrm{Lip}_0(X),$$

and

$$R_k(u)(x^*) = x^*(u), \quad x^* \in X^*, u \in E_k.$$

Given $1 \le j \le k$, let $T_k^j\colon \mathbb{R}^{k-1} \to X$ be the linear map given by $T_k^j(e_i) = x_i$ if $i \le j-1$ and $T(e_i) = x_{i+1}$ if $j \le i \le k-1$. The fundamental theorem of calculus and Fubini's theorem yield

$$R_k(x_j)(f) = \int_{[0,1]^{k-1}} (f(T_k^j(\xi) + x_j) - f(T_k^j(\xi))\, d\xi, \quad f \in \mathrm{Lip}_0(f). \tag{14.25}$$

Let $j \le m \le n$ and consider the linear map $T_{m,n}\colon \mathbb{R}^{n-m} \to X$ determined by

$$T_{m,n}(e_i) = x_{m+i}, \quad 1 \le i \le n-m.$$

Given $\xi = (t_1, \dots, t_{n-1})$ in $\mathbb{R}^{n-1}$ put $\xi_a = (t_1, \dots, t_{m-1})$ and $\xi_b = (t_m, \dots, t_{n-1})$. Then we have $T_n^j(\xi) - T_n^j(\xi_a) = T_{m,n}(\xi_b)$. Hence, by (14.25),

$$\begin{aligned}\|R_n(x_j)(f) - R_m(x_j)(f)\| &\le 2\,\mathrm{Lip}(f) \int_{[0,1]^{n-m}} \|T_{m,n}(\xi)\| d\xi \\ &\le \mathrm{Lip}(f) \sum_{i=m+1}^{n} \|x_i\|.\end{aligned}$$

Therefore, if $k \le m \le n$, for $f \in \mathrm{Lip}_0(f)$ and $u \in E_k$,

$$\|R_n(u)(f) - R_m(u)(f)\| \le \mathrm{Lip}(f) \|T_k^{-1}(u)\|_1 \sum_{i=m+1}^{n} \|x_i\|.$$

Then the sequence $(R_n|_{E_k})_{n=k}^{\infty}$ converges in the Banach space $\mathcal{B}(E_k, \mathrm{Lip}_0(X)^*)$ to an operator $P_k \colon E_k \to \mathrm{Lip}_0(X)^*$. It is clear that $P_{k+1}$ extends $P_k$ and that $\|P_k\| \le 1$. Since $E$ is dense in $X$, there is a unique $R \colon X \to \mathrm{Lip}_0(X)^*$ that extends each operator $P_k$. The operator $R$ inherits from $(R_k)_{k=1}^{\infty}$ the properties $\|R\| \le 1$ and $R(u)(x^*) = x^*(u)$ for all $u \in X$ and all $x^* \in X^*$. Moreover, by (14.25),

$$|R(x_j)(f)| \le 2 \sup_{x \in K} |f(x)|, \quad j \in \mathbb{N}, f \in \mathrm{Lip}_0(X),$$

the supremum being taken over $K = \{\sum_{i=1}^{\infty} t_i x_i \colon 0 \le t_i \le 1\}$. Consequently, if we let $T \colon c_{00} \to E$ be the linear bijection given by $T(e_i) = x_i$ for $i \in \mathbb{N}$, then

$$|R(u)(f)| \le 2\|T^{-1}(u)\|_1 \sup_{x \in K} |f(x)|, \quad u \in E, f \in \mathrm{Lip}_0(X). \tag{14.26}$$

Once we have built $R$, we may define $U \colon \mathrm{Lip}_0(X, Y) \to \mathcal{B}(X, Y^{**})$ by

$$U(g)(u)(y^*) = R(u)(y^* \circ g), \quad g \in \mathrm{Lip}_0(X, Y),\ u \in X,\ y^* \in Y^*.$$

We have that $U$ is a continuous linear operator with norm $\|U\| \le 1$. Let us prove that $U(g) \in \mathcal{B}(X, Y)$ for all $g \in \mathrm{Lip}_0(X, Y)$, i.e., that $U(g)(u) \in j_Y(Y)$ for all $u \in X$ and $g \in \mathrm{Lip}_0(X, Y)$.

Since $j_Y(Y)$ is closed in $Y^{**}$, it suffices to prove that $U(g)(u) \in j_Y(Y)$ for all $u \in E$ and $g \in \mathrm{Lip}_0(X, Y)$. By Corollary G.10 in the appendix we need only show that the restriction of $U(g)(u)$ to the unit ball of $B_{Y^*}$ is weak* continuous. Let $(y_\alpha^*)_{\alpha \in J}$ be a net in $B_{Y^*}$ that converges to $y^* \in B_{Y^*}$ in the weak* topology. We infer that $\lim_{\alpha \in J} y_\alpha^* = y^*$ uniformly on subsets of $Y$ that are compact with respect to the norm topology. Since $g$ is continuous and $K$ is a compact subset of $X$, $\lim_{\alpha \in J} y_\alpha^* \circ g = y^* \circ g$ uniformly on $K$. Appealing to (14.26), we obtain $\lim_{\alpha \in J} R(u)(y_\alpha^* \circ g) = R(u)(y \circ g)$, as desired.

Summarizing, we have obtained a linear operator $U\colon \mathrm{Lip}_0(X, Y) \to \mathcal{B}(X, Y)$ such that $\|U\| \le 1$ and

$$y^*(U(g)(u)) = R(u)(y^* \circ g), \quad g \in \mathrm{Lip}_0(X, Y),\ u \in X,\ y^* \in Y. \tag{14.27}$$

Let us conclude the proof of the theorem. Let $g \in \mathrm{Lip}(X, Y)$. We may and do assume that $g(0) = 0$. Applying (14.27) with $y^* = x^* \circ Q$, where $x^*$ is an arbitrary functional in $X^*$, we obtain

$$x^*(Q(U(g)(u))) = R(u)(x^* \circ Q \circ g) = R(u)(x^*) = x^*(u)$$

for every $u \in X$ and $x^* \in X^*$. Thus $S = U(g)$ satisfies $Q \circ S = I_X$. Finally, we notice that $\|S\| \le \|U\| \,\mathrm{Lip}(g) \le \mathrm{Lip}(g)$. □

**Corollary 14.3.4.** *Let $X$ be a separable Banach space and let $Z$ be a closed subspace of $X$. If there exists a Lipschitz lifting $f\colon X/Z \to X$ of the quotient map $q\colon X \to X/Z$, then there exists also a linear lifting $T\colon X/Z \to X$ with $\|T\| \le \mathrm{Lip}(f)$.*

### *14.3.1 Linearization of Lipschitz Retractions*

We have seen that the derivative $D_f(x_0)$ of a Lipschitz embedding $f\colon X \to Y$ at a point $x_0 \in X$ provides a linear embedding of $X$ into $Y$ (Theorem 14.2.15). It turns out that under certain assumptions the position of the image $D_f(x_0)(X)$ in $Y$ can be specified. In what follows we will say that a Lipschitz map $r\colon Y \to Z$ from a metric space $Y$ onto a subset $Z$ of $Y$ is a *Lipschitz retraction* if it is the identity on $Z$. When such a Lipschitz retraction exists, $Z$ is called a *Lipschitz retract* of $Y$.

Note that a Banach space $X$ is Lipschitz isomorphic to a Lipschitz retract of a Banach space $Y$ if and only if there exist Lipschitz maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f = I_X$, in which case $f$ is a Lipschitz isomorphism of $X$ onto $f(X)$ and $f \circ g$ is a Lipschitz retraction of $Y$ onto $f(X)$. In what follows, as is customary, it will be convenient to identify a Banach space $X$ with its canonical image inside its bidual $X^{**}$, and we will say that $X$ is complemented in $X^{**}$ to signify that there is $P\colon X^{**} \to X$ linear and continuous such that $P \circ j_X = I_X$, where $j_X$ is the canonical embedding of $X$ into $X^{**}$.

**Proposition 14.3.5 (Heinrich and Mankiewicz [124]).** *Let $f$ be a Lipschitz embedding of a Banach space $X$ into a Banach space $Y$. Assume that $f$ is Gâteaux differentiable at a point $x_0 \in X$. If $f(X)$ is a Lipschitz retract of $Y$ and $X$ is complemented in $X^{**}$, then there exists a Lipschitz retraction from $Y$ onto $D_f(x_0)(X)$.*

*Proof.* By a simple translation argument, without loss of generality we may assume that $x_0 = 0$ and $f(0) = 0$. We will write $D_f$ instead of $D_f(x_0)$. Let $g\colon Y \to X$ be a Lipschitz map such that $g \circ f = I_X$. For all $y_1, y_2 \in Y$ and all positive integers $n$ we then have

$$\|ng(n^{-1}y_1) - ng(n^{-1}y_2)\| \le \mathrm{Lip}(g)\|y_1 - y_2\|. \tag{14.28}$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. From (14.28) and the assumption $f(0) = 0$ it follows that the sequence $(ng(n^{-1}y))_{n=1}^{\infty}$ is norm-bounded for each $y \in Y$, hence weak* relatively compact in $X^{**}$. Using Banach–Alaoglu's theorem, we can define $\tilde{g}: Y \to X^{**}$ by

$$\tilde{g}(y) = \text{weak}^*\text{-}\lim_{\mathcal{U}} ng(n^{-1}y), \qquad y \in Y.$$

Since balls are closed in the weak* topology, (14.28) gives

$$\|\tilde{g}(y_2) - \tilde{g}(y_1)\| \leq \text{Lip}(g)\|y_1 - y_2\|, \quad y_1, y_2 \in Y,$$

i.e., $\tilde{g}$ is Lipschitz and $\text{Lip}(\tilde{g}) \leq \text{Lip}(g)$. We shall see next that $\tilde{g} \circ D_f = j_Y$.

If for a fixed $u \in X$ we let $y = D_f(u)$, then

$$\begin{aligned} \|ng(n^{-1}y) - u\| &= n\|ng(n^{-1}y) - g(f(n^{-1}u))\| \\ &\leq n\,\text{Lip}(g)\|n^{-1}y - f(n^{-1}u)\| \\ &= \text{Lip}(g)\|y - nf(n^{-1}u)\|. \end{aligned}$$

By definition, $\lim_n(y - nf(n^{-1}u)) = 0$. Using again that balls are weak* closed, we get $\|\tilde{g}(y) - j_Y(u)\| \leq \varepsilon$ for every $\varepsilon > 0$. Therefore $\tilde{g}(y) = j_Y(u)$, as desired.

Now, it is obvious that the map $h := P \circ \tilde{g}$ is Lipschitz with $h \circ D_f = I_X$, where $P$ is a linear projection from $X^{**}$ onto $X$. That is, $D_f(X)$ is a Lipschitz retract of $Y$. □

Next we want to move one step further and achieve the linear complementability of the subspace $D_f(x_0)(X)$ in $Y$. The following result of Lindenstrauss [191] will be our main tool to linearize certain Lipschitz retractions.

**Theorem 14.3.6.** *Let $Y$ be a separable infinite-dimensional Banach space and let $Z$ be a closed subspace of $Y$. If there is a Lipschitz retraction from $Y$ onto $Z$ and $Y$ is complemented in $Y^{**}$, then there is a linear projection from $Y$ onto $Z$.*

The main step in the proof of Theorem 14.3.6 will be the linearization of Lipschitz maps on finite-dimensional spaces contained in Lemma 14.3.7.

**Lemma 14.3.7.** *Suppose $E$ and $Z$ are Banach spaces with $E$ finite-dimensional. Let $f: E \to Z$ be a Lipschitz map and let $E_0$ be a subspace of $E$ such that $f|_{E_0}$ is linear. Then, there is $T: E \to Z^{**}$ linear such that $T|_{E_0} = j_Z \circ f|_{E_0}$ and $\|T\| \leq \text{Lip}(f)$, where $j_Z$ denotes the canonical embedding of $Z$ into its second dual $Z^{**}$.*

*Proof.* Pick $\varphi \in \mathcal{C}^{(\infty)}(E_0, \mathbb{R})$ such that $\varphi \geq 0$, $\int_{E_0} \varphi\, d\lambda = 1$, and $\varphi(-x) = \varphi(x)$ for all $x \in E_0$. We replace $f$ with the function

$$g(x) = \int_{E_0} f(x - \xi)\varphi(\xi)d\lambda(\xi), \qquad x \in E.$$

By Lemma 14.2.4, $g$ is Lipschitz on $E$ with $\mathrm{Lip}(g) \leq \mathrm{Lip}(f)$, and $\partial_u g: E \to Z$ is linear and continuous for every $u \in E_0$. Since, by hypothesis, $S := f|_{E_0}$ is linear, for $x \in E_0$ we have

$$g(x) = \int_{E_0} S(x - \xi)\varphi(\xi)\, d\lambda(\xi) = \int_{E_0} (S(x) - S(\xi))\varphi(\xi)\, d\lambda(\xi) = S(x).$$

We decompose algebraically $E = E_0 \oplus E_1$. Let $k = \dim(E_1)$. Pick a Lebesgue measure $\mu$ on $E_1$ and a nonnegative function $\psi \in \mathcal{C}^{(\infty)}(E_1, \mathbb{R})$ such that $\int_{E_1} \psi\, d\mu = 1$. For every $n \in \mathbb{N}$ consider

$$g_n(x) = 2^{nk} \int_{E_1} g(x - \xi)\psi(2^n \xi)\, d\mu(\xi), \qquad x \in E.$$

By Lemma 14.2.5, each $g_n$ is Lipschitz with $\mathrm{Lip}(g_n) \leq \mathrm{Lip}(g) \leq \mathrm{Lip}(f)$, $\partial_u g_n(x)$ exists for all $x \in E$ and all $u \in E_1$, and $\partial_u g_n: E \to Z$ is continuous. Taking derivatives under the integral sign gives

$$\partial_u g_n(x) = 2^{nk} \int_{E_1} \partial_u g(x - \xi)\psi(2^n \xi)\, d\mu(\xi), \qquad \forall x \in E,\ \forall u \in E_0.$$

Moreover, for every $u \in E_0$, $\partial_u g_n$ is continuous on $E$. Hence $g_n$ belongs to $\mathcal{C}^{(1)}(E, Z)$, and therefore it is differentiable at every $x \in E$, that is, there exists $D_{g_n}(x): E \to Z$ for all $x \in E$. Let $T_n := D_{g_n}(0)$. We have

$$T_n(u) = 2^{nk} \int_{E_1} \partial_u g(0 - \xi)\psi(2^n \xi)\, d\mu(\xi), \qquad \forall\, u \in E_0.$$

Since $\partial_u g$ is continuous, by Lemma 14.2.3, the limit $\lim_n T_n(u)$ exists and

$$\lim_{n\to\infty} T_n(u) = \partial_u g(0) = \lim_{t\to 0} \frac{g(0 + tu) - g(0)}{t} = \lim_{t\to 0} \frac{S(tu) - S(0)}{t} = S(u).$$

Note that

$$\|T_n(u)\| \leq \mathrm{Lip}(g_n)\|u\| \leq \mathrm{Lip}(f)\|u\|$$

for each $n \in \mathbb{N}$ and for $u \in E$. Hence the set $\{T_n(u)\}_{n=1}^{\infty}$ is weak* relatively compact in $Z^{**}$ for every $u \in E$. Thus we can define a linear operator $T: E \to Z^{**}$ by $T(u) = \text{weak*-}\lim_{\mathcal{U}} j_Z(T_n(u))$, where $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. We have $\|T\| \leq \mathrm{Lip}(f)$, and for $u \in E_0$,

$$j_Z \circ S(u) = w^*\text{-}\lim_{n\to\infty} j_Z(T_n(u)) = T(u).$$

□

*Proof of Proposition 14.3.6.* Let $r: Y \to Z$ be a Lipschitz retraction. Choose an increasing sequence $(E_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of $Y$ such that $\overline{\cup_n E_n} = Y$ and $\overline{\cup_n (E_n \cap Z)} = Z$. For each $n$ the restriction of $r$ to $E_n \cap Z$ is the identity map. Lemma 14.3.7 applied to $r|_{E_n}$ yields a linear map $T_n: E_n \to Z^{**}$ such that $T_n|_{E_n \cap Z} = j_Z|_{E_n \cap Z}$ and $\|T_n\| \le \mathrm{Lip}(r|_{E_n}) \le \mathrm{Lip}(r)$.

Notice that if $(x_n)_{n=1}^{\infty}$ and $(\tilde{x}_n)_{n=1}^{\infty}$ are sequences with $x_n, \tilde{x}_n$ in $E_n$ for all $n$, and $\lim_{n\to\infty}(x_n - \tilde{x}_n) = 0$, then $\lim_{n\to\infty}(T_n(x_n) - T_n(\tilde{x}_n)) = 0$. Moreover, if $(x_n)_{n=1}^{\infty}$ converges to some $x$, then $(T_n(x_n))_{n=1}^{\infty}$ is bounded and therefore weak* relatively compact (by Banach–Alaouglu's theorem). So for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we can safely define a bounded linear operator $T: Y \to Z^{**}$ by $T(x) = \text{weak*-}\lim_{\mathcal{U}} T_n(x_n)$, where $x_n \in E_n$ for each $n$ and $x = \lim_{n\to\infty} x_n$.

Given $z \in Z$, pick a sequence $(z_n)_{n=1}^{\infty}$ in $E_n \cap Z$ such that $z = \lim_{n\to\infty} z_n$. Clearly, $\lim_{n\to\infty} j_Z(z_n) = j_Z(z)$ in norm, so that $\lim_{\mathcal{U}} j_Z(z_n) = j_Z(z)$ in $Z^{**}$ with respect to the weak* topology. Then for $z \in Z$,

$$T(z) = w^*\text{-}\lim_{\mathcal{U}} T_n(z_n) = w^*\text{-}\lim_{\mathcal{U}} j_Z(z_n) = j_Z(z),$$

and $\|T\| \le \sup_n \|T_n\| \le \mathrm{Lip}(r)$.

To conclude the proof we consider $Q: Z^{**} \to Z$ such that $Q \circ j_Z = I_Z$, and the map $P = Q \circ T$ is the desired linear projection of $Y$ onto $Z$. □

### 14.3.2 *Banach Spaces Determined by Their Lipschitz Structure*

The question now is how to use the additional information that $f: X \to Y$ is a Lipschitz isomorphism. If we assume that $X$ and $Y$ are reflexive or, more generally, have (RNP), we can at least use Theorem 14.2.13 to find a point $x_0 \in X$ such that $f$ has Gâteaux derivative $S: X \to Y$ at $x_0$. Similarly, we can apply the same result to find a point $y_0 \in Y$ where $f^{-1}$ has Gâteaux derivative $T: Y \to X$ at $y_0$. If $X$ and $Y$ were finite-dimensional, we could arrange to have $y_0 = f(x_0)$ and deduce by the chain rule that $ST = I_Y$ and $TS = I_X$. But in infinite dimensions this is no longer possible. There are two problems. First, the notion of Gâteaux differentiability is not strong enough to give a chain rule (see Problem 14.4). The second is that there is no guarantee that we can ensure that $y_0 = f(x_0)$. The key is that if $\widetilde{\Omega_f}$ is the set of points in $X$ at which $f$ is not Gâteaux differentiable, then $\widetilde{\Omega_f}$ is Haar-null but $f(\widetilde{\Omega_f})$ need not be Haar-null in $Y$ (see [23, page 149]). Thus the concept of a Haar-null set is dependent on the linear structure and not just on the metric structure.

The effect of these problems is that we can obtain positive results for the Lipschitz isomorphism problem only by using some fancy footwork. We often require deep results from the linear theory to complete our arguments. In this section we continue exploiting the classic 1982 paper of Heinrich and Mankiewicz [124], which gave much of the groundwork for the present state of art of the theory.

**Theorem 14.3.8.** *Let $X$ and $Y$ be separable Banach spaces such that $Y$ has the Radon-Nikodym property and $X$ is complemented in $X^{**}$. If $X$ is Lipschitz isomorphic to a Lipschitz retract of $Y$, then $X$ is linearly isomorphic to a complemented subspace of $Y$.*

*Proof.* Let $f: X \to Y$ and $g: Y \to X$ be Lipschitz maps such that $g \circ f = I_X$. Theorem 14.2.13 ensures the existence of points in $X$ where $f$ is Gâteaux differentiable. Let $D_f: X \to Y$ be the Gâteaux derivative of $f$ at one of these points. Then $D_f$ is a linear isomorphism onto a closed subspace $Z = D_f(X)$ of $Y$, and by Proposition 14.3.5, $Z$ is a Lipschitz retract of $Y$. Now Theorem 14.3.6 ensures that $Z$ is linearly complemented in $Y$. □

**Corollary 14.3.9.** *Let $X$ and $Y$ be separable reflexive Banach spaces and suppose that $X$ and $Y$ are Lipschitz isomorphic. Then $X$ is isomorphic to a complemented subspace of $Y$ and $Y$ is isomorphic to a complemented subspace of $X$.*

Corollary 14.3.9 combined with Pełczyński's decomposition trick (see Section 2.2) yields our major positive result on Problem 14.3.1:

**Theorem 14.3.10 (Heinrich and Mankiewicz [124]).** *Let $X$ and $Y$ be separable reflexive Banach spaces. Assume that the pair $(X, Y)$ satisfies Pełczyński's decomposition scheme. If $X$ and $Y$ are Lipschitz isomorphic, then they are linearly isomorphic.*

Taking into account that the properties of separability and superreflexivity are preserved under Lipschitz isomorphisms (see Corollary 14.2.31; cf. Problem 14.9), we obtain the following corollary:

**Corollary 14.3.11.** *Let $1 < p < \infty$. If a Banach space $X$ is Lipschitz isomorphic to $\ell_p$ [respectively, $L_p$], then $X$ is linearly isomorphic to $\ell_p$ [respectively, $L_p$].*

Notice that we are able to get a positive answer to Problem 14.3.1 for only a very small class of separable Banach spaces. In particular, these methods do not apply to $\ell_1$ or $c_0$.

In the case of $\ell_1$ the techniques we have used *almost* work. The space $\ell_1$ has (RNP), and so if $Y$ is Lipschitz isomorphic to $\ell_1$, we can use Theorem 14.2.15 to deduce that $Y$ is linearly isomorphic to a subspace of $\ell_1$. Thus we have the following result:

**Theorem 14.3.12.** *If $X$ is Lipschitz isomorphic to $\ell_1$ and is a dual space, then $X$ is linearly isomorphic to $\ell_1$.*

*Proof.* The extra hypothesis on $X$ guarantees that it is complemented in its bidual. By Theorem 14.3.8 we obtain that $X$ is complemented in $\ell_1$. Since $\ell_1$ is a prime space, $X$ must be isomorphic to $\ell_1$. □

However, the following problem is open:

**Problem 14.3.13.** *If $X$ is Lipschitz isomorphic to $\ell_1$, is $X$ linearly isomorphic to $\ell_1$?*

In the case of $c_0$, the classical differentiation results are quite useless to deduce linear embeddings from Lipschitz embeddings, since this space does not have (RNP). However, using different techniques that rely on the Gorelik principle (see Section 14.6), Godefroy, Kalton, and Lancien showed in [106] that if a Banach space $X$ is Lipschitz isomorphic to a *subspace* of $c_0$, then $X$ is linearly isomorphic to a subspace of $c_0$. Combining this with deep results from linear theory, they proved that if a Banach space $X$ is Lipschitz isomorphic to $c_0$, then $X$ is linearly isomorphic to $c_0$.

## 14.4 Linear Inverses to Nonlinear Isometric Embeddings

The Mazur–Ulam theorem (Theorem 14.1.3) applies only to surjective isometries. What happens if we have an isometric embedding? In 1968 this question was addressed by Figiel, who proved the existence of a contractive linear left inverse to an isometric embedding [93]. The linearization argument we will use in the proof of Figiel's theorem is a basic differentiation result on real-valued convex functions defined on a finite-dimensional space. To make this section as fully self-contained as possible, we provide the generic smoothness results of convex functions on $\mathbb{R}^n$.

### *14.4.1 Derivatives of Convex Functions on Finite-Dimensional Spaces*

We shall devote this section to studying the differentiability of convex functions defined on a finite-dimensional space with an eye to its applications to the differentiability of the norm. Recall that a map $f: \mathcal{C} \to \mathbb{R}$, where $\mathcal{C}$ is a convex subset of a vector space $E$, is *convex* if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y),$$

for every $x$ and $y$ in $\mathcal{C}$ and every $\alpha \in [0,1]$. In the 1-dimensional case we have that every convex function $f: I \to \mathbb{R}$ on an interval $I = (a,b)$ of the real line is continuous. Moreover, the left and right derivatives of $f$ exist at every point of $I$ and satisfy

$$\frac{f(t)-f(s)}{t-s} \leq f'_-(t) \leq f'_+(t) \leq \frac{f(u)-f(t)}{u-t}, \quad a < s < t < u < b.$$

Consequently, such an $f$ is differentiable outside a countable set. A straightforward application of these basic 1-dimensional ideas to functions of several variables is that for every convex map $f: E \to \mathbb{R}$, the left and right directional derivatives of $f$ exist at every $x \in E$ for all $u \in E$. Moreover, for $t > 0$ we have

$$\frac{f(x) - f(x - tu)}{t} \leq \partial_u^- f(x) \leq \partial_u^+ f(x) \leq \frac{f(x + tu) - f(x)}{t}. \tag{14.29}$$

The following lemma is just another elementary differentiability property.

**Lemma 14.4.1.** *A convex function $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at a point $x$ if and only if*

$$\lim_{t \to 0^+} \frac{f(x+t) + f(x-t) - 2f(x)}{t} = 0.$$

**Lemma 14.4.2.** *Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then for all $x \in \mathbb{R}^n$ and all $t > 0$,*

$$\sup_{\|h\|_1 \leq t} |g(x+h) - g(x)| = \max_{\substack{1 \leq i \leq n \\ \epsilon = \pm 1}} \left(g(x + \epsilon t e_i) - g(x)\right).$$

*Proof.* Pick $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ and $t > 0$. Convexity shows that

$$\sup_{\|h\|_1 \leq t} g(x+h) - g(x) = \max_{\substack{1 \leq i \leq n \\ \epsilon = \pm 1}} \left(g(x + \epsilon t e_i) - g(x)\right).$$

Moreover, if $\mathcal{C}$ is a nonempty symmetric convex subset of $\mathbb{R}^n$ and $G\colon \mathcal{C} \to \mathbb{R}$ is a convex function such that $G(0) = 0$ and is bounded from above on $\mathcal{C}$, then

$$\sup_{h \in K} G(h) = \sup_{h \in K} |G(h)|.$$

Using this with $G(h) = g(x+h) - g(x)$, we obtain

$$\sup_{\|h\|_1 \leq t} |g(x+h) - g(x)| = \max_{\substack{1 \leq i \leq n \\ \epsilon = \pm 1}} \left(g(x + \epsilon t e_i) - g(x)\right).$$

□

When $E = \mathbb{R}^n$, it is useful to consider the directional derivatives of a map $f\colon \mathbb{R}^n \to \mathbb{R}$ at a point $x = (x_1, \ldots, x_n)$ in the direction of the vectors of the canonical basis $\{e_1, \ldots, e_n\}$,

$$\frac{\partial f}{\partial x_i}(x) = \partial_{e_i} f(x), \qquad i = 1, \ldots, n.$$

**Proposition 14.4.3.** *A convex function $f\colon \mathbb{R}^n \to \mathbb{R}$ is Fréchet differentiable at a point $x \in \mathbb{R}^n$ if and only if the $n$ partial derivatives of $f$ at $x$ exist and are finite. In particular, $f$ is Fréchet differentiable at $x$ if and only if it is Gâteaux differentiable at $x$.*

*Proof.* Suppose the $n$ partial derivatives of $f$ exist at $x \in \mathbb{R}^n$. Define the linear map $T: \mathbb{R}^n \to \mathbb{R}$ by

$$T(h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) h_i, \qquad h = (h_1, \ldots, h_n) \in \mathbb{R}^n.$$

Notice that the function $g: \mathbb{R}^n \to \mathbb{R}$ given by $g(y) = f(y) - T(y)$ is convex. By Lemma 14.4.2, if we put $\|h\|_1 = t$, we have

$$\begin{aligned}
\frac{1}{t}|f(x+h) - f(x) - T(h)| &= \frac{1}{t}|g(x+h) - g(x)| \\
&\leq \frac{1}{t} \max_{\substack{1 \leq i \leq n \\ \epsilon = \pm 1}} (g(x + \epsilon t e_i) - g(x)) \\
&= \max_{\substack{1 \leq i \leq n \\ \epsilon = \pm 1}} \left( \frac{f(x + \epsilon t e_i) - f(x)}{t} - \epsilon \frac{\partial f}{\partial x_i}(x) \right),
\end{aligned}$$

and this last quantity tends to 0 as $t \to 0^+$ by hypothesis. □

**Corollary 14.4.4.** *A convex function $f: E \to \mathbb{R}$ is Fréchet differentiable at a point $x \in E$ if and only if $\partial_u^- f(x) = \partial_u^+ f(x)$ for all $u \in E$, i.e., all the directional derivatives of $f$ at $x$ exist.*

**Proposition 14.4.5.** *Suppose $E$ is a finite-dimensional normed space. If $f: E \to \mathbb{R}$ is a convex function, then it is continuous on $E$ and Fréchet differentiable at every point of a dense subset of $E$.*

*Proof.* Without loss of generality we may and do assume that $E$ is $\mathbb{R}^n$ equipped with the norm $\|\cdot\|_1$. The continuity of a convex function $f: \mathbb{R}^n \to \mathbb{R}$ when $n > 1$ follows from Lemma 14.4.2 and the 1-dimensional case. For $k \in \mathbb{N}$, $1 \leq i \leq n$, and $t > 0$, let

$$V_{k,i} = \bigcup_{t>0} \left\{ x \in \mathbb{R}^n : \frac{f(x + te_i) + f(x - te_i) - 2f(x)}{t} < \frac{1}{k} \right\}.$$

The sets $V_{k,i}$ are open, since each is the union of open sets. By Lemma 14.4.1,

$$\Delta_i := \bigcap_{k \geq 1} V_{k,i} = \left\{ x \in \mathbb{R}^n : \frac{\partial f}{\partial x_i}(x) \text{ exists} \right\}.$$

By Proposition 14.4.3 the set of differentiability points of $f$ is precisely

$$\Omega_f = \bigcap_{1 \leq i \leq n} \Delta_i.$$

Since every convex function of one real variable is differentiable outside a countable set, the sets $\Delta_i$ are dense in $\mathbb{R}^n$. Since every set $\Delta_i$ is a countable intersection of open sets, from Baire's category theorem we infer that the set $\Omega_f$ is dense in $\mathbb{R}^n$. □

Since a norm $\|\cdot\|$ on a vector space $E$ is a convex function, if $E$ is finite-dimensional, then Proposition 14.4.5 tells us that the set of points $\Omega_{\|\cdot\|}$ where $\|\cdot\|\colon E\to\mathbb{R}$ is differentiable is dense in $E$. The definition of derivative of a norm is the expected one.

**Definition 14.4.6.** A norm $\|\cdot\|$ on a finite-dimensional vector space $E$ is said to be *differentiable* at a point $x\in E$ if the *directional derivative of the norm at $x$ in the direction of $u$*, given by the limit

$$\lim_{t\to 0}\frac{\|x+tu\|-\|x\|}{t} := D_{\|\cdot\|}(x)(u), \tag{14.30}$$

exists for all $u\in E$. In this case, the bounded linear functional $D_{\|\cdot\|}(x)\colon E\to\mathbb{R}$ is called the *derivative of the norm at $x$.*

**Lemma 14.4.7.** *Suppose $(E,\|\cdot\|)$ is a finite-dimensional normed space.*

*(i) If $x\in\Omega_{\|\cdot\|}$, then*

$$\|D_{\|\cdot\|}(x)\| = D_{\|\cdot\|}(x)\left(\frac{x}{\|x\|}\right) = 1.$$

*In other words, $D_{\|\cdot\|}(x)$ is a norm-one linear functional on $E$ that attains its norm at $x/\|x\|$.*

*(ii) If $x\in E$ and $(x_n)_{n=1}^\infty$ is a sequence in $\Omega_{\|\cdot\|}$ that converges to $x$, then*

$$\lim_{n\to\infty} D_{\|\cdot\|}(x_n)(x) = \|x\|. \tag{14.31}$$

*Proof.* (i) For $u\in E$ and $t>0$, by the reverse triangle inequality,

$$\left|\frac{\|x+tu\|-\|x\|}{t}\right| \le \frac{\|x+tu-x\|}{|t|} = \|u\|.$$

Making $t\to 0^+$ gives $|D_{\|\cdot\|}(x)(u)|\le\|u\|$, so that $\|D_{\|\cdot\|}(x)\|\le 1$.

If we plug $u=x/\|x\|$ in (14.30), we obtain

$$D_{\|\cdot\|}(x)\left(\frac{x}{\|x\|}\right) = \lim_{t\to 0^+}\frac{\left\|x+t\frac{x}{\|x\|}\right\|-\|x\|}{t} = \|x\|\lim_{t\to 0^+}\frac{\left(1+\frac{t}{\|x\|}\right)-1}{t} = 1.$$

(ii) Let $t>0$ and $n\in\mathbb{N}$. One the one hand, by the triangle inequality,

$$\frac{\|x_n+tx\|-\|x_n\|}{t} \le \|x\|.$$

On the other hand, also by the triangle inequality,

$$\begin{aligned}\frac{\|x_n + tx\| - \|x_n\|}{t} &\geq -\|x_n - x\| + \frac{\|x_n + tx_n\| - \|x_n\|}{t}\\ &= -\|x_n - x\| + \|x_n\|\\ &\geq \|x\| - 2\|x_n - x\|.\end{aligned}$$

Letting $t$ tend to 0, we get

$$\|x\| - 2\|x_n - x\| \leq D_{\|\cdot\|}(x_n)(x) \leq \|x\|.$$

From here we conclude the proof using the Squeeze theorem. □

**Lemma 14.4.8.** *A norm $\|\cdot\|$ on a finite-dimensional space $E$ is differentiable at a norm-one vector $x \in E$ if and only if there is a unique $x^*$ in $E^*$ such that $x^*(x) = \|x^*\| = 1$.*

*Proof.* Let $x$ be in the unit sphere $S_E$ of $E$. Consider $x^* \in E^*$ with $\|x^*\| = x^*(x) = 1$ (such an $x^*$ always exists as a consequence of the Hahn–Banach theorem; see Corollary E.4 in the appendix). For every $t > 0$ and $u \in E$,

$$1 + tx^*(u) = x^*(x + tu) \leq \|x + tu\|.$$

That is,

$$x^*(u) \leq \frac{\|x + tu\| - \|x\|}{t}.$$

Similarly,

$$x^*(u) \geq \frac{\|x - tu\| - \|x\|}{-t}.$$

Letting $t$ tend to 0 gives

$$D^-_{\|\cdot\|}(x)(u) \leq x^*(u) \leq D^+_{\|\cdot\|}(x)(u).$$

Thus, if the norm $\|\cdot\|$ is differentiable al $x$, we have $x^*(u) = D_{\|\cdot\|}(x)(u)$ for all $u \in E$, i.e., $x^* = D_{\|\cdot\|}(x)$.

Conversely, if $\|\cdot\|$ fails to be differentiable at a point $x \in E \setminus \{0\}$, by Corollary 14.4.4, there is a direction $u \in E \setminus \{0\}$ such that $D^-_{\|\cdot\|}(x)(u) < D^+_{\|\cdot\|}(x)(u)$. Obviously, $x$ and $u$ are linearly independent. Using (14.29) and the triangle inequality, we have

$$
\begin{aligned}
-\|u\| \le \frac{\|x+tu\|-\|x\|}{t} &\le D^{-}_{\|\cdot\|}(x)(u), \quad \text{if } \ t<0, \\
D^{+}_{\|\cdot\|}(x)(u) \le \frac{\|x+tu\|-\|x\|}{t} &\le \|u\|, \quad \text{if } \ t>0.
\end{aligned}
\tag{14.32}
$$

Pick $\alpha$ real such that $D^{-}_{\|\cdot\|}(x)(u) \le \alpha \le D^{+}_{\|\cdot\|}(x)(u)$ and define a linear map $T_\alpha\colon [x,u] \to \mathbb{R}$ on the 2-dimensional subspace of $E$ spanned by $x$ and $u$ by $T_\alpha(x) = \|x\|$ and $T_\alpha(u) = \alpha$. Let $t \in \mathbb{R}$. Equation (14.32) yields

$$T_\alpha(x+tu) = \|x\| + \alpha t \le \|x+tu\|,$$

$$|T_\alpha(u)| = |\alpha| \le \|u\|,$$

and

$$-\|x+tu\| \le \|x\| - |t|\|u\| \le \|x\| - |t||\alpha| \le \|x\| + t\alpha = T_\alpha(x+tu).$$

Consequently, $\|T_\alpha\| = 1$. Use the Hahn–Banach theorem to extend $T_\alpha$ to a norm-one functional on $E$. Since different $\alpha$'s lead to different functionals, we are done. □

The following result is an improvement of the *only if* part of Lemma 14.4.8.

**Lemma 14.4.9.** *Let $(E, \|\cdot\|)$ be a finite-dimensional normed space. Pick $x \in E$ with $\|x\| = 1$ a point of differentiability of the norm. Then $D_{\|\cdot\|}(x)$ is the only 1-Lipschitz map $\varphi\colon E \to \mathbb{R}$ such that $\varphi(\alpha x) = \alpha$ for all $\alpha \in \mathbb{R}$.*

*Proof.* By Lemma 14.4.7 (i), $D_{\|\cdot\|}(x)\colon E \to R$ is 1-Lipschitz and $D_{\|\cdot\|}(x)(\alpha x) = \alpha$ for all $\alpha \in \mathbb{R}$. Conversely, let $\varphi : E \to \mathbb{R}$ be a 1-Lipschitz map such that $\varphi(\alpha x) = \alpha$ for all $\alpha \in \mathbb{R}$. Pick $y \in E$. For all $\alpha \ne 0$, one has

$$1 = |\alpha\varphi(y) - \alpha\varphi\big((\varphi(y) + 1/\alpha)x\big)| \le \|x - \alpha(y - \varphi(y)x)\|.$$

Therefore the right-hand-side function attains its minimum at $\alpha = 0$. Differentiation gives $D_{\|\cdot\|}(x)(y - \varphi(y)x) = 0$ and thus $D_{\|\cdot\|}(x) = \varphi$. □

### *14.4.2 The Structure of into Isometries*

Mazur–Ulam's theorem states that every onto isometry between normed spaces mapping 0 to 0 is linear. In 1968, Figiel [93] described the pattern of into isometries between Banach spaces by showing that if $\Phi\colon X \to Y$ is an isometric embedding between Banach spaces such that $Y$ is the closed linear span of $\Phi(X)$ and $\Phi(0) = 0$, then there is a unique quotient map $T\colon Y \to X$, and $\Phi$ is a lifting of $T$. The linearization of Lipschitz liftings of quotient maps obtained in Theorem 14.3.3

has to be understood in this context too. Combining Theorem 14.3.3 with Figiel's theorem, we will deduce another important linearization result by Godefroy and Kalton [105] that establishes that the *separable* subspace structure of a Banach space is determined by its isometric subset structure.

**Theorem 14.4.10 (Figiel [93]).** *Let $\Phi$ be an isometry from a Banach space $X$ into a Banach space $Y$ such that $Y = [\Phi(X)]$ and with $\Phi(0) = 0$. Then there exists a unique linear (onto) operator $T\colon Y \to X$ with $\|T\| = 1$ and $T \circ \Phi = I_X$.*

*Proof.* We will break down the proof into several cases. First off, we consider the one-dimensional case. Let $\Phi\colon \mathbb{R} \to Y$ be an isometry such that $\Phi(0) = 0$. For each $k \in \mathbb{N}$ there exists $x_k^* \in Y^*$ of norm 1 such that

$$x_k^*(\Phi(k) - \Phi(-k)) = 2k.$$

It is easily seen that $x_k^*(\Phi(t)) = t$ for all $t \in [-k, k]$. By the Banach–Alaoglu theorem there exists $x^* \in Y^*$ of norm 1 such that $x^*(\Phi(t)) = t$ for all $t \in \mathbb{R}$, and this linear form $x^*$ does the job.

Take now $\Phi\colon X \to Y$ as in the hypothesis with $X$ finite-dimensional. Pick any $x \in X$ where the norm $\|\cdot\|$ is differentiable. By the one-dimensional case, there exists $f_x^* \in X^*$ of norm 1 such that $f_x^*(\Phi(tx)) = t$ for all $t \in \mathbb{R}$. Lemma 14.4.9 shows that $f_x^* \circ \Phi = D_{\|\cdot\|}(x)$.

Proposition 14.4.5 and (14.31) yield that for every $z \in X\backslash\{0\}$ there is $x' \in \Omega_{\|\cdot\|}$ such that $D_{\|\cdot\|}(x')(z) \neq 0$. It follows that we can find points $x_1, x_2, \ldots, x_n$ in $\Omega_{\|\cdot\|}$ such that the set of linear forms $(D_{\|\cdot\|}(x_i))_{i=1}^n$ is a basis of $X^*$. We denote by $(z_j)_{j=1}^n$ the dual basis in $X$, so that $D_{\|\cdot\|}(x_j)(z_j) = \delta_{i,j}$. For each $1 \leq i \leq n$, there exists $f_{x_i}^* \in Y^*$ such that $D_{\|\cdot\|}(x_i) = f_{x_i}^* \circ \Phi$. We define $T\colon Y \to X$ by

$$T(y) = \sum_{i=1}^{n} f_{x_i}^*(y) z_i.$$

The map $T$ is linear and continuous, and $T \circ \Phi = I_X$. The uniqueness of such $T$ is a consequence of the fact that $[\Phi(X)] = Y$. Moreover, for all $x' \in \Omega_{\|\cdot\|}$,

$$f_{x'}^* = D_{\|\cdot\|}(x') \circ T, \tag{14.33}$$

since these continuous linear forms coincide on the dense set $\langle\Phi(X)\rangle$. If we pick any $y \in Y$ and we apply (14.31) to $z = T(y)$, it follows from (14.33) that $\|z\| \leq \|y\|$ and thus $\|T\| = 1$.

Assume $X$ is infinite-dimensional and separable. In this case we can write $X = \overline{\cup_{k\geq 1} E_k}$, where $(E_k)_{k\geq 1}$ is an increasing sequence of finite-dimensional subspaces. Let $F_k = \langle\Phi(E_k)\rangle$. The proof in the finite-dimensional case yields a unique continuous linear map $T_k : F_k \to E_k$ such that $T_k(\Phi(x)) = x$ for all $x \in E_k$, and moreover $\|T_k\| = 1$. The uniqueness implies that we can consistently define

$T : \bigcup_{k\geq 1} F_k \to X$ by $T(y) = T_k(y)$ if $y \in F_k$, and $\|T\| = 1$, since $\|T_k\| = 1$ for all $k$. Finally, our assumption implies that $Y = \overline{\bigcup_{k\geq 1} F_k}$ and $T$ can be extended to $Y$, since it takes values in the complete space $X$.

A similar argument works for every Banach space $X$, which we write as the union of the directed set of its finite-dimensional subspaces. □

Mazur–Ulam's theorem is now an immediate consequence.

**Corollary 14.4.11.** *Every onto isometry $\Phi : X \to Y$ between Banach spaces such that $\Phi(0) = 0$ is linear.*

*Proof.* Theorem 14.4.10 applied to $\Phi$ shows that $\Phi = T^{-1}$ is a linear isometry. □

**Theorem 14.4.12 (Godefroy and Kalton [105]).** *Let $X$ be a separable Banach space. If there exists an isometry $\Phi$ from $X$ into a Banach space $Y$, then $Y$ contains a closed linear subspace that is linearly isometric to $X$.*

*Proof.* We may and do assume that $\Phi(0) = 0$ and that $[\Phi(X)] = Y$. By Theorem 14.4.10, there is a norm-one quotient map $Q : Y \to X$ such that $Q \circ \Phi = I_X$. We can therefore apply Theorem 14.3.3 with $g = \Phi$, and this shows the existence of $S: X \to Y$ with $\|S\| = 1$ and $Q \circ S = I_X$. It is now clear that $S$ is a linear isometry from $X$ into $Y$. □

*Remark 14.4.13.* (a) In this proof it should be observed that the space $S(X)$ is contractively complemented in $[\Phi(X)]$ by the projection $P = S \circ Q$. The existence of a nonlinear isometric injection (mapping 0 to 0) from $X$ to $Y$ is therefore a quite restrictive condition on the pair of Banach spaces $(X, Y)$. We also mention the article [78], where it is shown that there exists a metric compact set $K$ such that every Banach space containing an isometric copy of $K$ has a subspace that is (linearly) isometric to $\mathcal{C}[0, 1]$.

(b) As already happened with Theorem 14.3.3, the hypothesis of separability in Theorem 14.4.12 is needed again. Indeed, the result fails to hold, for instance, if $X$ is a nonseparable Hilbert space (see [105]).

## 14.5 Uniform Homeomorphisms Between Banach Spaces

We consider now the uniform homeomorphism problem for Banach spaces. An immediate consequence of Theorem 14.2.27 is that if $X$ and $Y$ are uniformly homeomorphic Banach spaces, then $X$ is crudely finitely representable in $Y$ and vice versa. Hence, in general, local properties of Banach spaces are preserved under uniform homeomorphism. For global properties (such as reflexivity) this is no longer true, as we will see below.

Note also that in full generality a uniform homeomorphism does not imply a linear isomorphism even for separable spaces! For instance, there exists a pair $(X, Y)$ of separable uniformly homeomorphic Banach spaces such that $X$ is reflexive and $Y$ contains a subspace that is linearly isomorphic to $L_1$ [269]:

**Theorem 14.5.1 (Ribe [269]).** *Let* $(p_n)_{n=1}^{\infty}$ *in* $(1, \infty)$ *be a strictly decreasing sequence such that* $\lim_n p_n = 1$. *Let* $X = \ell_2(L_{p_n})$ *denote the* $\ell_2$*-sum of the spaces* $L_{p_n}[0, 1]$. *Then the spaces* $X$ *and* $X \oplus L_1$ *are uniformly homeomorphic.*

Therefore, in contrast to superreflexivity (see Problem 14.12), reflexivity is not preserved under uniform homeomorphisms, and so this category is quite different from the Lipschitz category. Ribe's construction was slightly generalized by Aharoni and Lindenstrauss [3] to create two uniformly homeomorphic superreflexive Banach spaces, namely $X = \ell_q(\ell_{p_n})$ and $Y = X \oplus \ell_p$, where $1 < p < \infty$, $p_n \to p$, $q \neq p$, and $p_n \neq p$, which are not Lipschitz isomorphic. Subsequently, Johnson, Lindenstrauss, and Schechtman [141] produced a Banach space (a variant of a Tsirelson space) that is uniformly homeomorphic to exactly two nonisomorphic Banach spaces. More recently, Kalton [159] has shown that for every $1 < p \neq 2 < \infty$ there are two uniformly homeomorphic subspaces of $\ell_p$ that are not linearly isomorphic, and that similarly, $c_0$ has two uniformly homeomorphic subspaces that are not isomorphic.

Another general question related to uniform homeomorphisms is whether the uniform structure of a Banach space is determined by the structure of its discrete subsets. But a striking recent construction by Kalton in [158] shows that there exist two separable Banach spaces that are net equivalent but not uniformly homeomorphic.

Let us now turn to positive results. The most natural place to start is in the study of the $\ell_p$ and $L_p$ spaces. From Theorem 14.2.27 (or Corollary 14.2.29) we deduce that a Banach space that is uniformly homeomorphic to $\ell_2$ is linearly isomorphic to $\ell_2$ (because separability is preserved by uniform homeomorphisms). This is one of the earliest results of the subject, originally due to Enflo (with a different proof) [86]. Note, however, that the assumption of being uniformly homeomorphic to a Hilbert space cannot be relaxed to the mere existence of a uniform *embedding* into a Hilbert space. To see this, take, for example, $\ell_1$, which does not linearly or coarsely Lipschitz embed in $\ell_2$ (by Corollary 14.2.29) but does uniformly embed in $\ell_2$ (see [23, Section 8.2]).

The case $\max\{p, q\} > 2$ in the following theorem was proved in 1964 by Lindenstrauss [191], while the case $\max\{p, q\} \leq 2$ was done in 1969 by Enflo [84]. With the benefit of hindsight both cases will follow easily from Proposition 14.1.21 and Corollary 14.2.29. However, the techniques originally used to prove Theorem 14.5.2 are still of considerable importance.

**Theorem 14.5.2 (Enflo–Lindenstrauss).** *Let* $1 \leq p < \infty$. *If the spaces* $L_p(\mu_1)$ *and* $L_q(\mu_2)$ *are uniformly homeomorphic, then either they are of the same finite dimension or* $p = q$.

*Proof.* Suppose $1 \leq p < q < \infty$ and that both $L_p(\mu_1)$ and $L_q(\mu_2)$ are of infinite dimension. If $L_p(\mu_1)$ and $L_q(\mu_2)$ are uniformly homeomorphic, by Proposition 14.1.21, $L_p(\mu_1)$ coarse Lipschitz embeds into $L_q(\mu_2)$ and vice versa. Using Corollary 14.2.29, we infer that $L_p(\mu_1)$ and $L_q(\mu_2)$ have the same type and cotype, which happens only when $p = q$. □

After this, the key question is whether $\ell_p$ and $L_p$ are uniformly homeomorphic when $1 \leq p < \infty$ and $p \neq 2$. This was also established piecemeal:

**Theorem 14.5.3 (Bourgain–Enflo–Gorelik).** *For $1 \leq p < \infty$ with $p \neq 2$ the spaces $\ell_p$ and $L_p$ are not uniformly homeomorphic.*

Here the case $p = 1$ is due to Enflo in the 1970s (unpublished; see a proof in [23, Theorem 10.13]); the case $1 < p < 2$ was established in 1987 by Bourgain [29], and the case $2 < p < \infty$ was not settled until 1994 by Gorelik [112].

### 14.5.1 The Coarse Lipschitz and Uniform Structures of $\ell_p$

In 1996, Johnson, Lindenstrauss, and Schechtman [141] achieved a major breakthrough in the nonlinear theory by showing that if $1 < p \neq 2 < \infty$, the spaces $\ell_p$ have a unique uniform structure. Recall that if $p = 1$, we do not even know whether the result holds for Lipschitz isomorphism (Problem 14.3.13). The original proof from [141] relied on the Gorelik principle, which we will present in Section 14.6. We are not going to follow that approach here. The key step in the proof is to show that a Banach space that is uniformly homeomorphic to $\ell_p$, $1 < p \neq 2 < \infty$, does not contain an isomorphic copy of $\ell_2$. Observe that if a Banach space $X$ is uniformly homeomorphic to $\ell_p$, then $X$ coarse Lipschitz embeds into $\ell_p$, and so to see that $\ell_2$ does not embed linearly into $X$, it suffices to prove that $\ell_2$ does not coarse Lipschitz embed into $\ell_p$.

This leads us to the general problem of what can be said about coarse Lipschitz maps $f: \ell_p \to \ell_q$ when $1 \leq p \neq q < \infty$. At the very least we want to conclude that $f$ cannot be a coarse Lipschitz embedding. Recall from the linear theory that the cases $q < p$ and $q > p$ are quite different. If $q < p$, then every bounded linear operator $T: \ell_p \to \ell_q$ is compact (Pitt's Theorem 2.1.4), while if $q > p$, we can deduce only that $T$ is strictly singular (Theorem 2.1.9). This difference persists in the analysis of nonlinear maps. We will discuss a few techniques that can be used here.

First we discuss the *approximate midpoint method*. This geometric tool is a classical technique in nonlinear theory that was developed by Enflo to show that $\ell_1$ and $L_1$ are not uniformly homeomorphic. The basic idea is simple and goes back to the original proof of the Mazur–Ulam theorem. If $X$ is a metric space, and $x, y \in X$, then a *metric midpoint* of $x$ and $y$ in $X$ is a point $z$ such that

$$d(z, x) = d(z, y) = \frac{1}{2}d(x, y).$$

If $f: X \to Y$ is Lipschitz and $x, y \in X$ are such that $d(f(x), f(y)) = \mathrm{Lip}(f)d(x, y)$, then $f$ maps metric midpoints of $x$ and $y$ to metric midpoints of $f(x)$ and $f(y)$. In general such points $x, y$ may not exist, so we resort to the definition of approximate midpoints.

**Definition 14.5.4.** Let $X$ be a metric space. Given two points $x, y \in X$ and $\delta > 0$, the *set of approximate metric midpoints* between $x$ and $y$ with error $\delta$ is the set

$$\mathrm{Mid}(x, y, \delta) = \left\{ z \in X : \max\{d(x, z), d(y, z)\} \leq (1 + \delta)\frac{d(x, y)}{2} \right\}.$$

The following result was formulated in [163] (see also [23, Lemma 10.11]). Roughly speaking, this proposition says that pairs of points that *stretch* a coarse Lipschitz map $f$ (in the sense that $f$ almost attains its coarse Lipschitz constant there) are such that their approximate metric midpoints are sent into approximate metric midpoints of their images.

**Proposition 14.5.5.** *Let $X$ and $Y$ be Banach spaces, and let $f: X \to Y$ be a coarse Lipschitz map. If $\mathrm{Lip}_\infty(f) > 0$, then for every $\theta > 0$, $\epsilon > 0$, and $0 < \delta < 1$, there exist $x, y \in X$ with $\|x - y\| > \theta$ and*

$$f(\mathrm{Mid}(x, y, \delta)) \subseteq \mathrm{Mid}(f(x), f(y), (1 + \epsilon)\delta).$$

*Proof.* Suppose $\theta, \delta, \epsilon$ are given. For $\eta > 0$ as small as we wish we pick $\theta' > \theta$ such that $\mathrm{Lip}_{\theta'}(f) < (1 + \eta)\,\mathrm{Lip}_\infty(f)$. Then we choose $x, y \in X$ such that $\|x - y\| \geq 2\theta'(1 - \delta)^{-1}$ and

$$\|f(x) - f(y)\| \geq \frac{1}{1 + \eta}\mathrm{Lip}_\infty(f)\|x - y\| \geq \frac{1}{(1 + \eta)^2}\mathrm{Lip}_{\theta'}(f)\|x - y\|.$$

Let $u \in \mathrm{Mid}(x, y, \delta))$. Then $\|y - u\| \geq \frac{1-\delta}{2}\|x - y\| \geq \theta'$, and so

$$\begin{aligned} \|f(y) - f(u)\| &\leq \mathrm{Lip}_{\theta'}(f)\|y - u\| \\ &\leq \mathrm{Lip}_{\theta'}(f)\frac{1 + \delta}{2}\|x - y\| \\ &\leq (1 + \eta)^2\frac{1 + \delta}{2}\|f(x) - f(y)\|. \end{aligned}$$

The same estimate holds for $\|f(x) - f(u)\|$. By appropriate choice of $\eta$ we obtain the proposition. □

Now the key fact is that we have some fairly precise information about the size of approximate midpoint sets in $\ell_p$. This is done in our next lemma, which can be found in [163], although we follow G. Lancien's adaptation from [185].

**Lemma 14.5.6.** *Suppose $1 \leq p < \infty$. Let $(e_i)_{i=1}^\infty$ be the canonical basis of $\ell_p$, and for $N \in \mathbb{N}$, denote by $E_N$ the closed linear span of $\{e_i : i > N\}$. Let $x, y \in \ell_p$, $\delta \in (0, 1)$, $u = \frac{x+y}{2}$, and $v = \frac{x-y}{2}$. Then:*

*(i) There exists $N \in \mathbb{N}$ such that $u + \delta^{1/p}\|v\|B_{E_N} \subset \mathrm{Mid}(x, y, \delta)$.*
*(ii) There is a compact subset $K$ of $\ell_p$ such that $\mathrm{Mid}(x, y, \delta) \subset K + 2\delta^{1/p}\|v\|B_{\ell_p}$.*

*Proof.* Given $0 < \eta < 1$, pick $N \in \mathbb{N}$ such that $\sum_{i=1}^{N} |v_i|^p \geq (1 - \eta^p)\|v\|_p^p$.

(i) We may clearly assume that $p > 1$. Let now $z \in E_N$ be that $\|z\|^p \leq \delta\|v\|^p$. Then, by choosing $\eta$ small enough,

$$\|x - (u + z)\|^p = \|v - z\|^p \leq \|v\|^p + (\|z\| + \eta\|v\|)^p \leq (1 + \delta)^p\|v\|^p.$$

The computation is the same for $\|y-(u+z)\| = \|v+z\|$. Hence $u+z \in \mathrm{Mid}(x, y, \delta)$.

(ii) Assume $u + z \in \mathrm{Mid}(x, y, \delta)$. Let us write $z = z^{(1)} + z^{(2)}$ with $z^{(1)} \in [e_i \colon 1 \leq i \leq N]$ and $z^{(2)} \in E_N$. Since $\|v - z\|, \|v + z\| \leq (1 + \delta)\|v\|$, by convexity we obtain that $\|z^{(1)}\| \leq \|z\| \leq (1 + \delta)\|v\|$. Therefore, $u + z^{(1)}$ belongs to the compact set $K = u + (1 + \delta)\|v\|B_{F_n}$. Convexity yields also that

$$\max\{|v_i|^p, |z_i|^p\} \leq \frac{1}{2}\left(|v_i - z_i|^p + |v_i + z_i|^p\right), \qquad \forall\, i \geq 1.$$

Summing over all $i$'s gives

$$(1 - \eta^p)\|v\|^p + \|z^{(2)}\|^p \leq \frac{1}{2}\left(\|v - z\|^p + \|v + z\|^p\right).$$

Therefore, if $\eta$ is chosen small enough,

$$\|z^{(2)}\| \leq \big((1 + \delta)^p - (1 - \eta^p)\big)\|v\|^p \leq 2^p\delta\|v\|^p.$$

□

Combining Proposition 14.5.5 and Lemma 14.5.6, we obtain the following:

**Proposition 14.5.7.** *Let $1 \leq p < q < \infty$ and suppose that $f \colon \ell_q \to \ell_p$ is a coarse Lipschitz map. Then for every $t > 0$ and $\epsilon > 0$ there exist $u \in \ell_q$, $\tau > t$, $N \in \mathbb{N}$, and a compact subset $K$ of $\ell_p$ such that*

$$f(u + \tau B_{E_N}) \subset K + \epsilon\tau B_{\ell_p},$$

*where $E_N = [e_i \colon i > N]$.*

*Proof.* We assume that $\mathrm{Lip}_\infty(f) > 0$, since if $\mathrm{Lip}_\infty(f) = 0$, the conclusion is clear. We choose a small $\delta > 0$ (to be specified later). Then we pick $\theta$ large enough (also to be detailed later) that $\mathrm{Lip}_\theta(f) \leq 2\,\mathrm{Lip}_\infty(f)$. By Proposition 14.5.5 there are $x, y \in \ell_q$ with $\|x - y\|_q \geq \theta$ such that $f(\mathrm{Mid}(x, y, \delta)) \subset \mathrm{Mid}(f(x), f(y), 2\delta)$. Let $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$, and $\tau = \delta^{1/q}\|v\|$. By Lemma 14.5.6 there is $N \in \mathbb{N}$ such that $u + \tau B_{E_N} \subset \mathrm{Mid}(x, y, \delta)$, and there exists a compact subset $K$ of $\ell_p$ such that

$$\mathrm{Mid}(f(x), f(y), 2\delta) \subset K + (2\delta)^{1/p}\|f(x) - f(y)\|B_{\ell_p}.$$

But if $\delta$ is chosen small enough, we will have

$$\begin{aligned}(2\delta)^{1/p}\|f(x)-f(y)\|_p &\le 2\,\mathrm{Lip}_\infty(f)(2\delta)^{1/p}\|x-y\|_q\\ &= 4\,\mathrm{Lip}_\infty(f)2^{1/p}\delta^{1/p-1/q}\tau\\ &\le \epsilon\tau.\end{aligned}$$

An appropriate choice of a large $\theta$ will ensure that $\tau > \frac{1}{2}\delta^{1/q}\theta > t$, as wished. □

**Corollary 14.5.8.** *If $1 \le p < q < \infty$, $\ell_q$ does not coarse Lipschitz embed into $\ell_p$.*

*Proof.* Let $f\colon \ell_q \to \ell_p$ be a coarse Lipschitz map. With the notation of the previous proposition, we can find a sequence $(u_n)_{n=1}^\infty$ in $u + \tau B_{E_N}$ such that $\|u_n - u_m\|_q \ge \tau$ for $n \ne m$. Then $f(u_n) = k_n + \epsilon\tau v_n$ with $k_n \in K$ and $v_n \in B_{\ell_p}$. Since $K$ is compact, by extracting a subsequence we may assume that $\|f(u_n) - f(u_m)\|_p \le 3\epsilon\tau$. Since $\epsilon$ can be chosen arbitrarily small and $\tau$ arbitrarily large, we infer that $f$ cannot be a coarse Lipschitz embedding. □

Proving that $\ell_q$ does not coarse Lipschitz embed into $\ell_p$ when $1 \le q < p < \infty$ demands something different. We follow an approach from [185] based on recent work by Kalton and Randrianarivony [163] that requires the introduction of special metric spaces.

Let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$, and let $k \in \mathbb{N}$. Put

$$G_k(\mathbb{M}) = \{\overline{n} = (n_1, \dots, n_k)\colon \{n_i\}_{i=1}^k \subset \mathbb{M} \text{ and } n_1 < n_2 < \cdots < n_k\}.$$

Then we equip $G_k(\mathbb{M})$ with the (Hamming) distance

$$d(\overline{n}, \overline{m}) = d((n_1, \dots, n_k), (m_1, \dots, m_k)) = |\{j\colon n_j \ne m_j\}|.$$

We note that the diameter of this metric space, $\operatorname{diam} G_k(\mathbb{M})$, is $k$.

The key result is an estimate of the minimal distortion of any Lipschitz embedding of $(G_k(\mathbb{M}), d)$ in an $\ell_p$-like Banach space.

**Theorem 14.5.9 (Kalton and Randrianarivony [163]).** *Suppose $1 < p < \infty$. Let $Y$ be a reflexive Banach space with the property that if $y \in Y$ and $(y_n)_{n=1}^\infty$ is a weakly null sequence in $Y$, we have*

$$\limsup \|y + y_n\|^p \le \|y\|^p + \limsup \|y_n\|^p. \tag{14.34}$$

*Then if $\mathbb{M}$ is an infinite subset of $\mathbb{N}$, $\epsilon > 0$, and $f\colon G_k(\mathbb{M}) \to Y$ is a Lipschitz map, there exists an infinite subset $\mathbb{M}'$ of $\mathbb{M}$ such that*

$$\operatorname{diam} f(G_k(\mathbb{M}')) \le 2k^{1/p}\,\mathrm{Lip}(f) + \epsilon.$$

*Proof.* We prove by induction on $k$ the following statement $(P_k)$: for every Lipschitz map $f\colon G_k(\mathbb{M}) \to Y$ and $\epsilon > 0$ there exist an infinite subset $\mathbb{M}'$ of $\mathbb{M}$ and $u \in Y$ such that $\|f(\overline{n}) - u\| \leq \mathrm{Lip}(f)k^{1/p} + \epsilon$ for all $\overline{n} \in G_k(\mathbb{M}')$.

Assume $k = 1$. By weak compactness, there exist an infinite subset $\mathbb{M}_0$ of $\mathbb{M}$ and a point $u \in Y$ such that $f(n) \to u$ weakly as $n \to \infty$ through the set $\mathbb{M}_0$. It follows that

$$\|u - f(n)\| \leq \limsup_{m \in \mathbb{M}_0} \|f(m) - f(n)\| \leq \mathrm{Lip}(f), \qquad \forall n \in \mathbb{M}_0.$$

We then obtain $(P_1)$ by taking a further subset $\mathbb{M}'$ of $\mathbb{M}_0$.

Assume that $(P_{k-1})$ holds. Let $f\colon G_k(\mathbb{M}) \to Y$ be a Lipschitz map, and let $\epsilon > 0$. Using again weak compactness, we can find an infinite subset $\mathbb{M}_0$ of $\mathbb{M}$ such that $\text{weak-lim}_{n_k \in \mathbb{M}_0} f(\overline{n}, n_k) = g(\overline{n}) \in Y$ for all $\overline{n} \in G_{k-1}(\mathbb{M}_0)$. Clearly, the map $g\colon G_{k-1}(\mathbb{M}_0) \to Y$ satisfies $\mathrm{Lip}(g) \leq \mathrm{Lip}(f)$. Let $\eta > 0$. By the induction hypothesis we can find an infinite subset $\mathbb{M}_1$ of $\mathbb{M}_0$ and $u \in Y$ such that

$$\|g(\overline{n}) - u\| \leq \mathrm{Lip}(f)(k-1)^{1/p} + \eta$$

for all $\overline{n} \in G_{k-1}(\mathbb{M}_1)$. Now,

$$\begin{aligned}\limsup_{n_k \in \mathbb{M}_1} \|u - f(\overline{n}, n_k)\|^p &\leq \|u - g(\overline{n})\|^p + \limsup_{n_k \in \mathbb{M}_1} \|g(\overline{n}) - f(\overline{n}, n_k)\|^p \\ &\leq (\mathrm{Lip}(f)(k-1)^{1/p} + \eta)^p + \mathrm{Lip}(f)^p.\end{aligned}$$

It follows that if we pick $\eta$ small enough, then

$$\limsup_{n_k \in \mathbb{M}_1} \|u - f(\overline{n}, n_k)\| \leq \mathrm{Lip}(f)k^{1/p} + \frac{\epsilon}{2}.$$

Finally we can use Ramsey's theorem (Theorem 11.1.1) to obtain an infinite subset $\mathbb{M}' \subseteq \mathbb{M}_1$ such that

$$\Big|\|u - f(\overline{n})\| - \|u - f(\overline{m})\|\Big| \leq \frac{\epsilon}{2}, \qquad \forall\ \overline{n}, \overline{m} \in G_k(\mathbb{M}').$$

This conclude the inductive proof of $(P_k)$. □

This theorem can be regarded as an asymptotic smoothness condition; it has generalizations to spaces with asymptotic smoothness (see [163]). The assumption that $Y$ is reflexive is important. Note that the nonreflexive space $c_0$ satisfies the condition (14.34) for every $p$ (and indeed for $p = \infty$), but every separable metric space can be Lipschitz embedded into $c_0$ by Aharoni's theorem (Theorem 14.2.25).

**Corollary 14.5.10.** *If $1 \leq q < p < \infty$, $\ell_q$ does not coarse Lipschitz embed into $\ell_p$.*

*Proof.* Suppose that $\ell_q$ coarse Lipschitz embeds into $\ell_p$. Then, using homogeneity, there exist $f\colon \ell_q \to \ell_p$ and $B \geq 1$ such that

$$\|x-y\|_q \leq \|f(x)-f(y)\|_p \leq B\|x-y\|_q \tag{14.35}$$

for all $x, y \in \ell_q$ with $\|x-y\|_q \geq 1$. Let $(e_n)_{n=1}^{\infty}$ be the canonical $\ell_q$-basis and consider the map $\varphi\colon G_k(\mathbb{N}) \to \ell_q$ defined by $\varphi(\overline{n}) = e_{n_1} + \cdots + e_{n_k}$ for $\overline{n} = (n_1, \ldots, n_k)$. It is clear that $\mathrm{Lip}(\varphi) \leq 2$. Moreover, $\|\varphi(\overline{n}) - \varphi(\overline{m})\|_q \geq 1$ whenever $\overline{n} \neq \overline{m}$, so that $\mathrm{Lip}(f \circ \varphi) \leq 2B$. By Theorem 14.5.9 there is an infinite subset $\mathbb{M} \subseteq \mathbb{N}$ with $\mathrm{diam}(f \circ \varphi)(G_k(\mathbb{M})) \leq 6Bk^{1/p}$. But $\mathrm{diam}\,\varphi(G_k(\mathbb{M})) = (2k)^{1/q}$, contradicting (14.35) for $k$ large enough. □

We are now ready to put everything in place and prove our main result of this section.

**Theorem 14.5.11 (Johnson et al. [141]).** *Suppose a Banach space $X$ is uniformly homeomorphic to $\ell_p$ for some $1 < p < \infty$ with $p \neq 2$. Then $X$ is linearly isomorphic to $\ell_p$.*

*Proof.* Fix $1 < p \neq 2 < \infty$. The starting point of the proof amalgamates some of the most important results of this chapter to make sure that if $X$ is uniformly homeomorphic to $\ell_p$, it must be complemented in $L_p$.

Indeed, combining Theorem 14.1.24, Corollary 14.1.22, and Remark J.8 shows that given any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ there exist a measure space $(\Omega, \Sigma, \mu)$ and a Lipschitz isomorphism $g\colon X_{\mathcal{U}} \to L_p(\Omega, \Sigma, \mu)$. Let $\iota_X\colon X \to X_{\mathcal{U}}$ be the natural inclusion given by $x \mapsto (x)_{\mathcal{U}}$. The space $X$ is reflexive (in fact, it is superreflexive by Corollary 14.2.31); hence it is complemented in $X_{\mathcal{U}}$ (see J.7 in the appendix). Let $Q\colon X_{\mathcal{U}} \to X$ be bounded and linear with $Q \circ \iota_X = I_X$. Since $X$ is separable, there is a measure space $(\Omega', \Sigma', \mu')$ with $\Omega' \subseteq \Omega$, $\Sigma' \subseteq \Sigma$, and $\mu' = \mu|_{\Sigma'}$ such that $g(\iota_X(X)) \subseteq L_p(\Omega', \Sigma', \mu') \subseteq L_p(\Omega, \Sigma, \mu)$ and $L_p(\mu')$ is separable. The maps $f = g \circ \iota_X\colon X \to L_p(\mu')$ and $h = Q \circ g^{-1}|_{L_p(\mu')}\colon L_p(\mu') \to X$ are both Lipschitz and satisfy $h \circ f = I_X$. In other words, $X$ is Lipschitz isomorphic to a Lipschitz retract of $L_p(\mu')$. By Theorem 14.3.8, $X$ is linearly isomorphic to a complemented subspace of $L_p(\mu')$. To reach the initial statement of this proof one just needs to recall that $L_p(\mu')$ is isomorphic to either $L_p$ or $\ell_p$.

It then follows, by appealing to the linear theory and a theorem of Johnson and Odell [139] that asserts that every infinite-dimensional complemented subspace of $L_p$ $(1 < p \neq 2 < \infty)$ that does not contain an isomorphic copy of $\ell_2$ is isomorphic to $\ell_p$, that all we need to do to complete the proof is to show that $X$ cannot contain an isomorphic copy of $\ell_2$. Since $X$ coarse Lipschitz embeds into $\ell_p$ (using Proposition 14.1.21), we have only to show that there is no coarse Lipschitz embedding of $\ell_2$ into $\ell_p$ when $p \neq 2$. But this is just a particular case of Corollary 14.5.8 in the case that $p < 2$ and of Corollary 14.5.10 when $p > 2$. □

## 14.6 Lipschitz Invariance of Asymptotic Smoothness

This section illustrates the idea that asymptotic properties of Banach spaces are good candidates for being invariant under Lipschitz isomorphisms or uniform homeomorphisms. This idea is already present in the work of Johnson, Lindenstrauss, and Schechtman [141], and it was confirmed later on, for instance in the work of Godefroy, Kalton, and Lancien [106, 107], in Kalton's articles [157–160], and in the paper [21] by Baudier, Kalton, and Lancien. We will restrict ourselves to the invariance under Lipschitz isomorphisms of the *modulus of asymptotic uniform smoothness*, introduced by Milman [221]. Let us first define this.

**Definition 14.6.1.** Let $X$ be a Banach space. If $\|x\| = 1$, $\tau > 0$, and $Y$ is a closed finite-codimensional subspace of $X$, we put

$$\overline{\rho}(\tau, x, Y) = \sup_{y \in S_Y} \|x + \tau y\| - 1,$$

where $S_Y$ denotes the unit sphere of $Y$. Then we let

$$\overline{\rho}(\tau, x) = \inf_{Y} \overline{\rho}(\tau, x, Y),$$

where the infimum is taken over all closed finite-codimensional subspaces. Finally, we put

$$\overline{\rho}(\tau) = \sup_{x \in S_X} \overline{\rho}(\tau, x).$$

This function $\overline{\rho}$ (or $\overline{\rho}_X$ if the space $X$ needs to be specified) is called the *modulus of asymptotic uniform smoothness* of $X$. A Banach space $X$ is said to be *asymptotically uniformly smooth* if

$$\lim_{\tau \to 0} \overline{\rho}(\tau)/\tau = 0.$$

*Example 14.6.2.* (a) If $X = \ell_p$ equipped with its natural norm, then

$$\overline{\rho}_{\ell_p}(\tau) = (1 + \tau^p)^{1/p} - 1, \quad \tau > 0,$$

and so $\ell_p$ is asymptotically uniformly smooth if $1 < p < \infty$. The natural norm actually has an optimal (i.e., minimal) modulus of asymptotic uniform smoothness among all equivalent norms.

(b) The space $X = c_0$ is also asymptotically uniformly smooth, and $\overline{\rho}_{c_0}(\tau) = 0$ for all $\tau \in (0, 1]$. In fact, the existence of a norm with this property characterizes the subspaces of $c_0$ (see [106]).

The following lemma provides a practical way to compute the modulus $\overline{\rho}$.

**Lemma 14.6.3.** *Let $X$ be a Banach space with separable dual. For $\tau \in (0,1]$ and $x \in S_X$ let*

$$\eta(\tau,x) = \sup \left\{ \limsup_{n\to\infty} \|x + x_n\| - 1 \right\},$$

*where the supremum is taken over all sequences $(x_n)_{n=1}^{\infty}$ weakly convergent to 0 such that $\|x_n\| \le \tau$ for all $n$. Then for every $\tau \in (0,1]$,*

$$\eta(\tau,x) = \overline{\rho}(\tau,x) \quad \text{and} \quad \eta(\tau) := \sup_{x\in S_X} \eta(\tau,x) = \overline{\rho}(\tau).$$

*Proof.* Suppose $(x_n)_{n=1}^{\infty}$ converges weakly to 0 and $\|x_n\| \le \tau$ for all $n$. Let $Y$ be a closed subspace of $X$ of finite codimension. The distance $d(x_n, Y)$ from $x_n$ to $Y$ tends to 0, so given $\epsilon > 0$, for $n$ large enough there exists $y_n \in Y$ with $\|x_n - y_n\| < \epsilon$. Then $\|y_n\| < \tau + \epsilon$ and

$$\|x + x_n\| - 1 \le \|x + y_n\| - 1 + \|x_n - y_n\| \le \overline{\rho}(\tau + \epsilon, x, Y) + \epsilon.$$

Since the subspace $Y$ of finite codimension is arbitrary, for $n$ large enough we have

$$\|x_n + x\| - 1 \le \overline{\rho}(\tau + \epsilon, x) + \epsilon,$$

and since $\epsilon > 0$ is arbitrary, it follows that $\eta(\tau,x) \le \overline{\rho}(\tau,x)$.

Conversely, we have $\eta(\tau,x) \ge \overline{\rho}(\tau,x)$. Indeed, let $(x_j^*)$ be a dense sequence in $X^*$, and let

$$Y_n = \bigcap_{j=0}^{n} \operatorname{Ker}(x_j^*).$$

Given $\epsilon > 0$, there is $x_n \in Y_n$ such that $\|x_n\| \le \tau$ with

$$\|x + x_n\| - 1 = \epsilon \ge \overline{\rho}(\tau, x, Y_n) \ge \overline{\rho}(\tau, x).$$

It is easy to check that the sequence $(x_n)$ is weakly null. Since $\epsilon > 0$ is arbitrary, it follows that $\eta(\tau,x) \ge \overline{\rho}(\tau,x)$. Hence these two quantities are equal, and we obtain the last assertion by taking the supremum over $x \in S_X$. □

The next result, called the *Gorelik principle*, is the crucial topological tool we need in order to transfer asymptotically uniformly smooth norms by means of Lipschitz maps.

**Lemma 14.6.4 (The Gorelik Principle).** *Let $E$ and $X$ be Banach spaces. Suppose $\varphi\colon E \to X$ is a homeomorphism whose inverse $\varphi^{-1}$ is Lipschitz. Let $b$ and $c$ be positive constants such that $c > \operatorname{Lip}(\varphi^{-1}) \cdot b$, and let $E_0$ be a closed subspace of finite codimension in $E$. Then there exists a compact subset $K$ of $X$ such that*

$$bB_X \subset K + \varphi(2cB_{E_0}).$$

*Proof.* Put $a = \mathrm{Lip}(\varphi^{-1}) \cdot b$. We first construct a compact subset $\tilde{K}$ of the ball $cB_E$ such that if $\psi : \tilde{K} \to E$ is a continuous map with $\|x - \psi(x)\| \le a$ for every $x \in \tilde{K}$, then $\psi(\tilde{K}) \cap E_0 \neq \emptyset$.

We denote by $Q : E \to E/E_0$ the canonical quotient map. We set $C = aB_{E/E_0}$. By the Bartle–Graves selection theorem (see [20]) there exists a continuous map $s : C \to cB_E$ such that $Q \circ s = I_C$. We define $F : C \to C$ by

$$F(t) = Q[s(t) - \psi(s(t))] = t - Q(\psi(s(t))).$$

By Schauder's fixed point theorem (see Appendix F), there exists $t_0 \in C$ such that $F(t_0) = t_0$. But this means that $Q(\psi(s(t_0))) = 0$; hence $\psi(s(t_0)) \in E_0$. Therefore the compact set $\tilde{K} = s(C)$ works.

Pick now any $x_0 \in bB_X$. We let $\psi(e) = \varphi^{-1}(x_0 + \varphi(e))$. We have

$$e - \psi(e) = \varphi^{-1}(\varphi(e)) - \varphi^{-1}(x_0 + \varphi(e)).$$

It follows that $\|e - \psi(e)\| \le a$. By the above, there is $k_0 \in \tilde{K}$ such that $\psi(k_0) \in E_0$. Set $\psi(k_0) = e_0$. Since $e_0 = (\psi(k_0) - k_0) + k_0$, the triangle inequality shows that $\|e_0\| \le 2c$. Moreover, $e_0 = \varphi^{-1}(x_0 + \varphi(k_0))$; hence $x_0 = \varphi(e_0) - \varphi(k_0)$. We deduce that the compact set $K = -\varphi(\tilde{K})$ yields the conclusion. □

*Remark 14.6.5.* Note that the proof of Lemma 14.6.4 also works under the assumption that $\varphi^{-1}$ is uniformly continuous, provided that

$$c > \sup\{\|\varphi^{-1}(x) - \varphi^{-1}(x')\| : \|x - x'\| \le b\}.$$

We are now ready to prove the transfer result. The assumption that $X$ and $Y$ have separable duals can actually be removed, and the result holds in full generality. Assuming it allows us to dispense with some technicalities.

**Theorem 14.6.6.** *Let $X$ and $Y$ be two Banach spaces with separable duals. We assume that $X$ is asymptotically uniformly smooth, and that there exists a Lipschitz isomorphism $f$ from $X$ onto $Y$. Then there is an asymptotically uniformly smooth equivalent norm on $Y$ whose modulus $\overline{\rho}_Y$ satisfies*

$$\overline{\rho}_Y(\tau/4\,\mathrm{dist}(f)) \le 2\overline{\rho}_X(\tau), \quad \forall \tau \in (0,1].$$

*Proof.* We may and do assume that $\mathrm{Lip}(f) = 1$ and put $\mathrm{Lip}(f^{-1}) = D$. We define a norm $|||\cdot|||_*$ on $Y^*$ by the formula

$$|||y^*|||_* = \sup\left\{ \frac{|y^*(f(x) - f(x'))|}{\|x - x'\|} : x, x' \in X, x \neq x' \right\}.$$

Since $f$ is a Lipschitz isomorphism from $X$ onto $Y$, this formula defines an equivalent norm on $Y^*$. We observe moreover that $|||\cdot|||_*$ is weak* lower semicontinuous, since

it is a supremum of weak* continuous functions. It follows easily from the Hahn–Banach theorem that an equivalent norm on $X^*$ is dual to a norm on $X$ if and only if it is weak* lower semicontinuous. Then $|||\cdot|||_*$ is the dual norm of an equivalent norm on $Y$ that we denote by $|||\cdot|||$. Since $f$ is 1-Lipschitz, this new norm is greater than or equal to the original norm on $Y$. Note that $|||\cdot|||$ could also be defined directly: indeed its unit ball is the closed convex hull of the set of all rates of change $\left\{\frac{f(x)-f(x')}{\|x-x'\|} : x, x' \in X, x \neq x'\right\}$. We claim that this norm satisfies the conditions stated in Theorem 14.6.6.

To that end, by Lemma 14.6.3 we need to show that

$$\eta_Y(\tau/4D) \leq 2\overline{\rho}_X(\tau) = 2\overline{\rho}(\tau),$$

where $\eta_Y = \eta$ is obtained from $|||\cdot|||$ along the lines of this lemma. Let $y \in Y$ with $|||y||| = 1$ and $(y_n)_{n=1}^\infty$ a sequence in $Y$ that converges weakly to 0 and such that $|||y_n||| \leq \tau/4D$ for all $n$. We have to show that

$$\limsup_{n\to\infty} |||y + y_n||| - 1 \leq 2\overline{\rho}(\tau).$$

For each $n$, we pick $y_n^* \in Y^*$ of norm 1 such that $y_n^*(y + y_n) = |||y + y_n|||$. We may and do assume that the sequence $(y_n^*)_{n=1}^\infty$ is weak* convergent to $y^*$ with $|||y^*|||_* \leq 1$ and that $\lim_n |||y^* - y_n^*|||_* = l$ exists. Pick $\epsilon > 0$ and $x \neq x'$ in $X$ such that

$$y^*(f(x) - f(x')) \geq (1-\epsilon)|||y^*|||_* \|x - x'\|.$$

We may assume that $x' = -x$ (hence $x \neq 0$) and $f(x') = -f(x)$, and thus

$$y^*(f(x)) \geq (1-\epsilon)|||y^*|||_* \|x\|.$$

Pick any $\beta > \overline{\rho}(\tau)$. By Lemma 14.6.3, there exists a subspace $X_0$ of finite codimension in $X$ such that if $z \in X_0$ and $\|z\| \leq \tau\|x\|$, then

$$\|x + z\| \leq (1+\beta)\|x\|.$$

Pick $b < \tau\|x\|/2D$ and let $c = \tau\|x\|/2$. Since $f^{-1}$ is $D$-Lipschitz (for the original norm, and thus also for the bigger norm $|||\cdot|||$), we can apply Lemma 14.6.4 for these values of $b$ and $c$ and conclude that there exists a compact set $K$ such that $bB_Y \subset K + f(2cB_{X_0})$.

We observe now that the sequence $(y_n^* - y^*)_{n=1}^\infty$ converges to 0 uniformly on the compact set $K$. It follows that there exists a sequence $(z_n)_{n=1}^\infty$ in $X_0$ such that $\|z_n\| \leq 2c = \tau\|x\|$ and $\lim_n \langle y_n^* - y^*, f(z_n)\rangle = -bl$.

Let $A_n = y_n^*(f(x) - f(z_n))$. We have

$$A_n \leq |||y_n^*|||_* \|x - z_n\| \leq (1+\beta)\|x\|.$$

Moreover,

$$A_n = y^*(f(x) - f(z_n)) + (y_n^* - y^*)(f(x)) - (y_n^* - y^*)(f(z_n)),$$

and since $(y_n^* - y^*)_{n=1}^{\infty}$ weak* converges to 0 and $f(-x) = -f(x)$, one has

$$A_n = 2y^*(f(x)) - y^*(f(z_n) - f(-x)) + bl + \epsilon(n),$$

with $\lim_{n\to\infty} \epsilon(n) = 0$. Since

$$y^*(f(z_n) - f(-x)) \leq |||y^*|||_* \|z_n + x\| \leq |||y^*|||_* (1+\beta)\|x\|,$$

il follows that

$$A_n \geq 2(1-\epsilon)|||y^*|||_* \|x\| - |||y^*|||_* (1+\beta)\|x\| + bl + \epsilon(n).$$

We can now combine the two inequalities on $A_n$ and let $n$ increase to infinity to obtain

$$(1+\beta)\|x\| \geq (1 - \beta - 2\epsilon)|||y^*|||_* \|x\| + bl.$$

Playing on $\beta$ and $b$ leads to

$$(1 + \overline{\rho}(\tau))\|x\| \geq (1 - \overline{\rho}(\tau) - 2\epsilon)|||y^*|||_* \|x\| + l\tau\|x\|/2D,$$

and since we can divide by $\|x\| \neq 0$ and $\epsilon > 0$ is arbitrary,

$$|||y^*|||_* \leq 1 + \frac{2\overline{\rho}(\tau)}{1 - \overline{\rho}(\tau)} - \frac{l\tau}{2D(1 - \overline{\rho}(\tau))}. \tag{14.36}$$

We have

$$|||y + y_n||| = y_n^*(y + y_n) = (y_n^* - y^*)(y) + (y_n^* - y^*)(y_n) + y^*(y + y_n),$$

and thus

$$\limsup_{n\to\infty} |||y+y_n||| \leq \frac{\tau}{4M} \lim_{n\to\infty} |||y_n^* - y^*|||_* + |||y^*|||_* = \frac{l\tau}{4D} + |||y^*|||_*. \tag{14.37}$$

If $\frac{l\tau}{4D} \leq 2\overline{\rho}(\tau)$, then since $|||y^*|||_* \leq 1$, it follows from (14.37) that

$$\limsup_{n\to\infty} |||y + y_n||| - 1 \leq 2\overline{\rho}(\tau).$$

If $\frac{l\tau}{4D} > 2\overline{\rho}(\tau)$, then from (14.36)

$$|||y^*|||_* \leq 1 - \frac{l\tau}{4D(1 - \overline{\rho}(\tau))} \leq 1 - \frac{l\tau}{4D},$$

and thus $\limsup_{n\to\infty} |||y + y_n||| \le 1$. Hence in both cases we have

$$\limsup_{n\to\infty} |||y + y_n||| - 1 \le 2\overline{\rho}(\tau)$$

and this concludes the proof. $\square$

It is easy to check that if $Y$ is a subspace of $c_0$ equipped with its natural norm, then $\rho_Y(\tau) = 0$ for all $\tau \in (0, 1]$. Hence Theorem 14.6.6 shows that if a Banach space $X$ is Lipschitz isomorphic to a subspace $Y$ of $c_0$, there is an equivalent norm on $X$ such that $\rho_X(\tau_0) = 0$ for some $\tau_0 > 0$. This is the critical step in the proof that the set of linear subspaces of $c_0$ is stable under Lipschitz isomorphisms, and that a Banach space that is Lipschitz isomorphic to $c_0$ is linearly isomorphic to that space [106]. It is not known whether this conclusion still holds for spaces that are uniformly homeomorphic to $c_0$ (see [107, p. 3915]).

## Problems

**14.1.** Let $\ell_\infty^2$ denote the space $\mathbb{R}^2$ equipped with the supremum norm.

(a) Show that the map $\phi : \mathbb{R} \to \ell_\infty^2$ defined by $\phi(t) = (t, \sin(t))$ is a nonlinear isometry from $\mathbb{R}$ into $\ell_\infty^2$.
(b) Show that if $\phi : \mathbb{R} \to \ell_\infty^2$ is an isometry such that $\phi(0) = 0$, then there exist a 1-Lipschitz map $g : \mathbb{R} \to \mathbb{R}$ and $\epsilon = 1$ such that for all $t$, $\phi(t) = (\epsilon t, g(t))$ or $\phi(t) = (g(t), \epsilon t))$ [*Hint*: Use the proof of Theorem 14.4.10.]

**14.2.**

(a) Let $\Phi$ be an isometry from a Banach space into a strictly convex Banach space (see Problem 12.3) such that $\Phi(0) = 0$. Show that $\Phi$ is a linear map.
(b) Suppose $Y$ is not strictly convex. Show that there exists an isometry $\phi$ from $\mathbb{R}$ into $Y$ with $\phi(0) = 0$ that is not linear.

**14.3.** A map $f : X \to Y$ between metric spaces is said to satisfy a *Hölder condition with exponent* $\alpha > 0$ (or that $f$ is $\alpha$-Hölder) if for some constant $C > 0$ and all $x, y \in X$,

$$d(f(x), f(y)) \le Cd^\alpha(x, y). \tag{14.38}$$

Of course, if $\alpha = 1$, this is simply the definition of a Lipschitz map. Show that if $X$ is metrically convex and $\alpha > 1$, then $f$ is constant.

**14.4.** Prove the following differentiation rules.

(a) If a map $f : X \to Y$ is Fréchet differentiable at a point $x$, then $f$ is continuous at $x$. However, this is not the case for Gâteaux differentiability (even in finite dimensions).

(b) If $f: X \to Y$ is continuous and linear, then $f$ is its own Fréchet derivative.
(c) Assuming the composition of functions $f \circ g$ is defined, that $g$ is Fréchet differentiable at $x$, and $f$ is Fréchet differentiable at $g(x)$, then $f \circ g$ is Fréchet differentiable at $x$ and the chain rule holds, i.e.,

$$D_{f \circ g}(x) = D_f(g(x)) \circ D_g(x).$$

(d) Assuming that the composition of functions $f \circ g$ is defined, that $g$ is Fréchet differentiable at $x$ and $f$ is Gâteaux differentiable at $g(x)$, then $f \circ g$ is Gâteaux differentiable at $x$ and the chain rule holds. The Gâteaux differentiability of $f$ at $g(x)$ is not sufficient to guarantee the Gâteaux differentiability of $f \circ g$ at $x$ unless $f$ is, for instance, Lipschitz.

**14.5.** Let $f: X \to Y$ be a map between Banach spaces.

(a) Prove the following version of the *mean value formula for the Gâteaux derivative*: If $f$ is Gâteaux differentiable on the interval $I$ connecting two points $x_0$ and $y_0$ in $X$, then

$$\|f(y_0) - f(x_0)\| \leq \sup_{x \in I} \|D_f(x)\| \|y_0 - x_0\|.$$

(b) Use (a) to deduce that if $f$ is Gâteaux differentiable in a neighborhood of a point $x_0 \in X$ and $D_f(x)$ is continuous at $x_0$, then $f$ is Fréchet differentiable at $x$.

**14.6.** Let $X$ be a Banach space. A function $f: X \to \mathbb{R}$ is Gâteaux differentiable at $x \in X$ if there exists $x^* \in X^*$ such that for all $h \in X$,

$$f(x + th) = f(x) + tx^*(h) + \epsilon_h(t),$$

with $\lim_{t \to 0} \epsilon_h(t) = 0$. Note that this limit is not required to be uniform on $h \in S_X$.

(a) Assume that $X$ is separable. Show that the set of points of Gâteaux differentiability of every continuous convex function $f: X \to \mathbb{R}$ is a dense $G_\delta$ subset of $X$. [*Hint*: Adapt the proof of Proposition 14.4.5.]

This result was first proved by Mazur [216], and it is the oldest, and perhaps the simplest, result on Gâteaux differentiability in infinite dimensions.
(b) Show that the norm is Gâteaux differentiable at $x$ of norm 1 if and only if there exists a unique $x^* \in X^*$ such that $\|x^*\| = x^*(x) = 1$.
(c) Determine the sets of points where the natural norms of $\ell_p (1 \leq p \leq \infty)$ and $c_0$ are Gâteaux differentiable.

**14.7.** Let $X$ be a Banach space. A map $f: X \to \mathbb{R}$ is Fréchet differentiable at $x \in X$ if there exists $x^* \in X^*$ such that

$$f(x + h) = f(x) + x^*(h) + \epsilon(h),$$

with $\lim_{\|h\| \to 0} \epsilon(h) = 0$.

(a) Show that the natural norm $\|\cdot\|_1$ on the space $\ell_1$ is a Lipschitz function from $\ell_1$ to $\mathbb{R}$ which is nowhere Fréchet differentiable.
(b) We define $E : \mathbb{R} \to \mathbb{R}$ by $E(t) = d(t, \mathbb{Z})$, and we consider the map $g : \mathbb{R} \to c_0$ defined by $g(t) = (2^{-n}E(2^n t))_{n\geq 0}$. Show that $g$ is a Lipschitz map that is nowhere differentiable.

*Remark.* It is well known that the function

$$h : \mathbb{R} \to \mathbb{R}, \quad t \mapsto h(t) = \sum_{n=0}^{\infty} 2^{-n}E(2^n t),$$

is continuous and nowhere differentiable. Changing the domain space or the range space provides similar nowhere smooth functions that are, however, Lipschitz.

**14.8.** Let $A$ be a measurable subset of the real line, and fix $a \in \mathbb{R}$. For all $x \geq a$, we define $g(x) = |A \cap [0, x]|$.

(a) Show that $g$ is a Lipschitz function from $[a, \infty)$ to $\mathbb{R}$.
(b) If $x > a$ is a point of differentiability of $g$, show that

$$g'(x) = \lim_{h\to 0+} \frac{|A \cap [x-h, x+h]|}{2h}.$$

(c) Deduce from (a) and (b) that almost every $x \in A$ is a Lebesgue point of $\chi_A$, in other words, for almost every $x \in A$ one has

$$\lim_{h\to 0+} \frac{|A \cap [x-h, x+h]|}{2h} = 1.$$

[*Hint*: Use $g(x) = \int_a^x g'(t)dt$ for all $x \geq a$.]

**14.9.** Prove that reflexivity is stable under Lipschitz isomorphisms.

**14.10. Free spaces.** Let $(X, d)$ be a *pointed metric space*, that is, a metric space equipped with a distinguished point denoted by 0. Let $\mathrm{Lip}_0(X)$ be the space of Lipschitz functions $f : X \to \mathbb{R}$ such that $f(0) = 0$. Put

$$\|f\|_L = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y\right\}.$$

For all $x \in X$ and $f \in \mathrm{Lip}_0(X)$, we let $f(x) = \langle \delta_X(x), f\rangle$.

(a) Show that $\|\cdot\|_L$ is a norm on $\mathrm{Lip}_0(X)$ for which $\mathrm{Lip}_0(X)$ is a Banach space, and that for all $x \in X$, one has $\delta_X(x) \in \mathrm{Lip}_0(X)^*$.
(b) Show that $\delta_X : X \to \mathrm{Lip}_0(X)^*$ is an isometric embedding from $X$ into $\mathrm{Lip}_0(X)^*$.
(c) The Banach space $\mathcal{F}(X) = [\delta_X(X)]$ generated by the set $\delta_X(X)$ is called the *free space over* $X$ (see [105]). Show that if $X$ and $Y$ are two pointed metric spaces and $F : X \to Y$ is a Lipschitz map such that $F(0) = 0$, then there is a unique continuous linear map $\hat{F} : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that $\hat{F} \circ \delta_X = \delta_Y \circ F$.

(d) Show that if $X$ and $Y$ are Lipschitz isomorphic, the Banach spaces $\mathrm{Lip}_0(X)$ and $\mathrm{Lip}_0(Y)$ are linearly isomorphic.
(e) Show that $\mathcal{F}(X)$ is an isometric predual of $\mathrm{Lip}_0(X)$.
(f) We now assume that $X$ is a Banach space equipped with the distance induced by the norm. Show that there exists a linear onto map $\beta_X: \mathcal{F}(X) \to X$ with $\|\beta_X\| = 1$ such that $\beta_X \circ \delta_X = I_X$. [*Hint*: Show that the restriction to $X^* \subset \mathrm{Lip}_0(X)$ of any element of $\mathcal{F}(X)$ belongs to $X$. ]
(g) We assume that $X$ is a separable Banach space. Show that there exists a continuous linear map $R_X: X \to \mathcal{F}(X)$ with $\|R_X\| = 1$ such that $\beta_X R_X = I_X$. Show that $X$ is isometric to a contractively complemented subspace of $\mathcal{F}(X)$.
(h) Let $X, Y, Z$ be separable Banach spaces, $T: X \to Y$ and $S: Z \to Y$ continuous linear operators, and $f: X \to Z$ a Lipschitz map such that $T = S \circ f$. Show that there exists a linear continuous map $L$ such that $T = SL$. [ *Hint*: Take $L = \beta_Z \hat{f} R_X$.]

**14.11.** Let $X$ be a Banach space, and $Y \subset X$ a closed subspace. We denote by $Q : X \to X/Y$ the canonical quotient map. A map $p : X \to Y$ is called a *quasi-additive projection* if $p(0) = 0$ and $p(x + y) = p(x) + y$ for every $x \in X$ and $y \in Y$.

(a) Show that $p \circ p = p$.
(b) Show that there exists a quasi-additive Lipschitz projection $p : X \to Y$ if and only if there exists a Lipschitz map $f: X/Y \to X$ such that $Q \circ f = I_{X/Y}$.
(c) Let $X$ be a separable Banach space. Show that a closed subspace $Y$ of $X$ is complemented in $X$ if and only if there exists a quasi-additive Lipschitz projection from $X$ onto $Y$.

**14.12.** Show that if $Y$ is superreflexive and there exists a coarse Lipschitz embedding from $X$ into $Y$, then $X$ is superreflexive.

**14.13.** Let $E$ be a Banach space, $e^* \in E^*$ of norm 1, and $E_0 = \mathrm{Ker}(e^*)$. Given $\epsilon \in (0, 1)$, we pick $x \in E$ such that $\|x\| \le 1$ and $e^*(x) > 1 - \epsilon$. Set $L = \{tx: |t| \le 1\}$.

(a) Let $\varphi: L \to E$ be a continuous map such that $\|\varphi(l) - l\| \le 1 - \epsilon$ for every $l \in L$. Show that $\varphi(L) \cap E_0 \neq \emptyset$.
(b) Deduce from (a) a proof of the Gorelik principle (Lemma 14.6.4) that does not use the Bartle–Graves selectors or Schauder's fixed point theorem, in the case that $E_0$ is of codimension 1 in $E$.

**14.14.** The *modulus of smoothness* $\rho$ of a Banach space $X$ is defined by the formula

$$\rho(\tau) = \sup\left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1: x, y \in X, \|x\| = \|y\| = 1 \right\}, \qquad \tau > 0.$$

We then say that $X$ is *uniformly smooth* if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$.

Show that if $X$ is uniformly smooth, then $X$ is asymptotically uniformly smooth, and moreover, one has $\overline{\rho}(\tau) \le 2\rho(\tau)$ for every $\tau > 0$.

**14.15.** Let $X$ be a separable Banach space. We denote by $\|\cdot\|^*$ the dual norm to the norm of $X$. Show that $X$ is asymptotically uniformly smooth if and only if for every $\epsilon \in (0, 1)$, there exists $\Delta(\epsilon) > 0$ such that if $(x_n^*)_{n=1}^{\infty}$ is a weak* convergent sequence with $\|x_n^*\|^* \leq 1$ and $\|x_n^* - x_k^*\|^* \geq \epsilon$ for all $n \neq k$, then $\|\lim(x_n^*)\|^* \leq 1 - \Delta(\epsilon)$.

**14.16.** Following Kalton [154], a Banach space $X$ has *Property* $(M)$ if for every $(u, v) \in X^2$ with $\|u\| = \|v\|$, and every weakly null sequence $(x_n)_{n=1}^{\infty}$,

$$\limsup_{n\to\infty} \|u + x_n\| = \limsup_{n\to\infty} \|v + x_n\|.$$

Let $X$ be a Banach space with separable dual and assume $X$ has Property (M).

(a) Show that $X$ is asymptotically uniformly smooth [*Hint*: Use the fact that since $X^*$ is separable, the norm of $X$ has a point of Fréchet differentiability (see, e.g., [60, Theorem I. 5. 7].)
(b) Let $|\cdot|$ be an equivalent norm on $X$ such that $\|x\| \leq |x| \leq d\|x\|$ for all $x \in X$. Show that $\overline{\rho}_{\|\cdot\|}(\tau) \leq \overline{\rho}_{|\cdot|}(d\tau)$ for all $\tau > 0$.
(c) Let $Y$ be a Banach space Lipschitz isomorphic to $\ell_p$, where $p \in (1, +\infty)$. Show that there exists an equivalent asymptotically uniformly smooth norm on $Y$ such that $\overline{\rho}_Y(\tau) \leq C\tau^p$ for every $\tau \in (0, 1]$.
(d) We assume that $2 < p < \infty$. Deduce from (b) and (c) that $\ell_2$ is not isomorphic to a subspace of $Y$.

We refer to [26, 79] for more applications of Property $(M)$ to nonlinear results. It follows in particular from (d) and Proposition 6.4.2 that $L_p$ and $\ell_p$ are not Lipschitz isomorphic if $2 < p < \infty$, although they have the same local structure, and thus Theorem 14.2.27 does not suffice for reaching this conclusion.

# Chapter 15
# Important Examples of Banach Spaces

In this last, optional chapter, we construct some examples of Banach spaces that played an important role in the development of Banach space theory. These constructions are not elementary, so we have preferred to remove them from the main text.

We first discuss a generalization of James space constructed by James [128] and improved by Lindenstrauss [195]. They show that for every separable Banach space $X$ one can construct a separable Banach space $\mathcal{Z}$ such that $\mathcal{Z}^{**}/\mathcal{Z} \approx X$. Furthermore, $\mathcal{Z}^*$ has a shrinking basis.

We then turn to treelike constructions and use a tree method to construct Pełczyński's universal basis space [246], which was a fundamental example in basis theory. It shows that there is a Banach space $U$ with a basis $(e_n)_{n=1}^{\infty}$ such that every basic sequence in $U$ is equivalent to a *complemented* subsequence of $(e_n)_{n=1}^{\infty}$.

Finally, we turn to the James tree space $\mathcal{JT}$, which was constructed in connection with Rosenthal's theorem (Chapter 11, Theorem 11.2.1). It is clear that if $X$ is a Banach space with separable dual, $X$ cannot contain $\ell_1$. The *James tree space*, $\mathcal{JT}$, gives an example to show that the converse statement is not true. The key is that $\mathcal{JT}^{**}/\mathcal{JT}$ is shown to be a nonseparable Hilbert space, and this is sufficient to show that $\ell_1$ cannot embed into $\mathcal{JT}$.

## 15.1 A Generalization of the James Space

In this section we will give an exposition of the construction of a generalization of the James space whose idea originated in James's 1960 paper [128] but was given in final form by Lindenstrauss in 1971 [195].

We recall our convention that if $E$ is a subset of $\mathbb{N}$ (in particular, any interval of integers) and $\xi = (\xi(n))_{n=1}^{\infty} \in c_{00}$, we write $E\xi$ for the sequence $(\chi_E(n)\xi(n))_{n=1}^{\infty}$,



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i.e., the sequence whose coordinates are $E\xi(n) = \xi(n)$ if $n \in E$ and $E\xi(n) = 0$ otherwise. We also remind the reader that if $E, F$ are subsets of $\mathbb{N}$, we write $E < F$ to mean $m < n$ whenever $m \in E$ and $n \in F$.

Let $X$ be any separable Banach space and suppose $(x_n)_{n=1}^{\infty}$ is any sequence such that $\{\pm x_n\}_{n=1}^{\infty}$ is dense in the surface of the unit ball of $X$, $\{x \in X : \|x\| = 1\}$. We define a norm on $c_{00}$ by

$$\|\xi\|_{\mathcal{X}} = \sup \left( \sum_{j=1}^{n} \left\| \sum_{i \in I_j} \xi(i) x_i \right\|^2 \right)^{1/2},$$

where the supremum is taken over all $n \in \mathbb{N}$ and all intervals $I_1 < I_2 < \cdots < I_n$.

In the case $X = \mathbb{R}$ we may take $x_n = 1$ for all $n$, and then we recover the original James space $\mathcal{J}$ but with a different basis from the original one, as in Problem 3.11.

The following proposition is quite trivial to see and we leave its proof as an exercise to the reader.

**Proposition 15.1.1.**

*(i) Let $\mathcal{X}$ be the completion of $(c_{00}, \|\cdot\|_{\mathcal{X}})$. The canonical unit vectors $(e_n)_{n=1}^{\infty}$ form a monotone basis for $\mathcal{X}$. Hence $\mathcal{X}$ can be identified as the space of all sequences $\xi$ such that*

$$\|\xi\|_{\mathcal{X}} = \sup \left( \sum_{j=1}^{n} \left\| \sum_{i \in I_j} \xi(i) x_i \right\|^2 \right)^{1/2} < \infty.$$

*(ii) $(e_n)_{n=1}^{\infty}$ is boundedly complete. Hence $(e_n^*)_{n=1}^{\infty}$ is a monotone basis for a subspace $\mathcal{Y}$ of $\mathcal{X}^*$, and so $\mathcal{X}$ can be identified (isometrically in this case) with $\mathcal{Y}^*$.*

**Proposition 15.1.2.** *There is a norm-one operator $T : \mathcal{X} \to X$ defined by $Te_n = x_n$ for $n \in \mathbb{N}$. The operator $T$ is a quotient map.*

*Proof.* It is easy to see that $\xi \in X$ implies that $\sum_{j=1}^{\infty} \xi(j) x_j$ must converge and that

$$\left\| \sum_{j=1}^{\infty} \xi(j) x_j \right\| \le \|\xi\|_{\mathcal{X}}.$$

Thus $T$ is well defined and has norm one. Since $T(B_{\mathcal{X}})$ contains $(x_n)_{n=1}^{\infty}$, it follows that $\overline{T(B_{\mathcal{X}})}$ contains $B_X$, and hence $T$ is a quotient map. □

Therefore $T^* : X^* \to \mathcal{X}^*$, the adjoint of $T$, given by

$$\langle \xi, T^* x^* \rangle = \sum_{i=1}^{\infty} \xi(i) x^*(x_i),$$

is an isometric embedding.

**Lemma 15.1.3.** *$T^*(X^*) \cap \mathcal{Y} = \{0\}$, and $T^*X^* + \mathcal{Y}$ is norm closed.*

*Proof.* It is enough to note that if $x^* \in X^*$ and $\xi^* \in \mathcal{Y}$, then

$$\|T^*x^*\|_{\mathcal{X}} = \|x^*\| \leq \|T^*x^* + \xi^*\|_{\mathcal{X}^*}.$$

Once we have this, it follows that $T^*X^* + \mathcal{Y}$ splits as a direct sum. In fact,

$$\|x^*\| = \limsup_{n\to\infty} |x^*(x_n)|.$$

But

$$\lim_{n\to\infty} \xi^*(e_n) = 0,$$

and so

$$\limsup_{n\to\infty} |(T^*x^* + \xi^*)e_n| = \|x^*\|.$$

□

**Lemma 15.1.4.** *Suppose $m < n$ and that $\xi^* \in B_{\mathcal{X}^*}$. Then we can decompose $\xi^* = \eta^* + \zeta^* + \psi^*$, where*

$$\eta^*(e_j) = 0, \qquad 1 \leq j \leq m, \tag{15.1}$$

$$\zeta^*(e_j) = 0, \qquad n \leq j < \infty, \tag{15.2}$$

$$(\|\eta^*\|^2_{\mathcal{X}^*} + \|\zeta^*\|^2_{\mathcal{X}^*})^{\frac{1}{2}} + \|\psi^*\|_{\mathcal{X}^*} \leq 1, \tag{15.3}$$

*and for some $x^* \in B_{X^*}$ we have*

$$T^*x^*(e_j) = \psi^*(e_j), \qquad m \leq j \leq n. \tag{15.4}$$

*Proof.* The set of $\xi^* \in B_{\mathcal{X}^*}$ that satisfy (15.1)–(15.4) is clearly convex. It is also weak* closed. To see this, suppose that $\xi_k^* \to \xi^*$ weak*, where each $\xi_k^*$ has a decomposition as $\xi_k^* = \eta_k^* + \zeta_k^* + \psi_k^*$, and $\psi_k^*(e_j) = x_k^*(e_j)$ for $m \leq j \leq n$ with $x_k^* \in B_{X^*}$. Then we can always pass to a subsequence such that $(\eta_k^*)_{k=1}^\infty, (\zeta_k^*)_{k=1}^\infty, (\psi_k^*)_{k=1}^\infty$ and $(x_k^*)_{k=1}^\infty$ are weak* convergent.

Now consider the set $\mathcal{S}$ of all $\xi^*$ of the form

$$\xi^* = \sum_{k=1}^N I_k^*(T^*x_k^*),$$

where

$$\sum_{k=1}^{n} \|x_k^*\|^2 \leq 1$$

and given intervals $I_1 < I_2 < \cdots < I_n$, $I_k^*$ is the adjoint of $I_k$ regarded as an operator. Then $\mathcal{S} \subset B_{\mathcal{X}^*}$. But if $\xi \in \mathcal{X}$ with $\|\xi\|_{\mathcal{X}} = 1$, and if $\epsilon > 0$, we can find $I_1 < I_2 < \cdots < I_N$ such that

$$\left( \sum_{k=1}^{N} \Big\| \sum_{i \in I_k} \xi(i) x_i \Big\|^2 \right)^{1/2} > 1 - \epsilon.$$

Hence we can find $x_1^*, x_2^*, \ldots, x_n^*$ with $\sum_{j=1}^{n} \|x_j^*\|^2 \leq 1$ and

$$\sum_{k=1}^{N} x^* \Big( \sum_{i \in I_k} \xi(i) x_k^*(x_i) \Big) > 1 - \epsilon$$

or, equivalently,

$$\Big\langle \xi, \sum_{k=1}^{N} I_k^* T^* x_k^* \Big\rangle > 1 - \epsilon.$$

Thus the set $\mathcal{S}$ norms $\mathcal{X}$, and hence its weak* closed convex hull $\overline{\text{co}}^{w^*}(\mathcal{S})$ coincides with $B_{\mathcal{X}^*}$ by a simple Hahn–Banach argument.

It remains only to show that if $\xi^* \in \mathcal{S}$, then (15.1)–(15.4) hold. Suppose

$$\xi^* = \sum_{k=1}^{N} I_k^* x_k^*$$

with $\sum_{k=1}^{N} \|x_k^*\|^2 \leq 1$. If one of the intervals $I_k$ includes $[m, n]$, we just put $\eta^* = \zeta^* = 0$ and $\psi^* = \xi^*$. If not, we let

$$\eta^* = \sum_{m < I_k} I_k^* x_k^*$$

and $\zeta^* = \xi^* - \eta^*,\ \psi^* = 0$, and we are done. □

**Lemma 15.1.5.** $T^*(X^*) \oplus \mathcal{Y} = \mathcal{X}^*$.

*Proof.* Let us suppose that $\|\xi^*\|_{\mathcal{X}^*} = 1$ and let $d = d(\xi^*, \mathcal{Y}^* + T^*(X^*))$. For every pair $m \leq n$ we can write $\xi^* = \eta_{m,n}^* + \zeta_{m,n}^* + \psi_{m,n}^*$, so that (15.1)–(15.4) hold for $\eta^* = \eta_{m,n}^*, \zeta^* = \zeta_{m,n}$, and $\psi^* = \psi_{m,n}^*$.

We observe that $\zeta^*_{m,n} \in \mathcal{Y}$, and so

$$\|\eta^*_{m,n}\|_{\mathcal{X}} + \|\psi^*_{m,n}\|_{\mathcal{X}^*} \geq d.$$

Now

$$(\|\eta^*_{m,n}\|^2_{\mathcal{X}} + \|\zeta^*_{m,n}\|^2_{\mathcal{X}})^{\frac{1}{2}} - \|\eta^*_{m,n}\|^2_{\mathcal{X}} \leq 1 - d,$$

which yields

$$1 - (1 - \|\zeta^*_{m,n}\|^2_{\mathcal{X}^*})^{1/2} \leq 1 - d,$$

or, equivalently,

$$\|\zeta^*_{m,n}\|_{\mathcal{X}^*} \leq (1 - d^2)^{1/2}.$$

By compactness we can pick a subsequence $M = (n_k)_{k=1}^\infty$ such that, keeping $m$ fixed,

$$\lim_{k\to\infty} \eta^*_{m,n_k} = \eta^*_m, \ \lim_{k\to\infty} \zeta^*_{m,n_k} = \zeta^*_m, \ \lim_{k\to\infty} \psi^*_{m,n_k} = \psi^*_m$$

all exist in the weak* topology.

It follows that $\|\zeta^*_m\|_{\mathcal{X}^*} \leq (1 - d^2)^{1/2}$. It is also elementary to see by Alaoglu's theorem that there exists $x^* \in B_{X^*}$ such that $\psi^*(e_j) = T^*x^*(e_j)$ for $m \leq j < \infty$. Hence $\psi^* - T^*x^* \in \mathcal{Y}$, i.e., $\psi^* \in T^*X^* + \mathcal{Y}$. Therefore,

$$d \leq \|\eta^*_m\|_{\mathcal{X}^*} + \|\zeta^*_m\|_{\mathcal{X}^*} \leq \|\eta^*_m\|_{\mathcal{X}^*} + (1 - d^2)^{1/2},$$

and so

$$\|\eta^*_m\|_{\mathcal{X}^*} \geq d - (1 - d^2)^{1/2}.$$

This yields

$$\|\psi^*_m\|_{\mathcal{X}^*} \leq 1 - d + (1 - d^2)^{1/2}.$$

The next step is to let $m \to \infty$; by passing again to a subsequence, we can ensure that

$$\lim_{k\to\infty} \eta^*_{m_k} = \eta^*, \ \lim_{k\to\infty} \zeta^*_{m_k} = \zeta^*, \ \lim_{k\to\infty} \psi^*_{m_k} = \psi^*$$

all exist in the weak* topology. But it is clear from the construction that $\eta^* = 0$, so $\xi^* = \zeta^* + \psi^*$ and therefore

$$1 = \|\xi^*\|_{\mathcal{X}^*} \leq (1 - d) + 2(1 - d^2)^{1/2}.$$

Hence $5d^2 \leq 4$, or, equivalently, $d \leq 2/\sqrt{5} < 1$.

This is enough to show that $T^*(X^*) + \mathcal{Y} = \mathcal{X}^*$, since if not, there would exist $\xi^* \in B_{X^*}$ with $d(\xi^*, T^*(X^*) + \mathcal{Y}) > 2/\sqrt{5}$. □

**Theorem 15.1.6.** *For every separable Banach space X there is a separable Banach space $\mathcal{Z}$ such that $\mathcal{Z}^{**}/\mathcal{Z}$ is isomorphic to X. Furthermore, $\mathcal{Z}^*$ has a shrinking basis.*

*Remark 15.1.7.* The fact that $\mathcal{Z}^*$ has a basis implies that $\mathcal{Z}$ has a basis: this is a deep result of Johnson, Rosenthal, and Zippin [142] that is beyond the scope of this book.

*Proof.* We take $\mathcal{Z} = \ker T$ in the above construction. We show that $\mathcal{X}$ can then be identified canonically with $\mathcal{Z}^{**}$. More precisely, we show that under the pairing between $\mathcal{X}$ and $\mathcal{Y}$ we can identify $\mathcal{Y}$ with $\mathcal{Z}^*$. The identification is not isometric, however.

Clearly, if $\eta^* \in \mathcal{Y}$, then $\eta^*|_{\mathcal{Z}} \in \mathcal{Z}^*$. Conversely, suppose $\zeta^* \in \mathcal{Z}^*$. By the Hahn–Banach theorem there exists $\xi^* \in \mathcal{X}^*$ such that $\xi^*|_{\mathcal{Z}} = \zeta^*$. By Lemma 15.1.5 there is a unique $x^* \in X^*$ such that $\eta^* = \xi^* - T^*x^* \in \mathcal{Y}$. Then $\eta^*|_{\mathcal{Z}} = \zeta^*$. Note that

$$\|\zeta^*\|_{\mathcal{Z}^*} \le \|\eta^*\|_{\mathcal{Y}} \le \|\xi^*\|_{\mathcal{X}^*} + \|x^*\| \le 2\|\zeta^*\|_{\mathcal{Z}^*}.$$

This completes the proof, since $\mathcal{Z}^{**}/\mathcal{Z}$ is isomorphic to $\mathcal{X}/\ker T$, i.e., to $X$. □

**Corollary 15.1.8.**

*(a) If X is a separable dual space, then there is a Banach space Z with a shrinking basis such that $Z^{**} \approx Z \oplus X$.*
*(b) If X is a separable reflexive space, then there is a Banach space Z with a boundedly complete basis such that $Z^{**} \approx Z \oplus X$.*

*Proof.* (*a*) If $X = Y^*$, construct $\mathcal{Z}$ as above, so that $\mathcal{Z}^{**}/\mathcal{Z} \approx Y$ and then $\mathcal{Z}^{***}/\mathcal{Z}^* \approx X$. Let $Z = \mathcal{Z}^*$.

(*b*) In this case take $Z = \mathcal{Z}^{**}$. □

## 15.2 Constructing Banach Spaces via Trees

Let $\mathcal{F}\mathbb{N}$ denote the family of all finite subsets of $\mathbb{N}$. We introduce an ordering on $\mathcal{F}\mathbb{N}$: given $A = \{m_1, m_2, \dots, m_j\}$ and $E = \{n_1, n_2, \dots, n_k\}$ in $\mathcal{F}\mathbb{N}$, we write $A \preceq E$ if we have $j \le k$ and $m_i = n_i$ for $1 \le i \le j$. This means that $A$ is the initial part of $E$. We will write $A \prec E$ if $A \preceq E$ and $A \ne E$.

The partially ordered set $(\mathcal{F}\mathbb{N}, \preceq)$ is an example of a *tree*. This means that for each $A \in \mathcal{F}\mathbb{N}$ the set $\{E : E \preceq A\}$ is both finite and totally ordered, and is empty for exactly one choice of $A$, namely, $A = \emptyset$; the empty set is then the *root* of the tree.

We will actually find it more convenient to consider the partially ordered set $\mathcal{F}^*\mathbb{N}$ of all *nonempty* sets in $\mathcal{F}\mathbb{N}$. This is not a tree, since it has infinitely many roots (i.e., the singletons); it is perhaps a forest.

A *segment* in $\mathcal{F}^*\mathbb{N}$ is a subset of $\mathcal{F}^*\mathbb{N}$ of the form $S = S(A_0, A_1) = \{E : A_0 \subset E \subset A_1\}$. A subset $\mathcal{A}$ of $\mathcal{F}^*\mathbb{N}$ is called *convex* (for the partial order $\preceq$) if given $A_0, A_1 \in \mathcal{A}$ we also have $S(A_0, A_1) \subset \mathcal{A}$.

A *branch* $B$ is a maximal totally ordered subset: this is easily seen to be a sequence $(A_n)_{n=1}^\infty$ of the form

$$A_n = \{m_1, \dots, m_n\}, \qquad n = 1, 2, \dots,$$

where $(m_n)_{n=1}^\infty$ is a subsequence of $\mathbb{N}$.

It will be convenient to introduce a coding, or labeling, of $\mathcal{F}^*\mathbb{N}$ by the natural numbers as follows. For $A = \{m_1, \dots, m_n\}$ we define

$$\psi(A) = 2^{m_1-1} + 2^{m_2-1} + \cdots + 2^{m_n-1}.$$

$\psi : \mathcal{F}^*\mathbb{N} \to \mathbb{N}$ is thus a bijection such that $A \preceq E \implies \psi(A) \le \psi(E)$.

We can thus transport $\preceq$ to $\mathbb{N}$ and define

$$m \preceq n \Leftrightarrow \psi(m) \preceq \psi(n).$$

We then consider $(\mathbb{N}, \preceq)$, and we can similarly define segments, convex sets, and branches in this partially ordered set. Note that intervals $I = [m, n]$ for the usual order on $\mathbb{N}$ are convex for the ordering $\preceq$ .

The key idea of our construction is that we want to make a norm on $c_{00} = c_{00}(\mathbb{N})$ that agrees with certain prescribed norms on $c_{00}(B)$ for every branch $B$. For this we require certain compatibility assumptions.

Let us suppose that for every branch $B$ in $(\mathbb{N}, \preceq)$ we are given a norm $\|\cdot\|_B$ on $c_{00}(B)$ and that the family of norms $\|\cdot\|_B$ satisfy the following conditions:

$$\|S\xi\|_B \le \|\xi\|_B, \qquad S \subset B,\ S \text{ an initial segment}, \tag{15.5}$$

and

$$\|\xi\|_B = \|\xi\|_{B'}, \qquad x \in c_{00}(B) \cap c_{00}(B'). \tag{15.6}$$

Condition (15.5) simply asserts that $(e_n)_{n\in B}$ is a monotone basis of the completion $X_B$ of $c_{00}(B)$. The second condition asserts that the family of norms is consistent on the intersections. We are next going to construct norms on $c_{00}$ such that $(e_n)_{n=1}^\infty$ is a monotone basis and whose restrictions to each complete branch $B$ reduce isometrically to the norms $\|\cdot\|_B$.

Our first, simplest definition will not solve our problem, but it leads to an interesting example. We define

$$\|\xi\|_{\mathcal{X}} = \sup_{B\in\mathcal{B}} \|B\xi\|, \qquad \xi \in c_{00}, \tag{15.7}$$

where $\mathcal{B}$ is the collection of all branches. Let $\mathcal{X}$ denote the completion of $c_{00}$ under this norm.

The following proposition is quite trivial, and we omit the proof.

**Proposition 15.2.1.** *In the space $\mathcal{X}$ we have:*

*(i)* $(e_n)_{n=1}^{\infty}$ *is a monotone basis.*
*(ii)* $\|B\xi\| \leq \|\xi\|$ *for each $B \in \mathcal{B}$, and so $X_B$ is complemented in $\mathcal{X}$.*

Now let us try to use this. Let us suppose that $X$ is a Banach space with a normalized monotone basis $(x_n)_{n=1}^{\infty}$. Consider the branch generated by the increasing sequence $(m_j)_{j=1}^{\infty}$, i.e., consisting of the sets $A_j = \{m_1, \ldots, m_j\}$ for $j = 1, 2, \ldots$. We define

$$\Big\| \sum_{j=1}^{N} \xi(j) e_{\psi(A_j)} \Big\|_B = \Big\| \sum_{j=1}^{N} \xi(j) x_{m_j} \Big\|_X.$$

Obviously the restriction that $(x_n)_{n=1}^{\infty}$ is monotone can be circumvented by simply renorming $X$. It is clear that we have the following:

**Proposition 15.2.2.** *If $X$ is a Banach space with a basis $(x_n)_{n=1}^{\infty}$, then there is a Banach space $\mathcal{X}$ with a basis $(e_n)_{n=1}^{\infty}$ such that for every increasing sequence $(m_j)_{j=1}^{\infty}$ the subsequence $(x_{m_j})_{j=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ is equivalent to a complemented subsequence $(e_{n_j})_{j=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$.*

## 15.3 Pełczyński's Universal Basis Space

We are in a position to prove the following surprising result due to Pełczyński [246] from 1969; our proof uses ideas of Schechtman [280]. We have seen by the Banach–Mazur theorem (Theorem 1.4.4) that every separable Banach space embeds in $\mathcal{C}[0, 1]$; however, very few spaces embed as a complemented subspace (for example, $\mathcal{C}[0, 1]$ has no complemented reflexive subspaces, as we saw in Proposition 5.7.4). It is therefore rather interesting that we can construct a separable Banach space $U$ with a basis such that every separable Banach space with a basis is isomorphic to a complemented subspace of $U$; moreover, there is exactly one such space. At the time of Pełczyński's paper, the basis problem was unsolved, and so it was not clear whether it might be that every separable Banach space was isomorphic to a complemented subspace of $U$; indeed, there was hope that this space might lead to some resolution of the basis problem. Later, Johnson, and Szankowski [140] showed, using the negative solution of the approximation property, that there is no separable Banach space that contains a complemented copy of all separable Banach spaces.

**Theorem 15.3.1 (Pełczyński's universal basis space).** *There is a unique separable Banach space $U$ with a basis and with the property that every Banach space with a basis is isomorphic to a complemented subspace of $U$.*

*Proof.* To prove the existence of $U$ it suffices to construct a Banach space $X$ with a basis $(x_n)_{n=1}^{\infty}$ such that every normalized basic sequence (in any Banach space) is equivalent to a complemented subsequence of $(x_n)_{n=1}^{\infty}$. Then the existence of $U$ follows from Proposition 15.2.2.

To construct $X$ we first find a sequence $(f_n)_{n=1}^\infty$ that is dense in the surface of the unit ball of $\mathcal{C}[0,1]$. We define a norm on $c_{00}$ by

$$\|\xi\|_X = \sup_k \Big\| \sum_{j=1}^k \xi(k) f_k \Big\|_{\mathcal{C}[0,1]}, \qquad \xi \in c_{00}.$$

The space $X$ is the completion of $(c_{00}, \|\cdot\|_X)$.

One readily checks that the canonical basis $(e_n)_{n=1}^\infty$ is a monotone basis of $X$.

The space $\mathcal{C}[0,1]$ is universal for separable spaces, and if $(g_j)_{j=1}^\infty$ is a basic sequence in $\mathcal{C}[0,1]$ and $\epsilon > 0$, we can find an increasing sequence $(m_j)_{j=1}^\infty$ such that

$$\sum_{j=1}^\infty \|g_j - f_{m_j}\| < \epsilon.$$

Taking $\epsilon$ small enough, we can ensure that $(f_{m_j})_{j=1}^\infty$ is a basic sequence equivalent to $(g_j)_{j=1}^\infty$. But then $(e_{m_j})_{j=1}^\infty$ is equivalent to $(f_{m_j})_{j=1}^\infty$. This yields the existence of $U$.

Uniqueness is an exercise in the Pełczyński decomposition technique. It is clear that $\ell_2(U)$ also has a basis, and so $\ell_2(U)$ is isomorphic to a complemented subspace of $U$. Hence for some $Y$ we have

$$U \approx Y \oplus \ell_2(U) \approx Y \oplus \ell_2(U) \oplus \ell_2(U) \approx U \oplus \ell_2(U) \approx \ell_2(U).$$

If $V$ is any other space with the same properties, then $V$ is isomorphic to a complemented subspace of $U$, and $U$ is isomorphic to a complemented subspace of $V$. Hence, by Theorem 2.2.3, $U \approx V$. □

Notice that the basis of $U$ that we implicitly constructed above has the property that every normalized basic sequence in any Banach space is equivalent to a complemented subsequence.

There is an unconditional basis form of the universal basis space, also constructed by Pełczyński.

**Theorem 15.3.2.** *There is a unique Banach space $U_1$ with an unconditional basis $(u_n)_{n=1}^\infty$ and with the property that every Banach space with an unconditional basis is isomorphic to a complemented subspace of $U_1$.*

*Proof.* Suppose $X$ is the space constructed in the preceding proof. Then we can define a norm on $c_{00}$ by

$$\|\xi\|_{U_1} = \sup_{\epsilon_j = \pm 1} \Big\| \sum_{j=1}^\infty \epsilon_j \xi(j) e_j \Big\|_X.$$

We leave to the reader the remaining details. See [246, 280]. □

## 15.4 The James Tree Space

It is clear that if $X$ is a separable Banach space with separable dual, then $X$ cannot contain a copy of $\ell_1$. The aim of this section is to give the example promised in Chapter 11 (Remark 11.2.3) of a separable Banach space that does not contain a copy of $\ell_1$, but has nonseparable dual.

Let us start by introducing a definition that will be useful in the remainder of the section.

**Definition 15.4.1.** A basis $(x_n)_{n=1}^\infty$ with biorthogonal functionals $(x_n^*)_{n=1}^\infty$ in a Banach space $X$ is said to satisfy a *lower* 2*-estimate on blocks* if there is a constant $C$ such that whenever $I_1, \dots, I_n$ are disjoint intervals of integers,

$$\sum_{j=1}^{n} \Big\| \sum_{k\in I_j} x_k^*(x) x_k \Big\|^2 \le C\|x\|^2.$$

We say that $(x_n)_{n=1}^\infty$ satisfies an *exact lower* 2*-estimate on blocks* if we may take $C = 1$.

**Proposition 15.4.2.** *Suppose a basis $(x_n)_{n=1}^\infty$ of a Banach space $X$ satisfies a lower* 2*-estimate on blocks. Then,*

*(i) The formula*

$$|||x||| = \max\left\{ \|x\|, \sup\Big( \sum_{j=1}^{n} \| \sum_{k\in I_j} x_k^*(x) x_k \|^2 \Big)^{1/2} \right\}, \qquad x \in X,$$

*defines an equivalent norm on $X$ with respect to which the basis satisfies an exact lower* 2*-estimate on blocks.*
*(ii) $(x_n)_{n=1}^\infty$ is boundedly complete.*

*Thus, $X = [x_n]_{n=1}^\infty$ is isomorphic to the dual of the space $Y = [x_n^*]_{n=1}^\infty$.*

*Proof.* We leave the verification of $(i)$ to the reader. To show $(ii)$, suppose

$$\sup_n \Big\| \sum_{k=1}^{n} a_k x_k \Big\| < \infty$$

but the series $\sum_{k=1}^\infty a_k x_k$ does not converge. Then we may find disjoint intervals $I_1 < I_2 < \cdots$ such that

$$\Big\| \sum_{k\in I_j} a_k x_k \Big\| \ge \delta > 0, \qquad j = 1, 2, \dots.$$

But then if $I_1, \dots, I_n \subset \{1, 2, \dots, N\}$,

$$n^{\frac{1}{2}}\delta \le C\Big\| \sum_{k=1}^{N} a_k x_k \Big\|,$$

and we get a contradiction. $\square$

*Remark 15.4.3.* In the particular case in Proposition 15.4.2 that $(x_n)_{n=1}^{\infty}$ satisfies an exact lower 2-estimate on blocks, then the basis $(x_n)_{n=1}^{\infty}$ is monotone, and hence $X$ is isometrically identified with $Y^*$.

In order to provide the aforementioned example, we need to modify our construction of $\mathcal{X}$. Returning to our conditions on the branch norms $\|\cdot\|_B$ in Section 15.2, we shall impose one further condition in addition to (15.5) and (15.6). We shall assume that for any disjoint segments $S_1, \dots, S_n$,

$$\sum_{j=1}^{n} \|S_j\xi\|_B^2 \le \|\xi\|_B^2, \qquad \xi \in c_{00}(B). \tag{15.8}$$

Thus we are assuming that for every branch $B$, the basis $(e_n)_{n\in B}$ of $X_B$ satisfies an exact lower 2-estimate on blocks (for the obvious ordering). This, in turn, means by Proposition 15.4.2 that each such basis is boundedly complete and that $X_B$ can be identified isometrically with the dual of the space $Y_B = [e_n^*]_{n\in B}$.

Notice that for every segment $S$, by (15.6) all the branch norms $\|\cdot\|_B$ for which $S$ is contained in $B$ agree on thus if $\xi \in c_{00}$, the value of $\|S\xi\|$ is well defined for every segment $S$. We put

$$\|\xi\|_{\mathcal{X}} = \sup\left\{ \Big( \sum_{j=1}^{n} \|S_j\xi\|^2 \Big)^{1/2} : S_1, \dots, S_n \text{ disjoint segments} \right\},$$

and let $\mathcal{X}$ be the completion of $c_{00}$ with this norm.

We shall say that two subsets $E, F \subset \mathbb{N}$ are *mutually incomparable* (for the order $\preceq$) if $m \in E$ and $n \in F$ imply that neither $m \preceq n$ nor $n \preceq m$ can hold. It is easy to see that the union of a family of mutually incomparable convex sets is again convex.

**Proposition 15.4.4.** *The norm $\|\cdot\|_{\mathcal{X}}$ has the following properties:*

*(i) For every $B \in \mathcal{B}$,*

$$\|\xi\|_B = \|\xi\|_{\mathcal{X}}, \qquad \xi \in c_{00}(B).$$

*(ii) If $E_1, \dots, E_n$ are disjoint and convex,*

$$\sum_{j=1}^{n} \|E_j\xi\|_{\mathcal{X}}^2 \le \|\xi\|_{\mathcal{X}}^2, \qquad \xi \in c_{00}.$$

*(iii) The basis $(e_n)_{n=1}^{\infty}$ of $\mathcal{X}$ satisfies an exact lower 2-estimate on blocks.*
*(iv) If $E_1, \dots, E_n$ are convex and mutually incomparable, then*

$$\sum_{j=1}^{n} \|E_j \xi\|_{\mathcal{X}}^2 = \Big\| \sum_{j=1}^{n} E_j \xi \Big\|_{\mathcal{X}}^2 \le \|\xi\|_{\mathcal{X}}^2, \qquad \xi \in c_{00}.$$

*Proof.* $(i)$ follows directly from (15.8).

$(ii)$ Given $\epsilon > 0$, pick disjoint segments $(S_{jk})_{k=1}^{m_n}$ for $j = 1, 2, \dots, n$ such that

$$\sum_{j=1}^{n} \sum_{k=1}^{m_n} \|S_{jk} E_j \xi\|^2 \ge \sum_{j=1}^{n} \|E_j \xi\|_{\mathcal{X}}^2 - \epsilon.$$

Let $S'_{jk} = E_j \cap S_{jk}$. Then the family of segments $(S'_{jk})_{j=1,k=1}^{n,m_n}$ is disjoint, so

$$\sum_{j=1}^{n} \sum_{k=1}^{m_n} \|S'_{jk} \xi\|^2 \le \|\xi\|_{\mathcal{X}}^2.$$

Hence

$$\sum_{j=1}^{n} \|E_j \xi\|_{\mathcal{X}}^2 - \epsilon \le \|\xi\|_{\mathcal{X}}^2.$$

Since $\epsilon > 0$ is arbitrary, we are done.

$(iii)$ Intervals are convex.

$(iv)$ In this case, for $\epsilon > 0$ pick disjoint segments $S_1, \dots, S_m$ such that

$$\sum_{k=1}^{m} \Big\| S_k \sum_{j=1}^{n} E_j \xi \Big\|^2 \ge \Big\| \sum_{j=1}^{n} E_j \xi \Big\|_{\mathcal{X}}^2 - \epsilon.$$

Let $S'_{jk} = E_j \cap S_k$. The assumption that the $E_j$'s are mutually incomparable implies that for each $k$, $S'_{jk}$ is nonempty for at most one $j$. Thus,

$$\sum_{k=1}^{m} \Big\| S_k \sum_{j=1}^{n} E_j \xi \Big\|^2 = \sum_{k=1}^{m} \sum_{j=1}^{n} \|S'_{jk} \xi\|^2 \le \sum_{j=1}^{n} \|E_j \xi\|_{\mathcal{X}}^2.$$

Hence,

$$\Big\| \sum_{j=1}^{n} E_j \xi \Big\|_{\mathcal{X}}^2 - \epsilon \le \sum_{j=1}^{n} \|E_j \xi\|_{\mathcal{X}}^2.$$

Since $\epsilon > 0$, this establishes an inequality

$$\Big\| \sum_{j=1}^{n} E_j \xi \Big\|_{\mathcal{X}}^2 \leq \sum_{j=1}^{n} \|E_j \xi\|_{\mathcal{X}}^2.$$

The reverse inequality follows from (*ii*).

Finally, since $E_1, \dots, E_n$ are incomparable, the union $\cup_{j=1}^{m} E_j$ is also convex, and by (*ii*),

$$\Big\| \sum_{j=1}^{n} E_j \xi \Big\|_{\mathcal{X}} \leq \|\xi\|_{\mathcal{X}}.$$

□

*Remark 15.4.5.* By (*iii*) of Proposition 15.4.4, we see that the basis $(e_n)_{n=1}^{\infty}$ of $\mathcal{X}$ is boundedly complete and that $\mathcal{X}$ can be isometrically identified with the dual of $\mathcal{Y} = [e_n^*]_{n=1}^{\infty} \subset \mathcal{X}^*$.

For $n \in \mathbb{N}$ let $T_n = \{m : \ n \preceq m\}$ and $T_n^+ = \{m : \ n \prec m\}$.

**Lemma 15.4.6.** *Suppose* $\xi \in c_{00}$ *is supported on* $[1, N]$ *and* $\eta \in c_{00}$ *is supported on* $[N+1, \infty)$. *Then*

$$\|\xi + \eta\|_{\mathcal{X}} \leq (\|\xi\|_{\mathcal{X}}^2 + \|\eta\|_{\mathcal{X}}^2)^{\frac{1}{2}} + N^{\frac{1}{2}} \sup_{m \geq N+1} \|T_m \eta\|_{\mathcal{X}}.$$

*Proof.* Let $\delta = \sup_{m \geq N+1} \|T_m \eta\|_{\mathcal{X}}$. Suppose $\epsilon > 0$ and pick disjoint segments $(S_j)_{j=1}^{m}$ such that

$$\|\xi + \eta\|_{\mathcal{X}}^2 < \sum_{j=1}^{m} \|S_j(\xi + \eta)\|^2 + \epsilon.$$

We may assume the segments $(S_j)_{j=1}^{m}$ are such that $S_j \subset [1, N]$ for $1 \leq j < k$, $S_j \subset [N+1, \infty)$ for $l < j \leq m$, and that $S_j$ meets both $[1, N]$ and $[N+1, \infty)$ for $k \leq j \leq l$, where $0 \leq k \leq l+1 \leq m+1$ (taking account of the possibilities that each collection might be empty!).

Then

$$\sum_{l < j \leq m} \|S_j(\xi + \eta)\|^2 \leq \|\eta\|_{\mathcal{X}}^2.$$

But if $k \leq j \leq l$,

$$\|S_j(\xi + \eta)\| \leq \|S_j \xi\| + \|S_j \eta\| \leq \|S_j \xi\| + \delta.$$

Thus,

$$\left(\sum_{1\le j\le l}\|S_j(\xi+\eta)\|^2\right)^{1/2} \le \left(\sum_{1\le j\le l}\|S_j(\xi)\|^2\right)^{1/2} + (l-k+1)^{\frac{1}{2}}\delta,$$

$$\le \|\xi\|_{\mathcal{X}} + N^{\frac{1}{2}}\delta,$$

since $l-k+1 \le N$, since the sets $S_j$ are disjoint. Hence,

$$\left(\sum_{j=1}^{m}\|S_j(\xi+\eta)\|^2\right)^{1/2} \le (\|\xi\|_{\mathcal{X}} + \|\eta\|_{\mathcal{X}}^2)^{\frac{1}{2}} + N^{\frac{1}{2}}\delta,$$

and this completes the proof. □

We now come to the main point of the construction. Let us recall that whenever $(X_i)_{i\in\mathcal{I}}$ is an uncountable family of Banach spaces, $\ell_\infty(X_i)_{i\in\mathcal{I}}$ is the Banach space of all $(x_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} X_i$ such that $(\|x_i\|)_{i\in\mathcal{I}}$ is bounded, with the norm

$$\|(x_i)_{i\in\mathcal{I}}\|_\infty = \sup_{i\in I}\|x_i\|_{X_i}.$$

Similarly, $\ell_2(X_i)_{i\in\mathcal{I}}$ is the Banach space of all $(x_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} X_i$ such that $(\|x_i\|_{X_i})_{i\in\mathcal{I}} \in \ell_2(\mathcal{I})$ with the norm

$$\|(x_i)_{i\in\mathcal{I}}\|_2 = \Big(\sum_{i\in\mathcal{I}}\|x_i\|_{X_i}^2\Big)^{1/2}.$$

**Proposition 15.4.7.** *Then $\mathcal{Y}^{**}/\mathcal{Y}$ is isometrically isomorphic to the space $\ell_2(Y_B^{**}/Y_B)_{B\in\mathcal{B}}$.*

*Proof.* Let us write $J_n = [n,\infty)$. Then if $\xi^* \in \mathcal{Y}^{**} = \mathcal{X}^*$, we have $\xi^* \in \mathcal{Y}$ if and only if $\lim_{n\to\infty}\|J_n^*\xi^*\| = 0$. Here we interpret $J_n$ as an operator on $\mathcal{X}$.

We will repeatedly use the following fact: If $(A_n)_{n=1}^\infty$ is a sequence of mutually incomparable convex sets, then $A = \cup_{n=1}^\infty$ is also convex and $A\mathcal{X} = \ell_2(A_n\mathcal{X})$; this follows directly from Proposition 15.4.4 (*iii*). Hence if $\xi^* \in \mathcal{X}^*$, we have

$$\|A^*\xi^*\| = \Big(\sum_{n=1}^\infty\|A_n\xi^*\|^2\Big)^{1/2}.$$

Define a linear operator $V : \mathcal{X}^* \to \ell_\infty(X_B^*)_{B\in\mathcal{B}}$ naturally by setting $V\xi^* = (\xi^*|_{X_B})_{B\in\mathcal{B}}$. Then $V$ is clearly a norm-one operator, and $V(\mathcal{Y}) \subset \ell_\infty(Y_B)_{B\in\mathcal{B}}$.

The first step is to show that $V^{-1}\big(\ell_\infty(Y_B)_{B\in\mathcal{B}}\big) = \mathcal{Y}$. Suppose $\xi^* \in \mathcal{X}^*$ and $\xi^*|_{X_B} \in Y_B$ for every $B \in \mathcal{B}$. This means that

$$\lim_{n\to\infty}\|(J_n\cap B)^*\xi^*\| = 0$$

for every branch $B$.

Fix a branch $B$. For each $n \in B$ let $T'_n = T_n^+ \setminus T_{n'}$, where $n'$ is the successor of $n$ in the branch. Then the sequence $(T'_n)_{n\in B}$ consists of mutually incomparable tree-convex sets. Hence

$$\|(\cup_{n\preceq m}T'_m)^*\xi^*\| = (\sum_{n\preceq m}\|(T'_m)^*\xi^*\|^2)^{\frac{1}{2}}, \qquad n \in B,$$

and so

$$\lim_{\substack{n\to\infty\\ n\in B}} \|(\cup_{n\preceq m}T'_m)^*\xi^*\| = 0.$$

Since $\cup_{n\preceq m}T'_m \cup (J_n \cap B) = T_n$, by the triangle inequality we have

$$\lim_{\substack{n\to\infty\\ n\in B}} \|T_n^*\xi^*\| = 0$$

for every branch $B$.

We next want to conclude that

$$\lim_{n\to\infty} \|T_n^*\xi^*\| = 0. \tag{15.9}$$

Indeed, if there exists $\epsilon > 0$ and infinitely many $n$ such that $\|T_n^*\xi^*\| \geq \epsilon$, then by the preceding reasoning we cannot find infinitely many $n$ belonging to one branch. Hence we can pass to an infinite subset $A$ such that if $m, n \in A$ with $m < n$, then it is not true that $m \preceq n$. Then the sets $\{T_n\}_{n\in A}$ are mutually incomparable. Hence

$$\sum_{n\in A}\|T_n^*\xi^*\|^2 < \infty,$$

and this gives a contradiction. Thus (15.9) holds.

Assuming (15.9), let $\delta_n = \sup_{m\geq n}\|T_m^*\xi^*\|$. Let us fix $m$ and $\epsilon > 0$. Then we may find $\xi \in c_{00}$ with $\|\xi\|_{\mathcal{X}} = 1$ and $\langle \xi, J_m^*\xi^*\rangle > (1-\epsilon)\|J_m^*\xi^*\|$. Choose $r$ such that $\xi(j) = 0$ for $j \geq r$. If $n \geq r$, let $A$ be the set of $k \geq n$ such that the predecessor of $k$ is less than or equal to $n$. There are at most $n$ such $k$. Then the sets $(T_k)_{k\in A}$ are mutually incomparable and convex, and $\cup_{k\in A}T_k = J_n$. For $0 < \epsilon < \frac{1}{2}$, identifying $J_n\mathcal{X}$ with the $\ell_2$-sum of the space $T_k\mathcal{X}$ for $k \in A$, we can find $\eta \in J_n\mathcal{X} \cap c_{00}$ with $\|\eta\|_{\mathcal{X}} = 1$ and

$$\langle \eta, J_n^*\xi^*\rangle > (1-\epsilon)\|J_n^*\xi^*\|$$

in such a way that

$$\|T_k\eta\| \leq 2\|T_n^*\xi^*\|\|J_n^*\xi^*\|^{-1}, \qquad k \in A.$$

Hence,

$$\sup_{k\in A} \|T_k\eta\| \le 2\delta_n \|J_n^*\xi^*\|^{-1}.$$

Therefore,

$$\begin{aligned}(1-\epsilon)\big(\|J_m^*\xi^*\| + \|J_n^*\xi^*\|\big) &\le \langle \xi + \eta, J_m^*\xi^*\rangle \\ &\le \|J_m^*\xi^*\|\|\xi + \eta\|_{\mathcal{X}} \\ &\le \|J_m^*\xi^*\|(2^{\frac{1}{2}} + r^{\frac{1}{2}} \sup_{l\ge r} \|T_l\eta\|) \\ &\le \|J_m^*\xi^*\|(2^{\frac{1}{2}} + r^{\frac{1}{2}} \sup_{l\ge n} \|T_l\eta\|) \\ &\le \|J_m^*\xi^*\|(2^{\frac{1}{2}} + 2r^{\frac{1}{2}}\delta_n \|J_n^*\xi^*\|^{-1}).\end{aligned}$$

Assume $\lim_{n\to\infty} \|J_n^*\xi^*\| > 0$. Then, letting $n \to \infty$, and then $\epsilon \to 0$, we have

$$\|J_m^*\xi^*\| + \lim_{n\to\infty} \|J_n^*\xi^*\| \le \sqrt{2}\|J_m^*\xi^*\|,$$

and so

$$\lim_{n\to\infty} \|J_n^*\xi^*\| \le (\sqrt{2} - 1)\|J_m^*\xi^*\|, \qquad m \in \mathbb{N}.$$

Letting $m \to \infty$ shows that $\lim_{n\to\infty} \|J_n^*\xi^*\| = 0$, giving a contradiction. This concludes the proof of the first step, i.e., $V^{-1}(\ell_\infty(Y_B)_{B\in\mathcal{B}}) = \mathcal{Y}$.

This yields a naturally induced one-to-one map

$$\tilde{V} : \mathcal{Y}^{**}/\mathcal{Y} \to \ell_\infty(Y_B^{**}/Y)_{B\in\mathcal{B}}.$$

Let us show that $\tilde{V}$ maps into $\ell_2(Y_B^{**}/Y)_{B\in\mathcal{B}}$. Let $Q$ be the quotient map of $\mathcal{Y}^{**}$ onto $\mathcal{Y}^{**}/\mathcal{Y}$, and $Q_B$ the corresponding quotient map of $Y_B^{**}$ onto $Y_B^{**}/Y$. If $B_1, \ldots, B_n$ are distinct complete branches and $\xi^* \in \mathcal{X}^*$, then we may pick $m$ large enough that the branches $B_j \cap J_m$ are disjoint. Since they are tree-convex, we have

$$\Big\| \sum_{j=1}^{n} (B_j \cap J_m)^*\xi^* \Big\|^2 = \sum_{j=1}^{m} \|(B_j \cap J_m)^*\xi^*\|^2 \le \|\xi^*\|^2,$$

which yields

$$\sum_{j=1}^{n} \|Q_{B_j}\xi^*|_{X_{B_j}}\|^2 \le \|\xi^*\|^2.$$

It follows that $\|\tilde{V}\| \le 1$ as an operator from $\mathcal{Y}^{**}/\mathcal{Y}$ into $\ell_2(Y_B^{**}/Y)_{B\in\mathcal{B}}$.

Finally we check that $\tilde{V}$ is an onto isometry. Suppose we have a finitely supported element $u = (u_B)_{B\in\mathcal{B}}$ in $\ell_2(Y_B^{**}/Y)_{B\in\mathcal{B}}$. For $\epsilon > 0$ pick $\xi_B^* \in Y_B^* = B^*(\mathcal{X}^*)$ with $\|\xi_B^*\| \le (1+\epsilon)\|\xi_B^*\|$ and $Q_B\xi_B^* = u_B$. Pick $m$ large enough that the branches $\{B \cap J_m :\ u_B \neq 0\}$ are disjoint. Then let $\xi^* = \sum_{u_B\neq 0} J_m^*\xi_B^*$; we have

$$\|\xi^*\| = (\sum_{u_B\neq 0} \|J_m^*\xi_B^*\|^2)^{\frac{1}{2}} \le (1+\epsilon)(\sum_{u_B\neq 0} \|u_B\|^2)^{\frac{1}{2}} = (1+\epsilon)\|u\|.$$

Since $\tilde{V}Q\xi^* = u$, $\tilde{V}$ is an onto isometry. □

In the following theorem we re-create an example due to James [133]. The space $\mathcal{X} = \mathcal{Y}^*$ is usually called the *James tree space* and it is denoted by $\mathcal{JT}$. James showed that $\ell_1$ does not embed into $\mathcal{JT}$ but that $\mathcal{JT}^*$ is not separable. Other examples were independently constructed by Lindenstrauss and Stegall [199]. The next theorem is, in fact, due to Lindenstrauss and Stegall [199]. A full account of James-type constructions can be found in [92].

**Theorem 15.4.8.** *There is a Banach space $\mathcal{Y}$ such that $\mathcal{Y}^*$ is separable and $\mathcal{Y}^{**}/\mathcal{Y}$ is isometric to $\ell_2(\mathcal{I})$, where $\mathcal{I}$ has the cardinality of the continuum.*

*Proof.* We use the space $\mathcal{J}$ but with the basis of Problem 3.11, which is a special case of the construction of Theorem 15.1.6. It is trivial to see that the basis $(e_n)_{n=1}^{\infty}$ of the space $\mathcal{X}$ constructed in Section 15.1 has an exact lower 2-estimate on blocks. To avoid confusion let us denote this norm now by $|||\cdot|||$.

Again we identify $(\mathbb{N}, \preceq)$ with $\mathcal{F}\mathbb{N}$. Let $B$ be the branch generated by the increasing sequence $(m_j)_{j=1}^{\infty}$, i.e., consisting of the sets $A_j = \{m_1, \ldots, m_j\}$. We define the branch norms on $c_{00}$ by

$$\Big\| \sum_{j=1}^{n} a_j e_{\psi(A_j)} \Big\|_B = \Big|\Big|\Big| \sum_{j=1}^{n} a_j e_j^* \Big|\Big|\Big|.$$

Letting our construction run its course, we see that each $Y_B^{**}/Y$ is isometric to $\mathbb{R}$. The result is then immediate. □

**Theorem 15.4.9.** *The space $\mathcal{Y}^* = \mathcal{JT}$ has nonseparable dual, but $\ell_1$ does not embed into $\mathcal{JT}$.*

*Proof.* Obviously, $\mathcal{JT}^*$ is nonseparable. Since $\mathcal{JT}$ is a dual space, it is complemented in its bidual, and so $\mathcal{JT}^{**} = \mathcal{JT} \oplus W$, where $W$ can be identified as the dual of the space $\mathcal{JT}^*/\mathcal{JT}_*$, and $\mathcal{JT}_*$ is the predual $\mathcal{Y}$ given by the construction. Hence, using Theorem 15.4.8, we conclude that $W = \ell_2(\mathcal{I})$ for an uncountable set $(\mathcal{I})$.

If $\ell_1$ embeds in $\mathcal{JT}$, then $\ell_1^{**} = \ell_\infty^*$ embeds in $\mathcal{JT}^{**}$. But $\ell_\infty = \mathcal{C}(K)$ for some uncountable compact Hausdorff space $K$, and hence using point masses, the space $\ell_1(\Gamma)$ embeds into $\mathcal{JT}^{**}$ for some uncountable set $\Gamma$. Let $T : \ell_1(\Gamma) \to \mathcal{JT} \oplus W$ be an embedding and assume it has the form $T = T_1 \oplus T_2$, where $T_1 : \ell_1(\Gamma) \to \mathcal{JT}$

and $T_2 : \ell_1(\Gamma) \to W$. Using the separability of $\mathcal{JT}$, we may find a sequence of basis vectors $(e_{\gamma_n})_{n=1}^{\infty}$ such that $(T_1 e_{\gamma_n})_{n=1}^{\infty}$ converges. Hence $\lim_{n\to\infty} \|T_1(e_{\gamma_{2n}} - e_{\gamma_{2n+1}})\| = 0$, so replacing the original sequence by a subsequence, we can assume that $(T_2(e_{\gamma_{2n}} - e_{\gamma_{2n+1}}))_{n=1}^{\infty}$ is a basic sequence equivalent to the canonical basis of $\ell_1$; this is absurd, since $W$ is a Hilbert space. □

In his 1974 paper [133], James showed that every infinite-dimensional subspace of $\mathcal{JT}$ contains a subspace isomorphic to a Hilbert space and thus deduced Theorem 15.4.9.

Going back to Theorem 15.4.8 and using Theorem 15.1.6, it is clear that we can also prove the following:

**Theorem 15.4.10.** *Let $X$ be any separable dual space. Then there is a Banach space $Z$ such that $Z^{**}/Z$ is isomorphic to $\ell_2(X)_{i\in\mathcal{I}}$ where $\mathcal{I}$ has the cardinality of the continuum.*

*Proof.* Let $X = Y^*$ and construct $\mathcal{Z}$ as in Section 15.1 such that $\mathcal{Z}^{**}/\mathcal{Z} \approx Y$. Using the canonical basis of $\mathcal{Z}$ as in Theorem 15.4.8 will give us a space $Z$ such that $Z^{**}/Z$ is isomorphic to $\ell_2(Y^*)_{i\in\mathcal{I}}$. □

# Appendix A
# Normed Spaces and Operators

A *normed space* $(X, \|\cdot\|)$ is a linear space $X$ endowed with a nonnegative function $\|\cdot\| : X \to \mathbb{R}$ called a *norm* satisfying

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\alpha x\| = |\alpha|\|x\| \quad (\alpha \in \mathbb{R}, x \in X)$;
(iii) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\| \quad (x_1, x_2 \in X)$.

A *Banach space* is a normed linear space $(X, \|\cdot\|)$ that is complete in the metric defined by $\rho(x, y) = \|x - y\|$. Here $B_X$ will denote the *closed unit ball of* $X$, that is, $\{x \in X : \|x\| \leq 1\}$. Similarly, the *open unit ball of* $X$ is $\{x \in X : \|x\| < 1\}$, and $S_X = \{x \in X : \|x\| = 1\}$ is the *unit sphere of* $X$.

**A.1. Completeness Criterion.** *A normed space* $(X, \|\cdot\|)$ *is complete if and only if the (formal) series* $\sum_{n=1}^{\infty} x_n$ *in* $X$ *converges in norm whenever* $\sum_{n=1}^{\infty} \|x_n\|$ *converges.*

A linear subspace $Y$ of a Banach space $(X, \|\cdot\|)$ is closed in $X$ if and only if $(Y, \|\cdot\|_Y)$ is a Banach space, where $\|\cdot\|_Y$ denotes the restriction of $\|\cdot\|$ to $Y$. If $Y$ is a subspace of $X$, so is its closure $\overline{Y}$.

Two norms $\|\cdot\|$ and $\|x\|_0$ on a linear space $X$ are *equivalent* if there exist positive numbers $c$, $C$ such that for all $x \in X$ we have

$$c\|x\|_0 \leq \|x\| \leq C\|x\|_0. \tag{A.1}$$

An *operator* between two Banach spaces $X$, $Y$ is a norm-to-norm continuous linear map. The following conditions are equivalent ways to characterize the continuity of a mapping $T : X \to Y$ with respect to the norm topologies of $X$ and $Y$:

(i) $T$ is bounded, meaning $T(B)$ is a bounded subset of $Y$ whenever $B$ is a bounded subset of $X$.
(ii) $T$ is continuous at 0.
(iii) There is a constant $C > 0$ such that $\|Tx\| \leq C\|x\|$ for every $x \in X$.



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(iv) $T$ is uniformly continuous on $X$.
(v) The quantity $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$ is finite.

The linear space of all continuous operators from a normed space $X$ into a Banach space $Y$ with the usual *operator norm*

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$$

is a Banach space, which will be denoted by $\mathcal{B}(X,Y)$. When $X = Y$, we will put $\mathcal{B}(X) = \mathcal{B}(X,X)$.

The set of all *functionals* on a normed space $X$ (that is, the continuous linear maps from $X$ into the scalars) is a Banach space, denoted by $X^*$ and called the *dual space of* $X$. The norm of a functional $x^* \in X^*$ is given by

$$\|x^*\| = \sup\{|x^*(x)| : x \in B_X\}.$$

Let $T: X \to Y$ be an operator. We say that $T$ is *invertible* if there exists an operator $S: Y \to X$ such that $TS$ is the identity operator on $Y$ and $ST$ is the identity operator on $X$. When this happens, $S$ is said to be the *inverse* of $T$ and is denoted by $T^{-1}$.

**A.2. Existence of Inverse Operator.** *Let $X$ be a Banach space. Suppose that $T \in \mathcal{B}(X)$ is such that $\|I_X - T\| < 1$ ($I_X$ denotes the identity operator on $X$). Then $T$ is invertible and its inverse is given by the Neumann series*

$$T^{-1}(x) = \lim_{n\to\infty} \Big(I_X + (I_X - T) + (I_X - T)^2 + \cdots + (I_X - T)^n\Big)(x), \quad x \in X.$$

An operator $T$ between two normed spaces $X$, $Y$ is an *isomorphism* if $T$ is a continuous bijection whose inverse $T^{-1}$ is also continuous. That is, an isomorphism between normed spaces is a linear homeomorphism. Equivalently, $T : X \to Y$ is an isomorphism if and only if $T$ is onto and there exist positive constants $c$, $C$ such that

$$c\|x\|_X \leq \|Tx\|_Y \leq C\|x\|_X$$

for all $x \in X$. In such a case the spaces $X$ and $Y$ are said to be *isomorphic*, and we write $X \approx Y$. We call $T$ an *isometric isomorphism* when $\|Tx\|_Y = \|x\|_X$ for all $x \in X$.

An operator $T$ is an *embedding of $X$ into $Y$* if $T$ is an isomorphism onto its image $T(X)$. In this case we say that $X$ *embeds in* $Y$ or that $Y$ contains an isomorphic copy of $X$. If $T: X \to Y$ is an embedding such that $\|Tx\|_Y = \|x\|_X$ for all $x \in X$, $T$ is said to be an *isometric embedding*.

**A.3. Extension of Operators by Density.** *Suppose that $M$ is a dense linear subspace of a normed linear space $X$, that $Y$ is a Banach space, and that $T : M \to Y$ is a bounded operator. Then there exists a unique continuous operator $\tilde{T} : X \to Y$ such that $\tilde{T}|_M = T$ and $\|\tilde{T}\| = \|T\|$. Moreover, if $T$ is an isomorphism or isometric isomorphism, then so is $\tilde{T}$.*

Given $T : X \to Y$, the operator $T^* : Y^* \to X^*$ defined as $T^*(y^*)(x) = y^*(T(x))$ for every $y^* \in Y^*$ and $x \in X$ is called the *adjoint of* $T$ and has the property that $\|T^*\| = \|T\|$.

An operator $T : X \to Y$ between the Banach spaces $X$ and $Y$ is said to be *compact* if $T(B_X)$ is relatively compact, that is, $\overline{T(B_X)}$ is a compact set in $Y$. The space of compact operators from $X$ to $Y$ is denoted by $\mathcal{K}(X, Y)$. If a linear operator $T: X \to Y$ is compact, then it is continuous.

An operator $T : X \to Y$ has *finite rank* if the dimension of its range $T(X)$ is finite.

**A.4. Schauder's Theorem.** *A bounded operator $T$ from a Banach space $X$ into a Banach space $Y$ is compact if and only if $T^* : Y^* \to X^*$ is compact.*

A bounded linear operator $P : X \to X$ is a *projection* if $P^2 = P$, i.e., $P(P(x)) = P(x)$ for all $x \in X$; hence $P(y) = y$ for all $y \in P(X)$. A subspace $Y$ of $X$ is *complemented* if there is a projection $P$ on $X$ with $P(X) = Y$. Thus complemented subspaces of Banach spaces are always closed.

**A.5. Property.** *Suppose $Y$ is a closed subspace of a Banach space $X$. If $Y$ is complemented in $X$, then $Y^*$ is isomorphic to a complemented subspace of $X^*$.*

Let us finish this section by recalling that the *codimension* of a closed subspace $Y$ of a Banach space $X$ is the dimension of the quotient space $X/Y$.

**A.6. Subspaces of Codimension One.** *Every two closed subspaces of codimension* 1 *in a Banach space $X$ are isomorphic.*

# Appendix B
# Elementary Hilbert Space Theory

An *inner product space* is a linear space $X$ over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ of $X$ equipped with a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ called an *inner product* or *scalar product* satisfying the following conditions:

(i) $\langle x, x \rangle \geq 0$ for all $x \in X$,
(ii) $\langle x, x \rangle = 0$ if and only if $x = 0$,
(iii) $\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle$ if $\alpha_1, \alpha_2 \in \mathbb{K}$ and $x_1, x_2, y \in X$,
(iv) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in X$. (The bar denotes complex conjugation.)

An inner product on $X$ gives rise to a norm on $X$ defined by $\|x\| = \sqrt{\langle x, x \rangle}$. The axioms of a scalar product yield the **Schwarz inequality**:

$$|\langle x, y \rangle| \leq \|x\| \|y\| \quad \text{for all } x \text{ and } y \in X,$$

as well as the **parallelogram law**:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \qquad x, y \in X. \tag{B.1}$$

A *Hilbert space* is an inner product space that is complete in the metric induced by the scalar product.

Given a Banach space $(X, \|\cdot\|)$, there is an inner product $\langle \cdot, \cdot \rangle$ such that $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space with norm $\|\cdot\|$ if and only if $\|\cdot\|$ satisfies (B.1). In this case the scalar product is uniquely determined by the formula

$$\langle x, y \rangle = \frac{\|x + y\|^2 - \|x - y\|^2}{4}, \quad x, y \in X.$$



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Two vectors $x$, $y$ in a Hilbert space $X$ are said to be *orthogonal*, and we write $x \perp y$, provided $\langle x, y\rangle = 0$. If $M$ is a subspace of $X$, we say that $x$ is orthogonal to $M$ if $\langle x, y\rangle = 0$ for all $y \in M$. The closed subspace $M^{\perp} = \{x \in X : \langle x, y\rangle = 0$ for all $y \in M\}$ is called the *orthogonal complement* of $M$.

A set $\mathcal{S}$ in $X$ is said to be an *orthogonal system* if every two distinct elements $x, y$ of $\mathcal{S}$ are *orthogonal*. The vectors in an orthogonal system are linearly independent. A set $\mathcal{S}$ is called *orthonormal* if it is orthogonal and $\|x\| = 1$ for each $x \in \mathcal{S}$.

Assume that $X$ is separable and let $\mathcal{C} = \{u_1, u_2, \dots\}$ be a dense subset of $X$. Using the *Gram–Schmidt procedure*, from $\mathcal{C}$ we can construct an orthonormal sequence $(v_n)_{n=1}^{\infty} \subset X$ that has the added feature of being *complete* (or *total*): $\langle x, v_k\rangle = 0$ for all $k$ implies $x = 0$. A *basis* of a Hilbert space is a complete orthogonal sequence.

Let $(v_k)_{k=1}^{\infty}$ be an orthonormal (not necessarily complete) sequence in a Hilbert space $X$. The inner products $(\langle x, v_k\rangle)_{k=1}^{\infty}$ are the *Fourier coefficients* of $x$ with respect to $(v_k)$.

Suppose that $x \in X$ can be expanded as a series $x = \sum_{k=1}^{\infty} a_k v_k$ for some scalars $(a_k)$. Then $a_k = \langle x, v_k\rangle$ for each $k \in \mathbb{N}$. In fact, for every $x \in X$, without any assumptions or knowledge about the convergence of the *Fourier series* $\sum_{k=1}^{\infty} \langle x, v_k\rangle v_k$, **Bessel's inequality** always holds:

$$\sum_{k=1}^{\infty} |\langle x, v_k\rangle|^2 \le \|x\|^2.$$

**B.1. Parseval's Identity.** *Let $(v_k)_{k=1}^{\infty}$ be an orthonormal sequence in an inner product space $X$. Then $(v_k)$ is complete if and only if*

$$\sum_{k=1}^{\infty} |\langle x, v_k\rangle|^2 = \|x\|^2 \quad \textit{for every } x \in X. \tag{B.2}$$

In turn, equation (B.2) is equivalent to saying that

$$x = \sum_{k=1}^{\infty} \langle x, v_k\rangle v_k$$

for each $x \in X$.

Bessel's inequality establishes that a necessary condition for a sequence of numbers $(a_k)_{k=1}^{\infty}$ to be the Fourier coefficients of an element $x \in X$ (relative to a fixed orthonormal system $(v_k)$) is that $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. The Riesz–Fischer theorem tells us that if $(v_k)$ is complete, this condition is also sufficient.

**B.2. The Riesz–Fischer Theorem.** *Let $X$ be a Hilbert space with complete orthonormal sequence $(v_k)_{k=1}^{\infty}$. Assume that $(a_k)_{k=1}^{\infty}$ is a sequence of real numbers such that $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. Then there exists an element $x \in X$ whose Fourier coefficients relative to $(v_k)$ are $(a_k)$.*

Thus from the isomorphic classification point of view, $\ell_2$ with the regular inner product of any two vectors $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$,

$$\langle a, b \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n},$$

is essentially the only separable Hilbert space. Indeed, combining B.1 with B.2, we obtain that the map from $X$ onto $\ell_2$ given by

$$x \mapsto (\langle x, v_k \rangle)_{k=1}^{\infty}$$

is a Hilbert space isomorphism (hence an isometry).

**B.3. Representation of Functionals on Hilbert Spaces.** *To every functional $x^*$ on a Hilbert space $X$ there corresponds a unique $x \in X$ such that $x^*(y) = \langle y, x \rangle$ for all $y \in X$. Moreover, $\|x^*\| = \|x\|$.*

Hilbert spaces are exceptional Banach spaces for many reasons. For instance, the Gram–Schmidt procedure and the fact that subsets of separable metric spaces are also separable yield that every subspace of a separable Hilbert space has an orthonormal basis. Another important property is that closed subspaces are always complemented, which relies on the existence of unique minimizing vectors:

**B.4. The Projection Theorem.** *Let $F$ be a nonempty, closed, convex subset of a Hilbert space $X$. For every $x \in X$ there exists a unique $\overline{y} \in F$ such that*

$$d(x, F) = \inf_{y \in F} \|x - y\| = \|x - \overline{y}\|.$$

*In particular, every nonempty, closed, convex set in a Hilbert space contains a unique element of smallest norm.*

If $F$ is a nonempty, closed, convex subset of a Hilbert space $X$, for every $x \in X$ the point $\overline{y}$ given by B.4, called the *projection of $x$ onto $F$*, is characterized by

$$\overline{y} \in F \quad \text{and} \quad \Re\langle x - \overline{y}, y - \overline{y} \rangle \leq 0 \quad \text{for all } y \in F.$$

The map $P_F : X \to F$ defined by $P_F(x) = \overline{y}$ is a contraction; that is,

$$\|P_F(x_1) - P_F(x_2)\| \leq \|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in X.$$

Therefore, it is continuous.

If $M$ is a closed subspace of $X$, then $P_M$ is a linear operator from $X$ onto $M$ and $P_M(x)$ is the unique $y \in X$ such that $y \in M$ and $x - y \in M^{\perp}$. We call $P_F$ the *orthogonal projection* from $X$ onto $M$. Thus, if $M$ is a closed subspace of a Hilbert space $X$, then $X = M \oplus M^{\perp}$.

# Appendix C
# Duality in $L_p(\mu)$: Results Related to Hölder's Inequality

Suppose $(\Omega, \Sigma, \mu)$ is a positive measure space. Given $1 \leq p \leq \infty$, let $1 \leq q \leq \infty$ be the conjugate exponent of $p$, i.e., $1/p + 1/q = 1$. Hölder's inequality establishes that if $f \in L_p(\mu)$ and $g \in L_q(\mu)$, then $fg \in L_1(\mu)$ and

$$\left| \int_\Omega fg \, d\mu \right| \leq \|f\|_p \|g\|_q.$$

One often needs to delve deeper into this inequality and use results concerning its optimality.

**C.1.** *Let $1 \leq p < \infty$. For every $f \in L_p(\mu)$ there is a function $g$ in the unit ball of $L_q(\mu)$ such that*

$$\|f\|_p = \int_\Omega fg \, d\mu.$$

From C.1 we get that $\|f\|_p$ (when it is finite and $p < \infty$) can be recovered from the action via the Lebesgue integral of the function $f$ on other measurable functions. This fact is true even when $p = \infty$ or $\|f\|_p = \infty$. To be precise, we have the following.

**C.2.** *Suppose that $\mu$ is a $\sigma$-finite measure and that $f$ is a measurable function. Then*

$$\|f\|_p = \sup \left\{ \int_\Omega fg \, d\mu \colon g \text{ simple such that } \|g\|_q \leq 1 \text{ and } fg \in L_1(\mu) \right\}.$$

Another consequence of Hölder's inequality is that for every $f \in L_p(\mu)$ we have a functional in $L_q(\mu)$ given by $g \mapsto \int_\Omega fg \, d\mu$ whose norm is not bigger than $\|f\|_p$. The Riesz representation theorem for this type of space establishes that every functional in $L_q(\mu)$ has the aforementioned form.


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**C.3. Riesz Representation Theorem.** *Let $1 < p \leq \infty$ and suppose that $x^* \in (L_q(\mu))^*$. Then there is $f \in L_p(\mu)$ with $\|f\|_p \leq \|x^*\|$ such that*

$$x^*(g) = \int_\Omega fg \, d\mu, \qquad g \in L_q(\mu).$$

Note that C.3 combined with Hölder's inequality provides a natural linear isometry between $L_p(\mu)$ and $(L_q(\mu))^*$ for $p > 1$.

# Appendix D
# Main Features of Finite-Dimensional Spaces

Suppose that $\mathcal{S} = \{x_1, \dots, x_n\}$ is a set of independent vectors in a normed space $X$ of any dimension. Using a straightforward compactness argument, it can be shown that there exists a constant $C > 0$ (depending only on $\mathcal{S}$) such that for every choice of scalars $\alpha_1, \dots, \alpha_n$ we have

$$C\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq |\alpha_1| + \cdots + |\alpha_n|.$$

This is the basic ingredient to obtain both D.1 and D.2.

**D.1. Operators on Finite-Dimensional Normed Spaces.** *Suppose that $T : X \to Y$ is a linear operator between the normed spaces $X$ and $Y$. If $X$ has finite dimension, then $T$ is bounded. In particular, every linear operator between normed spaces of the same finite dimension is an isomorphism.*

**D.2. Isomorphic Classification.** *Every two finite-dimensional normed spaces (over the same scalar field) of the same dimension are isomorphic.*

From D.2 one easily deduces the following facts:

- **Equivalence of norms.** *If $\|\cdot\|$ and $\|\cdot\|_0$ are two norms on a finite-dimensional vector space $X$, then they are equivalent. Consequently, if $\tau$ and $\tau_0$ are the respective topologies induced on $X$ by $\|\cdot\|$ and $\|\cdot\|_0$, then $\tau = \tau_0$.*
- **Completeness.** *Every finite-dimensional normed space is complete.*
- **Closedness of subspaces.** *The finite-dimensional linear subspaces of a normed space are closed.*

The **Heine–Borel Theorem** asserts that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded; combining this with D.2 we further deduce the following:

- **Compactness.** *Let $X$ be a finite-dimensional normed space and $A$ a subset of $X$. Then $A$ is compact if and only if $A$ is closed and bounded.*



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We know that the compact subsets of a Hausdorff topological space $X$ are closed and bounded. A general topological space $X$ is said to have the *Heine–Borel property* when the converse holds. The following lemma is not restricted to finite-dimensional, spaces and it is a source of interesting results in functional analysis, as for instance the characterization of the normed spaces that enjoy the Heine–Borel property, which we write as a corollary.

**D.3. Riesz's Lemma.** *Let $X$ be a normed space and $Y$ a closed proper subspace of $X$. Then for each real number $\theta \in (0, 1)$ there exists an $x_\theta \in S_X$ such that $\|y - x_\theta\| \geq \theta$ for all $y \in Y$.*

**D.4. Corollary.** *Let $X$ be a normed space. Then $X$ is finite-dimensional if and only if each closed bounded subset of $X$ is compact.*

Taking into account that in metric spaces compactness and sequential compactness are equivalent, we obtain the following:

**D.5. Corollary.** *Let $X$ be a normed space. Then $X$ is finite-dimensional if and only if every* bounded *sequence in $X$ has a convergent subsequence.*

# Appendix E
# Cornerstone Theorems of Functional Analysis

## E.1 The Hahn–Banach Theorem

**E.1. The Hahn–Banach Theorem (Real Case).** *Let $X$ be a real linear space, $Y \subset X$ a linear subspace, and $p : X \to \mathbb{R}$ a sublinear functional, i.e.,*

*(i) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$ ($p$ is subadditive), and*
*(ii) $p(\lambda x) \leq \lambda p(x)$ for all $x \in X$ and $\lambda \geq 0$ ($p$ is nonnegatively subhomogeneous).*

*Assume that we have a linear map $f : Y \to \mathbb{R}$ such that $f(y) \leq p(y)$ for all $y \in Y$. Then there exists a linear map $F : X \to \mathbb{R}$ such that $F|_Y = f$ and $F(x) \leq p(x)$ for all $x \in X$.*

**E.2. Normed-Space Version of the Hahn-Banach Theorem.** *Let $y^*$ be a bounded linear functional on a subspace $Y$ of a normed space $X$. Then there is $x^* \in X^*$ such that $\|x^*\| = \|y^*\|$ and $x^*|_Y = y^*$.*

Let us note that this theorem says nothing about the *uniqueness* of the extension unless $Y$ is a dense subspace of $X$. Note also that $Y$ need not be closed.

**E.3. Separation of Points from Closed Subspaces.** *Let $Y$ be a closed subspace of a normed space $X$. Suppose that $x \in X \setminus Y$. Then there exists $x^* \in X^*$ such that $\|x^*\| = 1$, $x^*(x) = d(x, Y) = \inf\{\|x - y\| : y \in Y\}$, and $x^*(y) = 0$ for all $y \in Y$.*

**E.4. Corollary.** *Let $X$ be a normed linear space and $x \in X$, $x \neq 0$. Then there exists $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*(x) = \|x\|$.*

**E.5. Separation of Points.** *Let $X$ be a normed linear space and $x, y \in X$, $x \neq y$. Then there exists $x^* \in X^*$ such that $x^*(x) \neq x^*(y)$.*



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**E.6. Corollary.** *Let $X$ be a normed linear space. For every $x \in X$ we have*

$$\|x\| = \sup\Big\{|x^*(x)| : x^* \in X^*, \|x^*\| \leq 1\Big\}.$$

**E.7. Corollary.** *Let $X$ be a normed linear space. If $X^*$ is separable, then so is $X$.*

## E.2 Baire's Category Theorem and Its Consequences

A subset $E$ of a metric space $X$ is *nowhere dense* in $X$ (or *rare*) if its closure $\overline{E}$ has empty interior. Equivalently, $X$ is nowhere dense in $X$ if and only if $X \setminus \overline{E}$ is (everywhere) dense in $X$. The sets of the *first category in $X$* (or, also, *meager in $X$*) are those that are the union of countably many sets each of which is nowhere dense in $X$. A subset of $X$ that is not of the first category is said to be of the *second category in $X$* (or *nonmeager in $X$*). This density-based approach to give a topological meaning to the size of a set is due to Baire. Nowhere dense sets would be the "very small" sets in the sense of Baire, whereas the sets of the second category would play the role of the "large" sets in the sense of Baire in a metric (or more generally in any topological) space.

**E.8. Baire's Category Theorem.** *Let $X$ be a* complete *metric space. Then the intersection of every countable collection of dense open subsets of $X$ is dense in $X$.*

Let $\{E_i\}$ be a countable collection of nowhere dense subsets of a complete metric space $X$. For each $i$ the set $U_i = X \setminus \overline{E_i}$ is dense in $X$; hence by Baire's theorem it follows that $\cap U_i \neq \emptyset$. Taking complements, we deduce that $X \neq \cup E_i$. That is, *a complete metric space $X$ cannot be written as a countable union of nowhere dense sets in $X$.* Therefore, nonempty complete metric spaces are of the second category in themselves.

A function $f$ from a topological space $X$ into a topological space $Y$ is *open* if $f(V)$ is an open set in $Y$ whenever $V$ is open in $X$.

**E.9. Open Mapping Theorem.** *Let $X$ and $Y$ be Banach spaces and let $T : X \to Y$ be a bounded linear operator.*

*(i) If $\delta B_Y = \{y \in Y : \|y\| < \delta\} \subseteq \overline{T(B_X)}$ for some $\delta > 0$, then $T$ is an open map.*
*(ii) If $T$ is onto, then the hypothesis of (i) holds. That is, every bounded operator from a Banach space onto a Banach space is open.*

**E.10. Corollary.** *If $X$ and $Y$ are Banach spaces and $T$ is a continuous linear operator from $X$ onto $Y$ that is also one-to-one, then $T^{-1} : Y \to X$ is a continuous linear operator.*

**E.11. Closed Graph Theorem.** *Let $X$ and $Y$ be Banach spaces. Suppose that $T : X \to Y$ is a linear mapping of $X$ into $Y$ with the following property: whenever $(x_n) \subset X$ is such that both $x = \lim x_n$ and $y = \lim Tx_n$ exist, it follows that $y = Tx$. Then $T$ is continuous.*

**E.12. Uniform Boundedness Principle.** *Suppose $(T_\gamma)_{\gamma\in\Gamma}$ is a family of bounded linear operators from a Banach space $X$ into a normed linear space $Y$. If* $\sup\{\|T_\gamma x\| : \gamma \in \Gamma\}$ *is finite for each $x$ in $X$, then* $\sup\{\|T_\gamma\| : \gamma \in \Gamma\}$ *is finite.*

**E.13. Banach–Steinhaus Theorem.** *Let $(T_n)_{n=1}^\infty$ be a sequence of continuous linear operators from a Banach space $X$ into a normed linear space $Y$ such that*

$$T(x) = \lim_n T_n(x)$$

*exists for each $x$ in $X$. Then $T$ is continuous.*

**E.14. Partial Converse of the Banach–Steinhaus Theorem.** *Let $(T_n)_{n=1}^\infty$ be a sequence of continuous linear operators from a Banach space $X$ into a normed linear space $Y$ such that* $\sup_n \|T_n\| < \infty$. *If $T : X \to Y$ is another operator, then the subspace*

$$\{x \in X\colon \|T_n(x) - T(x)\| \to 0\}$$

*is norm-closed in $X$.*

# Appendix F
# Convex Sets and Extreme Points

Let $S$ be a subset of a vector space $X$. We say that $S$ is *convex* if $\lambda x + (1 - \lambda)y \in S$ whenever $x, y \in S$ and $0 \leq \lambda \leq 1$. Notice that every subspace of $X$ is convex, and if a subset $S$ is convex, so is each of its translates $x + S = \{x + y : y \in S\}$. If $X$ is a normed space and $S$ is convex, then so is its norm-closure $\overline{S}$.

Given a real linear space $X$, let $F$ and $K$ be two subsets of $X$. A linear functional $f$ on $X$ is said to *separate $F$ and $K$* if there exists a number $\alpha$ such that $f(x) > \alpha$ for all $x \in F$ and $f(x) < \alpha$ for all $x \in K$. As an application of the Hahn–Banach theorem we have the following:

**F.1. Separation of Convex Sets.** *Let $X$ be a locally convex space and let $K$, $F$ be disjoint closed convex subsets of $X$. Assume that $K$ is compact. Then there exists a continuous linear functional $f$ on $X$ that separates $F$ and $K$.*

The *convex hull* of a subset $S$ of a linear space $X$, denoted $\operatorname{co}(S)$, is the smallest convex set that contains $S$. Obviously, such a set always exists by since $X$ is convex and the arbitrary intersection of convex sets is convex, and can be described analytically by

$$\operatorname{co}(S) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \ : \ (x_i)_{i=1}^{n} \subset S,\ \lambda_i \geq 0 \text{ and } \sum_{i=1}^{n} \lambda_i = 1; n \in \mathbb{N} \right\}.$$

If $X$ is equipped with a topology $\tau$, then $\overline{\operatorname{co}}^{\tau}(S)$ will denote the *closed convex hull of $S$*, i.e., the smallest $\tau$-closed convex set that contains $S$ (that is, the intersection of all $\tau$-closed convex sets that include $S$). The closed convex hull of $S$ with respect to the norm topology will be simply denoted by $\overline{\operatorname{co}}(S)$. Let us observe that in general, $\overline{\operatorname{co}}^{\tau}(S) \neq \overline{\operatorname{co}(S)}^{\tau}$ but that equality holds if $\tau$ is a vector topology on $X$.

If $S$ is convex, a point $x \in S$ is an *extreme point of $S$* if whenever $x = \lambda x_1 + (1 - \lambda)x_2$ with $0 < \lambda < 1$, then $x = x_1 = x_2$. Equivalently, $x$ is an extreme point of $S$ if and only if $S \setminus \{x\}$ is still convex. We let $\partial_e(S)$ denote the set of extreme points of $S$.



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**F.2. The Krein–Milman Theorem.** *Suppose $X$ is a locally convex topological vector space. If $K$ is a nonempty compact convex set in $X$, then $K$ is the closed convex hull of its extreme points. In particular, each convex nonempty compact subset of a locally convex topological vector space has an extreme point.*

**F.3. Milman's Theorem.** *Suppose $X$ is a locally convex topological vector space. Let $K$ be a nonempty closed and compact*[1] *set. If $u$ is an extreme point of $\overline{co}(K)$ then $u \in K$.*

**F.4. Schauder's Fixed Point Theorem.** *Let $K$ be a nonempty closed convex subset of a Banach space $X$. Suppose $T : X \to X$ is a continuous linear operator such that $T(K) \subset K$ and $T(K)$ is compact. Then there exists at least one point $x$ in $K$ such that $Tx = x$.*

[1]Notice that we are not assuming that $X$ has any topological separation properties. If $X$ is Hausdorff, then every compact subset of $X$ is automatically closed.

# Appendix G
# The Weak Topologies

Let $X$ be a normed vector space. The *weak topology of* $X$, usually called the $w$-topology or $\sigma(X, X^*)$-topology, is the weakest topology on $X$ such that each $x^* \in X^*$ is continuous. This topology is linear (addition of vectors and multiplication of vectors by scalars are continuous), and a base of neighborhoods of $0 \in X$ is given by the sets of the form

$$V_\epsilon(0; x_1^*, \dots, x_n^*) = \big\{x \in X : |x_i^*(x)| < \epsilon,\ i = 1, \dots, n\big\},$$

where $\epsilon > 0$ and $\{x_1^*, \dots, x_n^*\}$ is any finite subset of $X^*$. Obviously this defines a non-locally bounded, locally convex topology on $X$. One can also give an alternative description of the weak topology via the notion of convergence of nets: take a net $(x_\alpha)$ in $X$; we will say that $(x_\alpha)$ *converges weakly to* $x_0 \in X$, and we write $x_\alpha \overset{w}{\to} x_0$, if for each $x^* \in X^*$,

$$x^*(x_\alpha) \to x^*(x_0).$$

Next we summarize some elementary properties of the weak topology of a normed vector space $X$, noting that it is in the setting of infinite-dimensional spaces that the different natures of the weak and norm topologies become apparent.

- *If $X$ is infinite-dimensional, every nonempty weak open set of $X$ is unbounded.*
- *A subset $S$ of $X$ is norm-bounded if and only if $S$ is weakly bounded (that is, $\{x^*(a) : a \in S\}$ is a bounded set in the scalar field of $X$ for every $x^* \in X^*$).*
- *If the weak topology of $X$ is metrizable, then $X$ is finite-dimensional.*
- *If $X$ is infinite-dimensional, then the weak topology of $X$ is not complete.*
- *A linear functional on $X$ is norm-continuous if and only if it is continuous with respect to the weak topology.*



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- *Let $T : X \to Y$ be a linear map. Then $T$ is weak-to-weak continuous if and only if $x^* \circ T \in X^*$ for every $x^* \in X^*$.*
- *A linear map $T : X \to Y$ is norm-to-norm continuous if and only if $T$ is weak-to-weak continuous.*

**G.1. Mazur's Theorem.** *If $\mathcal{C}$ is a convex set in a normed space $X$, then the closure of $\mathcal{C}$ in the norm topology, $\overline{\mathcal{C}}$, coincides with $\overline{\mathcal{C}}^w$, the closure of $\mathcal{C}$ in the weak topology.*

**G.2. Corollary.** *If $Y$ is a linear subspace of a normed space $X$, then $\overline{Y} = \overline{Y}^w$.*

**G.3. Corollary.** *If $S$ is any subset of a normed space $X$, then $\overline{co}(S) = \overline{co}^w(S)$.*

**G.4. Corollary.** *Let $(x_n)$ be a sequence in a normed space $X$ that converges weakly to $x \in X$. Then there is a sequence of convex combinations of the $x_n$, $y_k = \sum_{i=k}^{N(k)} \lambda_i x_i$, $k = 1, 2, \ldots$, such that $\|y_k - x\| \to 0$.*

Let us turn now to the weak* topology on a dual space $X^*$. Let $j : X \to X^{**}$ be the natural embedding of a Banach space in its second dual, given by $j(x)(x^*) = x^*(x)$. As usual, we identify $X$ with $j(X) \subset X^{**}$. The weak* topology on $X^*$, called the $w^*$-topology or $\sigma(X^*, X)$-topology, is the topology induced on $X^*$ by $X$, i.e., it is the weakest topology on $X^*$ that makes all linear functionals in $X \subset X^{**}$ continuous.

Like the weak topology, the weak* topology is a locally convex Hausdorff linear topology, and a base of neighborhoods at $0 \in X^*$ is given by the sets of the form

$$W_\epsilon(0; x_1, \ldots, x_n) = \{x^* \in X^* : |x^*(x_i)| < \varepsilon \text{ for } i = 1, \ldots, n\},$$

for any finite subset $\{x_1, \ldots, x_n\} \in X$ and any $\epsilon > 0$. Thus by translation we obtain the neighborhoods of other points in $X^*$.

As before, we can equivalently describe the weak* topology of a dual space in terms of convergence of nets: we say that a net $(x_\alpha^*) \subset X^*$ *converges weak* to* $x_0^* \in X^*$, and we write $x_\alpha^* \overset{w^*}{\to} x_0^*$, if for each $x \in X$,

$$x_\alpha^*(x) \to x_0^*(x).$$

Of course, the weak* topology of $X^*$ is no bigger than its weak topology, and in fact, $\sigma(X^*, X) = \sigma(X^*, X^{**})$ if and only if $j(X) = X^{**}$ (that is, if and only if $X$ is reflexive). Notice also that when we identify $X$ with $j(X)$ and consider $X$ a subspace of $X^{**}$, this is not simply an identification of sets; actually,

$$(X, \sigma(X, X^*)) \overset{j}{\longrightarrow} (X, \sigma(X^{**}, X^*))$$

is a linear homeomorphism. Analogously to the weak topology, dual spaces are never $w^*$-metrizable or $w^*$-complete unless the underlying space is finite-dimensional. The most important feature of the weak* topology is the following

compactness property, basic to modern functional analysis, which was discovered by Banach in 1932 for separable spaces and was extended to the general case by Alaoglu in 1940.

**G.5. The Banach–Alaoglu Theorem.** *If $X$ is a normed linear space, then the set $B_{X^*} = \{x^* \in X^* : \|x^*\| \le 1\}$ is weak$^*$-compact.*

**G.6. Corollary.** *The closed unit ball $B_{X^*}$ of the dual of a normed space $X$ is the weak$^*$ closure of the convex hull of the set of its extreme points:*

$$B_{X^*} = \overline{co}^{w^*}\big(\partial_e(B_{X^*})\big).$$

If $X$ is a nonreflexive Banach space, then $X$ cannot be dense or weak dense in $X^{**}$. However, it turns out that $X$ must be weak$^*$ dense in $X^{**}$, as deduced from the next useful result, which is a consequence of the fact that the weak$^*$ dual of $X^*$ is $X$.

**G.7. Goldstine's Theorem.** *Let $X$ be a normed space. Then $B_X$ is weak$^*$ dense in $B_{X^{**}}$.*

**G.8. The Banach–Dieudonné Theorem.** *Let $\mathcal{C}$ be a convex subset of a dual space $X^*$. Then $\mathcal{C}$ is weak$^*$-closed if and only if $\mathcal{C} \cap \lambda B_{X^*}$ is weak$^*$-closed for every $\lambda > 0$.*

**G.9. Proposition.** *Let $X$ and $Y$ be normed spaces and suppose that $T : X \to Y$ is a linear mapping.*

*(i) If $T$ is norm-to-norm continuous, then its adjoint $T^* : Y^* \to X^*$ is weak$^*$-to-weak$^*$ continuous.*
*(ii) If $R : Y^* \to X^*$ is a weak$^*$-to-weak$^*$ continuous operator, then there is $T : X \to Y$ norm-to-norm continuous such that $T^* = R$.*

**G.10. Corollary.** *Let $y^{**} \in Y^{**}$ be such that $y^{**}|_{B_{Y^*}}$ is weak$^*$ continuous. Then $y^{**} \in Y$, i.e., there exists $y \in Y$ such that $y^{**} = j_Y(y)$.*

**G.11. Corollary.** *Suppose $X$, $Y$ are normed spaces. Then every weak$^*$-to-weak$^*$ continuous linear operator from $X^*$ to $Y^*$ is norm-to-norm continuous.*

Let us point out here that the converse of Corollary G.11 is not true in general.

# Appendix H
# Weak Compactness of Sets and Operators

A subset $A$ of a normed space $X$ is said to be *[relatively] weakly compact* if [the weak closure of] $A$ is compact in the weak topology of $X$.

**H.1. Proposition.** *If $K$ is a weakly compact subset of a normed space $X$ then $K$ is norm-closed and norm-bounded.*

**H.2. Proposition.** *Let $X$ be a Banach space. Then $B_X$ is weakly compact if and only if $X$ is reflexive.*

This proposition yields the first elementary examples of weakly compact sets, which we include in the next corollary.

**H.3. Corollary.** *Let $X$ be a reflexive space.*

*(i) If $A$ is a bounded subset of $X$, then $A$ is relatively weakly compact.*
*(ii) If $A$ is a convex, bounded, norm-closed subset of $X$, then $A$ is weakly compact.*
*(iii) If $T : X \to Y$ is a continuous linear operator, then $T(B_X)$ is weakly compact in $Y$.*

When $X$ is not reflexive, in order to check whether a given set is relatively weakly compact, we can employ the characterization provided by the following result.

**H.4. Proposition.** *A subset $A$ of a Banach space $X$ is relatively weakly compact if and only if it is norm-bounded and the $\sigma(X^{**}, X^*)$-closure of $A$ in $X^{**}$ is contained in $A$.*

The most important result on weakly compact sets is the Eberlein–Šmulian theorem, which we included in Chapter 1 (Theorem 1.6.3). This is indeed a very surprising result; when we consider $X$ endowed with the norm topology, in order that every bounded sequence in $X$ have a convergent subsequence, it is necessary and sufficient that $X$ be finite-dimensional. If $X$ is infinite-dimensional, the weak topology is not metrizable, and thus sequential extraction arguments would not seem



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to apply in order to decide whether a subset of $X$ is weakly compact. The Eberlein–Šmulian theorem, oddly enough, tells us that a bounded subset $A$ *is weakly compact if and only if every sequence in $A$ has a subsequence weakly convergent to some point of $A$*.

A bounded linear operator $T : X \to Y$ is said to be *weakly compact* if the set $T(B_X)$ is relatively weakly compact, that is, if $\overline{T(B_X)}$ is weakly compact. Since every bounded subset of $X$ is contained in some multiple of the unit ball of $X$, we have that $T$ is weakly compact if and only if it maps bounded sets into relatively weakly compact sets. Using the Eberlein–Šmulian theorem, one can further state that $T : X \to Y$ is weakly compact if and only if for every bounded sequence $(x_n)_{n=1}^{\infty} \subset X$, the sequence $(Tx_n)_{n=1}^{\infty}$ has a weakly convergent subsequence.

**H.5. Gantmacher's Theorem.** *Suppose $X$ and $Y$ are Banach spaces and let $T : X \to Y$ be a bounded linear operator. Then:*

*(i) $T$ is weakly compact if and only if the range of its double adjoint $T^{**} : X^{**} \to Y^{**}$ is in $Y$, i.e., $T^{**}(X^{**}) \subset Y$.*
*(ii) $T$ is weakly compact if and only if its adjoint $T^* : Y^* \to X^*$ is weak$^*$-to-weak continuous.*
*(iii) $T$ is weakly compact if and only if its adjoint $T^*$ is.*

The next remarks follow easily from what has been said in this section:

- Let $T: X \to Y$ be an operator. If $X$ or $Y$ is reflexive, then $T$ is weakly compact.
- The identity map on a nonreflexive Banach space is never weakly compact.
- A Banach space $X$ is reflexive if and only if $X^*$ is.

# Appendix I
# Basic Probability in Use

A *random variable* is a real-valued measurable function on some probability space $(\Omega, \Sigma, \mathbb{P})$. The *expectation* (or *mean*) of a random variable $f$ is defined by

$$\mathbb{E}f = \int_{\Omega} f(\omega)\, d\mathbb{P}(\omega).$$

A finite set of random variables $\{f_j\}_{j=1}^n$ on the same probability space is *independent* if

$$\mathbb{P}\bigcap_{j=1}^{n}\big(f_j \in B_j\big) = \prod_{j=1}^{n}\mathbb{P}(f_j \in B_j)$$

for all Borel sets $B_j$. Therefore, if $(f_j)_{j=1}^n$ are independent, then $\mathbb{E}\big(f_1 f_2 \cdots f_n\big) = \mathbb{E}(f_1)\mathbb{E}(f_2)\cdots\mathbb{E}(f_n)$. An arbitrary set of random variables is said to be independent if every finite subcollection of the set is independent.

If $f$ is a real random variable on some probability measure space $(\Omega, \Sigma, \mathbb{P})$, the *distribution* of $f\colon \Omega \to \mathbb{R}$ is the probability measure $\mu_f$ on $\mathbb{R}$ given by

$$\mu_f(B) = \mathbb{P}(f^{-1}B)$$

for every Borel set $B$ of $\mathbb{R}$. The random variable $f$ is called *symmetric* if $f$ and $-f$ have the same distribution.

Conversely, for each probability measure $\mu$ on $\mathbb{R}$ there exist real random variables $f$ with $\mu_f = \mu$, and the formula

$$\int_{\Omega} F(f(\omega))\, d\mathbb{P}(\omega) = \int_{-\infty}^{\infty} F(x)\, d\mu_f(x) \tag{I.1}$$

holds for every positive Borel function $F : \mathbb{R} \to \mathbb{R}$.



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The *characteristic function* $\phi_f$ of a random variable $f$ is the function $\phi_f : \mathbb{R} \to \mathbb{C}$ defined by $\phi_f(t) = \mathbb{E}(e^{itf})$. This is related to $\mu_f$ via the Fourier transform:

$$\hat{\mu}_f(-t) = \int_{\mathbb{R}} e^{itx} d\mu_f(x) = \phi_f(t).$$

In particular, $\phi_f$ *determines* $\mu_f$, i.e., if $f$ and $g$ are two random variables (possibly on different probability spaces) with $\phi_f = \phi_g$, then $\mu_f = \mu_g$. Here are some other basic useful properties of characteristic functions:

- $\phi_f(-t) = \overline{\phi_f(t)}$;
- $\phi_{cf+d}(-t) = e^{idt}\phi_f(ct)$, for $c, d$ constants;
- $\phi_{f+g} = \phi_f\phi_g$ if $f$ and $g$ are independent.

**I.1.** If $f_1, \ldots, f_n$ are independent random variables (not necessarily equally distributed) on some probability space, then we can exploit independence to compute the characteristic function of any linear combination $\sum_{j=1}^n a_j f_j$:

$$\mathbb{E}\big(e^{it\sum_{j=1}^n a_j f_j}\big) = \prod_{j=1}^n \mathbb{E}\big(e^{ita_j f_j}\big) = \prod_{j=1}^n \phi_{f_j}(a_j t). \tag{I.2}$$

Suppose we are given a probability measure $\mu$ on $\mathbb{R}$. The random variable $f(x) = x$ has distribution $\mu$ with respect to the probability space $(\mathbb{R}, \mu)$. Next consider the countable product space $\mathbb{R}^{\mathbb{N}}$ with the product measure $\mathbb{P} = \mu \times \mu \times \cdots$. Then $(\mathbb{R}^{\mathbb{N}}, \mathbb{P})$ is also a probability space, and the coordinate maps $f_j \colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$,

$$f_j(x_1, \ldots, x_n, \ldots) = x_j,$$

are identically distributed random variables on $\mathbb{R}^{\mathbb{N}}$ with distribution $\mu$. Moreover, the random variables $(f_j)_{j=1}^\infty$ are independent. Although we created the sequence of functions $(f_j)_{j=1}^\infty$ on $(\mathbb{R}^{\mathbb{N}}, \mathbb{P})$, we might just as well have worked on $([0,1], \mathcal{B}, \lambda)$. As we discuss in Section 5.1, there is a Borel isomorphism $\sigma : \mathbb{R}^{\mathbb{N}} \to [0,1]$ that preserves measure, that is,

$$\lambda(B) = \mathbb{P}(\sigma^{-1}B), \qquad B \in \mathcal{B},$$

and the functions $(f_j \circ \sigma^{-1})_{j=1}^\infty$ have exactly the same properties on $[0, 1]$.

This remark, in particular, allows us to pick an infinite sequence of independent identically distributed random variables on $[0, 1]$ with a given distribution.

**I.2. Gaussian Random Variables.** The *standard normal distribution* is given by the measure on $\mathbb{R}$,

$$d\mu_{\mathcal{G}} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

We will call any random variable with this distribution a *(normalized) Gaussian*. In this case we have

$$\hat{\mu}_{\mathcal{G}}(-t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx - x^2/2}\, dx = e^{-t^2/2},$$

so the characteristic function of a Gaussian is $e^{-t^2/2}$.

# Appendix J
# Generalities on Ultraproducts

The idea of ultraproducts in Banach spaces crystallized in the work of Dacunha-Castelle and Krivine [54]. Ultraproducts serve as an appropriate vehicle to study finite representability by infinite-dimensional methods. Let us recall, first, a few definitions.

**J.1. Filters.** If $\mathcal{I}$ is any infinite set, a *filter* on $\mathcal{I}$ is a nonempty subset $\mathcal{F}$ of $\mathcal{P}(\mathcal{I})$ satisfying the following properties:

- $\emptyset \notin \mathcal{F}$.
- If $A \subset B$ and $A \in \mathcal{F}$ then $B \in \mathcal{F}$.
- If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

Given a topological space $X$, a function $f : \mathcal{I} \to X$ is said to *converge to* $\xi$ *through* $\mathcal{F}$, and we write

$$\lim_{\mathcal{F}} f(x) = \xi,$$

if $f^{-1}(U) \in \mathcal{F}$ for every open set $U$ containing $\xi$.

We will be primarily interested in the case $\mathcal{I} = \mathbb{N}$, so that a function on $\mathbb{N}$ is simply a sequence.

**J.2. Examples of Filters on $\mathbb{N}$.**

*(a)* *For each $n \in \mathbb{N}$ we can define the filter $\mathcal{F}_n = \{A : \ n \in A\}$. Then a sequence $(\xi_k)_{k=1}^{\infty}$ converges to $\xi$ through $\mathcal{F}_n$ if and only if $\xi_n = \xi$.*

*(b)* *Let us consider the filter $\mathcal{F}_\infty = \{A\colon [n, \infty) \subset A \quad \text{for some} \quad n \in \mathbb{N}\}$. Then $\lim_{\mathcal{F}_\infty} \xi_n = \xi$ if and only if $\lim_{n\to\infty} \xi_n = \xi$. More generally, if $(\mathcal{I}, \leq)$ is a directed set, there is a minimum filter on $\mathcal{I}$ containing all sets of the form $\{i \in \mathcal{I}\colon i \geq j\}$ for $j \in \mathcal{I}$.*



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**J.3. Ultrafilters.** An *ultrafilter* $\mathcal{U}$ is a maximal filter with respect to inclusion, i.e., a filter that is not properly contained in any larger filter. By Zorn's lemma, every filter is contained in an ultrafilter. Ultrafilters are characterized by one additional property:

- If $A \in \mathcal{P}(\mathcal{I})$, then either $A \in \mathcal{U}$ or $\mathcal{I} \setminus A \in \mathcal{U}$.

**J.4. Convergence Through Ultrafilters.** *Let $\mathcal{U}$ be an ultrafilter, $X$ a topological space, and $f : \mathcal{U} \to X$ a function such that $f(\mathcal{U})$ is relatively compact. Then $f$ converges through $\mathcal{U}$. In particular, every bounded real-valued function converges through $\mathcal{U}$.*

*Proof.* Indeed, choose a compact subset $K$ in $X$ such that $f(x) \in K$ for all $x \in \mathcal{U}$ and suppose that $f$ does not converge through $\mathcal{U}$. Then for every $\xi \in K$ we can find an open set $U_\xi$ containing $\xi$ such that $f^{-1}(U_\xi) \notin \mathcal{U}$. Using compactness, we can find a finite set $\{\xi_1, \dots, \xi_n\} \subset K$ such that $K \subset \cup_{j=1}^n U_{\xi_j}$. Now $f^{-1}(X \setminus U_{\xi_j}) \in \mathcal{U}$ for each $j$, since it is an ultrafilter. But then the properties of filters imply that the intersection $\cap_{j=1}^n f^{-1}(X \setminus U_{\xi_j}) \in \mathcal{U}$; however, this set is empty, and we have a contradiction. □

**J.5. Principal and Nonprincipal (or Free) Ultrafilters.** Let us restrict again to $\mathbb{N}$. The filters $\mathcal{F}_n$ in Example J.2 are in fact ultrafilters; these are called the *principal ultrafilters*. Every other ultrafilter must contain $\mathcal{F}_\infty$; these are the *nonprincipal (or free) ultrafilters*.

The following are elementary properties of limits of sequences through a free ultrafilter $\mathcal{U}$ in a Banach space:

- $\lim_{\mathcal{U}}(x_n + y_n) = \lim_{\mathcal{U}} x_n + \lim_{\mathcal{U}} y_n$.
- If $(\alpha_n)$ is a bounded sequence of scalars and $\lim_{\mathcal{U}} x_n = 0$, then $\lim_{\mathcal{U}} \alpha_n x_n = 0$.
- If $\lim_{n\to\infty} x_n = x$, then $\lim_{\mathcal{U}} x_n = x$.

**J.6. Ultraproducts of Banach Spaces.** Suppose $X$ is a Banach space and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. We consider the $\ell_\infty$-product $\ell_\infty(X)$ and define on it a seminorm by

$$\|(x_n)_{n=1}^\infty\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_n\|.$$

Then $\|(x_n)_{n=1}^\infty\|_{\mathcal{U}} = 0$ if and only if $(x_n)_{n=1}^\infty$ belongs to the closed subspace $c_{0,\mathcal{U}}(X)$ of $\ell_\infty(X)$ of all $(x_n)_{n=1}^\infty$ such that $\lim_{\mathcal{U}} \|x_n\| = 0$. It is readily verified that $\|\cdot\|_{\mathcal{U}}$ induces the quotient norm on the quotient space $X_{\mathcal{U}} = \ell_\infty(X)/c_{0,\mathcal{U}}(X)$. This space is called an *ultraproduct* or *ultrapower* of $X$. The class representative in $X_{\mathcal{U}}$ of an element $(x_n)_{n=1}^\infty$ in $\ell_\infty(X)$ will be written $(x_n)_{\mathcal{U}}$.

It is, of course, possible to define ultraproducts using ultrafilters on sets $\mathcal{I}$ other than $\mathbb{N}$, and this is useful for nonseparable Banach spaces. For our purposes the natural numbers will suffice.

**J.7. Complementability of Reflexive Spaces in Their Ultrapowers.** Given a Banach space $X$ and a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, let $\iota_X : X \mapsto X_{\mathcal{U}}$ be the natural injection given by $x \mapsto (x)_{\mathcal{U}}$. Now consider the bounded linear operator $Q_X : X_{\mathcal{U}} \to X^{**}$ defined by $Q_X((x_n)_{\mathcal{U}}) = \lim_{\mathcal{U}} x_n$. We have $Q_X \circ \iota_X = j_X$, where $j_X$ denotes the canonical embedding of $X$ into its second dual $X^{**}$. Therefore if $X$ is complemented in $X^{**}$, then $X$ is complemented in $X_{\mathcal{U}}$. In particular, if $X$ is reflexive, then $X$ is complemented in $X_{\mathcal{U}}$.

**J.8. Remark.** One of the virtues of the ultraproduct technique is that passing from Banach spaces to their ultraproducts may preserve additional structures. For example, we know from Dacunha-Castelle and Krivine [54] that the property of being an $L_p(\mu)$ space for some $1 \leq p < \infty$ or some $\mathcal{C}(K)$ is stable under the formation of ultraproducts.

# Appendix K
# The Bochner Integral Abridged

Throughout this section, $(\Omega, \Sigma, \mu)$ will be a positive measure space, and $X$ will denote a Banach space.

**K.1. Strong Measurability**. A function $f\colon \Omega \to X$ is said to be *strongly measurable* if there is $g\colon \Omega \to X$ such that

(i) $f = g$ almost everywhere;
(ii) $g^{-1}(A) = \{\omega \in \Omega\colon g(\omega) \in A\} \in \Sigma$ for every open set $A \subset X$;
(iii) $g(\Omega) = \{g(\omega)\colon \omega \in \Omega\}$ is a separable subset of $X$.

If $f$ satisfies (i) and (ii), then the norm function $\|f\|$ coincides almost everywhere with some nonnegative measurable function, so that we can safely define

$$\|f\|_1 := \int_\Omega \|f(\omega)\| \, d\mu(\omega).$$

We will denote by $L_1(\mu, X)$ the normed space of all strongly measurable functions $f\colon \Omega \to X$ such that $\|f\|_1 < \infty$, modulo almost everywhere zero functions. A function in this space $L_1(\mu, X)$ is called *Bochner integrable*.

The Lebesgue dominated convergence theorem still holds in this setting:

**K.2. The Dominated Convergence Theorem.** *Let $(f_n)_{n=1}^\infty$ be a sequence of strongly measurable functions and let $f\colon \Omega \to X$ be a function. Suppose that*

*(a)* $\lim_n f_n = f$ *a.e., and*
*(b)* $\int_\Omega \sup_n \|f_n(\omega)\| \, d\mu(\omega) < \infty$.

*Then the functions $f$ and $f_n$ belong to $L_1(\mu, X)$ for all $n \in \mathbb{N}$, and*

$$\lim_{n\to\infty} f_n = f \text{ in } L_1(\mu, X).$$



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The dominated convergence theorem yields that $L_1(\mu, X)$ is a Banach space.

**K.3. Density of the Simple Functions in** $L_1(\mu, X)$. *Consider the space of simple Bochner-integrable functions,*

$$\mathcal{S}(\mu, X) = \left\{ \sum_{j=1}^{n} x_j \chi_{A_j} : x_j \in X, A_j \in \Sigma, \mu(A_j) < \infty, n \in \mathbb{N} \right\}.$$

*Then* $\mathcal{S}(\mu, X)$ *is a dense subspace of* $L_1(\mu, X)$.

*Proof.* Let $f \in L_1(\mu, X)$. Choose $g$ fulfilling (i), (ii), and (iii) in the definition of strong measurability. Pick a sequence $(x_n)_{n=1}^\infty$ in $X \setminus \{0\}$ such that $g(\Omega)$ is contained in the closure of $\{x_n : n \in \mathbb{N}\}$. Choose also a sequence of positive simple functions $(\varphi_k)_{n=1}^\infty$ such that $\varphi_k \le \|g\|$ and $\lim_k \varphi_k = \|g\|$. For every $k \in \mathbb{N}$ there is a partition of $\Omega$ into measurable sets $(A_{n,k})_{n=1}^\infty$ such that

$$\|g(\omega) - x_n\| \le \frac{1}{k}, \quad \omega \in A_{n,k}.$$

Define

$$f_k = \sum_{k=1}^{\infty} \frac{x_n}{\|x_n\|} \min\{\|x_n\|, 2\varphi_k\} \chi_{A_{n,k}}.$$

Then $\|f_k(\omega)\| \le 2\|g(\omega)\|$ for all $\omega \in \Omega$, and $\lim_k f_k(\omega) = g(\omega)$. Since the range of $f_k$ is countable, $f_k$ is strongly measurable. By the dominated convergence theorem, $f_k \in L_1(\mu, X)$ for every $k$, and $\lim_k \|f - f_k\|_1 = 0$. Given $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $\|f - f_k\|_1 \le \varepsilon/2$. If we write

$$f_k = \sum_{j=1}^{\infty} y_j \chi_{B_j},$$

where $y_j \in X$ and $B_j \in \Sigma$ are mutually disjoint sets with $\mu(B_j) < \infty$, we have that $\lim_N \|f_k - \sum_{j=1}^N y_j \chi_{B_j}\|_1 = 0$. Then, there is $N \in \mathbb{N}$ such that $s = \sum_{j=1}^N y_j \chi_{B_j} \in \mathcal{S}(\mu, X)$ satisfies $\|f_k - s\|_1 \le \varepsilon/2$. By the triangle inequality, $\|f - s\|_1 \le \varepsilon$. □

**K.4. Definition of the Bochner Integral**. The mapping

$$\mathcal{I} : \mathcal{S}(\mu, X) \to X, \qquad \sum_{j=1}^{n} x_j \chi_{A_j} \mapsto \sum_{j=1}^{n} x_j \mu(A_j),$$

is well defined and linear, and it has norm one. Hence $\mathcal{I}$ extends univocally to a norm-one linear operator on $L_1(\mu, X)$. The *Bochner integral* of $f$ is defined as

$$\int_{\Omega} f(\omega)\, d\mu(\omega) = \mathcal{I}(f), \quad f \in L_1(\mu, X).$$

The linearity and boundedness of this integral are immediate from the definition,

- $\int_\Omega (c_1 f + c_2 g)\, d\mu = c_1 \int_\Omega f\, d\mu + c_2 \int_\Omega g\, d\mu$, for every $f, g \in L_1(\mu, X)$, and scalars $c_1, c_2$;
- $\| \int_\Omega f\, d\mu \| \leq \int_\Omega \|f\|\, d\mu$ for every $f \in L_1(\mu, X)$.

The stability under composition with linear and bounded operators is also clear. The usual results about taking limits (and derivatives) under the integral sign are derived from the dominated convergence theorem, so that they remain valid. Finally, we show that the Lebesgue differentiation theorem also works.

**K.5. Lebesgue Differentiation Theorem for the Bochner Integral.** *Let $f\colon \mathbb{R}^n \to X$ be a strongly measurable function with*

$$\int_K \|f(\xi)\|_X\, d\xi < \infty$$

*for all $K \subset \mathbb{R}^n$ compact. Then the set of Lebesgue points of $f$, i.e., the set of $x \in \mathbb{R}^n$ such that*

$$\lim_{\delta \to 0+} \delta^{-n} \int_{|\xi| \leq \delta} \|f(x+\xi) - f(x)\|\, d\xi = 0,$$

*is the complement of a zero-measure set.*

*Proof.* Assume, without loss of generality, that $f(\mathbb{R}^n)$ is separable and pick $Z \subset X$ countable such that $f(\mathbb{R}^n)$ is contained in the closure of $Z$. Appealing to the Lebesgue differentiation theorem in the scalar case, the set of points $x \in \mathbb{R}^n$ for which

$$\lim_{\delta \to 0+} \delta^{-n} \int_{|\xi| \leq \delta} \|f(\xi + x) - z\|\, d\xi = \|f(x) - z\|, \qquad \text{for all } z \in Z,$$

is the complement of a zero-measure set. For $x$ in the above set we have

$$\limsup_{\delta \to 0+} \delta^{-n} \int_{|\xi| \leq \delta} \|f(x+\xi) - f(x)\|\, d\xi \leq 2\|f(x) - z\|, \qquad \text{for all } z \in Z.$$

Taking the infimum in $z \in Z$, we are done. □

# List of Symbols

## Blackboard Bold Symbols

| | |
|---|---|
| $\mathbb{N}$ | The natural numbers |
| $\mathbb{Q}$ | The rational numbers |
| $\mathbb{R}$ | The real numbers |
| $\mathbb{C}$ | The complex numbers |
| $\mathbb{T}$ | The unit circle in the complex plane, $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$ |
| $\mathbb{P}$ | A probability measure on some probability space $(\Omega, \Sigma, \mathbb{P})$ (Section 6.2) |
| $\mathbb{E}f$ | The expectation of a random variable $f$ (Section 6.2) |
| $\mathbb{E}(f \mid \Sigma')$ | The conditional expectation of $f$ on the $\sigma$-algebra $\Sigma'$ (Section 6.1) |

## Classical Banach Spaces

| | |
|---|---|
| $L_\infty(\mu)$ | The (equivalence class) of $\mu$-measurable essentially bounded real-valued functions $f$ with the norm $\lVert f\rVert_\infty := \inf\{\alpha > 0 : \mu(\lvert f\rvert > \alpha) = 0\}$ |
| $L_p(\mu)$ | The (equivalence class) of $\mu$-measurable real-valued functions $f$ such that $\lVert f\rVert_p := (\int \lvert f\rvert^p \, d\mu)^{1/p} < \infty$ |
| $L_p(\mathbb{T})$ | $L_p(\mu)$ when $\mu$ is the normalized Lebesgue measure on $\mathbb{T}$ |
| $L_p$ | $L_p(\mu)$ when $\mu$ is the Lebesgue measure on [0, 1] |
| $\mathcal{C}(K)$ | The continuous real-valued functions on the compact space $K$ |
| $\mathcal{C}_{\mathbb{C}}(K)$ | The continuous complex-valued functions on the compact space $K$ |
| $\mathcal{J}$ | The James space (Section 3.4) |
| $\mathcal{T}$ | Tsirelson's space (Section 11.3) |
| $\mathcal{JT}$ | The James tree space (Section 15.4) |
| $\mathcal{M}(K)$ | The finite regular Borel signed measures on the compact space $K$ |



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| | |
|---|---|
| $\ell_\infty$ | The collection of bounded sequences of scalars $x = (x_n)_{n=1}^\infty$, with the norm $\|x\|_\infty = \sup_n |x_n|$ |
| $\ell_\infty^n$ | $\mathbb{R}^n$ equipped with the $\|\cdot\|_\infty$ norm |
| $\ell_p$ | $L_p(\mu)$ when $\mu$ is the *counting measure* on $\mathcal{P}(\mathbb{N})$, that is, the measure defined by $\mu(A) = |A|$ for any $A \subset \mathbb{N}$. Equivalently, the collection of all sequences of scalars $x = (x_n)_{n=1}^\infty$ such that $\|x\|_p := (\sum_{n=1}^\infty |x_n|^p)^{1/p} < \infty$ |
| $\ell_p^n$ | $\mathbb{R}^n$ equipped with the $\|\cdot\|_p$ norm |
| $c$ | The convergent sequences of scalars under the $\|\cdot\|_\infty$ norm |
| $c_0$ | The sequences of scalars that converge to 0 endowed with the $\|\cdot\|_\infty$ norm |
| $c_{00}$ | The (dense) subspace of $c_0$ of finitely nonzero sequences |

## Important Constants

| | |
|---|---|
| $\mathsf{C}_{\mathsf{ag}}$ | The almost-greedy constant of an almost-greedy basis (Section 10.5) |
| $\mathsf{C}_{\mathsf{b}}$ | The bidemocracy constant of a bidemocratic basis (Section 10.6) |
| $\mathsf{C}_{\mathsf{d}}$ | The democracy constant of a democratic basis (Section 10.3) |
| $\mathsf{C}_{\mathsf{g}}$ | The greedy constant of a greedy basis (Section 10.4) |
| $C_q(X)$ | The cotype-$q$ constant of the Banach space $X$ (Section 6.2) |
| $\mathsf{C}_{\mathsf{qg}}$ | The quasi-greedy constant of a quasi-greedy basis (Section 10.2) |
| $\mathsf{K}_{\mathsf{b}}$ | The basis constant of a Schauder basis (Section 1.1) |
| $K_G$ | The best constant in Grothendieck's inequality (Section 8.1) |
| $\mathsf{K}_{\mathsf{s}}$ | The symmetric constant of a symmetric basis (Section 9.2) |
| $\mathsf{K}_{\mathsf{su}}$ | The suppression-unconditional constant of an unconditional basis (Section 3.1) |
| $\mathsf{K}_{\mathsf{u}}$ | The unconditional basis constant of an unconditional basis (Section 3.1) |
| $T_p(X)$ | The type-$p$ constant of the Banach space $X$ (Section 6.2) |

## Operator-Related Symbols

| | |
|---|---|
| $I_X$ | The identity operator on $X$ |
| $j$ (or $j_X$) | The canonical embedding of $X$ into its second dual $X^{**}$ |
| $\iota_X$ | The natural injection of a Banach space $X$ into its ultrapower $X_{\mathcal{U}}$ |
| $i_X$ | The natural isometric embedding of a Banach space $X$ into $\mathcal{C}(B_{X^*})$ (see inside the proof of Theorem 1.4.4) |
| $\ker(T)$ | The null space of $T$; that is, $T^{-1}(0)$ |
| $S_N$ | The $N$th partial sum projection associated to a Schauder basis (Section 1.1) |

| | |
|---|---|
| $P_A$ | The (linear and bounded) projection associated to an unconditional basis $(e_n)_{n=1}^{\infty}$ onto the closed subspace $[e_n: n \in A]$ |
| $T^*$ | The adjoint operator of $T$ |
| $T^2$ | The composition operator of $T$ with itself, $T \circ T$ |
| $\langle x, x^* \rangle$ | The action of a functional $x^*$ in $X^*$ on a vector $x \in X$, also represented by $x^*(x)$ |
| $T(X)$ | The range (or image) of an operator $T$ defined on $X$ |
| $T\|_E$ | The restriction of the operator $T$ to a subspace $E$ of the domain space |
| $\pi_p(T)$ | The $p$-absolutely summing norm of $T$ (Section 8.2) |
| $\mathcal{B}(X, Y)$ | The space of bounded linear operators $T: X \to Y$ |
| $\mathcal{K}(X, Y)$ | The space of compact operators $T: X \to Y$ |

## Distinguished Sequences of Functions

| | |
|---|---|
| $(h_n)_{n=1}^{\infty}$ | The Haar system (Section 6.1) |
| $(h_n^p)_{n=1}^{\infty}$ | The normalized Haar system in $L_p[0, 1]$ (Section 10.4) |
| $(r_n)_{n=1}^{\infty}$ | The Rademacher functions (Section 6.3) |
| $(\varepsilon_n)_{n=1}^{\infty}$ | A Rademacher sequence (Section 6.2) |

## Several Types of Derivatives

| | |
|---|---|
| $f'(t)$ | The derivative of a function $f$ of a real variable at a point $t$ |
| $D_f(x)$ | The Gâteaux or Fréchet derivative of a function $f: X \to Y$ between Banach spaces at a point $x \in X$ (Section 14.2.3) |
| $\nabla f(x)$ | The gradient of a function $f$ defined on $\mathbb{R}^n$ at a point $x$, i.e., $\nabla f(x) = (\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x))$, where $\frac{\partial f}{\partial x_i}(x) = D_f(x)(e_i)$ for $i = 1, \dots, n$ are the derivatives of $f$ at $x$ in the direction of the vectors of the canonical basis $e_i$ |
| $D_f^*(x)$ | The weak* derivative of a function $f: X \to Y$ between Banach spaces at a point $x$ (Section 14.2) |
| $D_{\\|\cdot\\|}(x)$ | The derivative of a norm $\\|\cdot\\|$ at a point $x$ (Section 14.4) |
| $\Omega_{\\|\cdot\\|}$ | The set of differentiability points of a norm $\\|\cdot\\|$ on a finite-dimensional space (Section 14.4) |

## Sets and Subspaces

| | |
|---|---|
| $B_X$ | The closed unit ball of a normed space $X$, i.e., $\{x \in X: \\|x\\| \leq 1\}$ |
| $\langle A \rangle$ | The linear span of a set $A$ |

| | |
|---|---|
| $[A]$ | The closed linear span of a set $A$; i.e., the norm-closure of $\langle A\rangle$ |
| $[x_n]$ | The norm-closure of $\langle x_n : n \in \mathbb{N}\rangle$ |
| $\overline{S}$ or $\overline{S}^{\|\cdot\|}$ | The closure of a set $S$ of a Banach space in its norm topology |
| $\overline{S}^{w}$ or $\overline{S}^{\text{weak}}$ | The closure of a set $S$ of a Banach space in its weak topology |
| $\overline{S}^{w^*}$ or $\overline{S}^{\text{weak}^*}$ | The closure of a set $S$ of a dual space in its weak* topology |
| $M^{\perp}$ | The annihilator of $M$ in $X^*$, i.e., the collection of all continuous linear functionals on the Banach space $X$ that vanish on the subset $M$ of $X$ |
| $\partial_e(S)$ | The set of extreme points of a convex set $S$ |
| $\tilde{A}$ or $X \setminus A$ | The complement of $A$ in $X$ |
| $\mathcal{P}A$ | The collection of all subsets of a (usually infinite) set $A$ |
| $\mathcal{P}_\infty A$ | The collection of all infinite subsets of a set $A$ |
| $\mathcal{F}A$ | The collection of all finite subsets of a set $A$ |
| $\mathcal{F}_r A$ | The collection of all finite subsets of a set $A$ of cardinality $r$ |
| $S_X$ | The unit sphere of a normed space $X$, i.e., $\{x \in X : \|x\| = 1\}$ |

## Abbreviations for Properties

| | |
|---|---|
| (BAP) | Bounded approximation property (Problems section of Chapter 1) |
| (DPP) | Dunford–Pettis property (Section 5.4) |
| (KMP) | Krein–Milman property (Section 5.5) |
| (MAP) | Metric approximation property (Problems section of Chapter 1) |
| (RNP) | Radon–Nikodym property (Section 5.5) |
| (u) | Pełczyński's property ($u$) (Section 3.5) |
| (UTAP) | Uniqueness of unconditional basis up to a permutation (Section 9.3) |
| wsc | Weakly sequentially complete space (Section 2.3) |
| (WUC) | Weakly unconditionally Cauchy series (Section 2.4) |

## Miscellaneous

| | |
|---|---|
| $\operatorname{sgn} x$ | $= \begin{cases} x/\|x\| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$ |
| $\lfloor x \rfloor$ (or $[x]$) | $= \max\{k \in \mathbb{Z} : k \leq x\}$ |
| $\lceil x \rceil$ | $= \min\{k \in \mathbb{Z} : x \leq k\}$ |
| $\chi_A$ | The characteristic function of a set $A$, $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A \end{cases}$ |
| $(a_n) \lesssim (b_n)$ | $a_n \leq C b_n \ \ \forall n \in \mathbb{N}$, for some nonnegative constant $C$ |
| $(a_n) \approx (b_n)$ | $c a_n \leq b_n \leq C a_n \ \ \forall n \in \mathbb{N}$, for some nonnegative constants $c, C$ |

| | |
|---|---|
| $X \approx Y$ | $X$ isomorphic to $Y$ |
| $\lvert \cdot \rvert$ | The absolute value of a real number, the modulus of a complex number, the cardinality of a finite set, or the Lebesgue measure of a set, depending on the context |
| $\delta_s$ | The Dirac measure at the point $s$, whose value at $f \in \mathcal{C}(K)$ is $\delta_s(f) = f(s)$ |
| $\delta_{jk}$ | The Kronecker delta: $\delta_{jk} = 1$ if $j = k$, and $\delta_{jk} = 0$ if $j \neq k$ |
| $X \oplus Y$ | Direct sum of $X$ and $Y$ |
| $X^2$ | $= X \oplus X$ |
| $\ell_p(X_n)$ | $= (X_1 \oplus X_2 \oplus \cdots)_p$, the infinite direct sum of the sequence of spaces $(X_n)_{n=1}^{\infty}$ in the sense of $\ell_p$ (Section 2.2) |
| $c_0(X_n)$ | $= (X_1 \oplus X_2 \oplus \cdots)_0$, the infinite direct sum of the sequence of spaces $(X_n)_{n=1}^{\infty}$ in the sense of $c_0$ (Section 2.2) |
| $\ell_\infty^n(X)$ | $= (X \oplus \cdots \oplus X)_\infty$, i.e., the space of all sequences $x = (x_1, \dots, x_n)$ such that $x_k \in X$ for $1 \leq k \leq n$, with the norm $\lVert x \rVert = \sup_{1 \leq k \leq n} \lVert x_k \rVert_X$ |
| $\ell_\infty(X_i)_{i\in\mathcal{I}}$ | The Banach space of all $(x_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} X_i$ such that $(\lVert x_i \rVert)_{i\in\mathcal{I}}$ is bounded, with the norm $\lVert (x_i)_{i\in\mathcal{I}} \rVert_\infty = \sup_{i\in\mathcal{I}} \lVert x_i \rVert_{X_i}$ |
| $d(x, A)$ | The distance from a point $x$ to the set $A$ in a normed space: $\inf_{a\in A} \lVert x - a \rVert$ |
| $d(X, Y)$ | The Banach–Mazur distance between two isomorphic Banach spaces $X$, $Y$ (Section 7.4) |
| $d_X$ | The Euclidean distance of $X$ (Equation (7.23)) |
| $\mathcal{E}$ | An ellipsoid in a finite-dimensional normed space (Section 13.1) |
| $\Delta$ | The Cantor set (Section 1.4) |

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